1. (7 pts) Express the multinomial coefficient $\binom{10}{4,3,2,1}$ as a product of binomial coefficients.

$\binom{10}{4}\binom{6}{3}\binom{3}{2}$. Other factorizations are possible.

2. (8 pts ea.) (a) Show that the number of $(n+1)$-element multisets with elements in $\{1,\dots,k+1\}$, and containing each element with multiplicity at least one, is equal to $\binom{n}{k}$.

It's the same as the number of $(n-k)$-element multisets from $[k+1]$, given by $\left\langle {k+1 \atop n-k} \right\rangle = \binom{n-k+k+1-1}{n-k} = \binom{n}{n-k} = \binom{n}{k}$.

(b) Find a closed-form expression (depending on k) for the ordinary generating function

$$\sum_{n=0}^{\infty} \binom{n}{k} x^n.$$

Part (a) implies that $\displaystyle\sum_{n=0}^{\infty} \binom{n}{k} x^{n+1} = \frac{x^{k+1}}{(1-x)^{k+1}}$.

Hence $\displaystyle\sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}$.

3. (8 pts) Prove that in every set X of 50 integers between 1 and 99, there are two elements $x, y \in X$ such that $x + y = 100$.

Let $Y = \{100 - x : x \in X\}$. Then Y is also a set of 50 elements from $[99]$. Hence $X \cap Y \neq \phi$, i.e. we can find $x \in X$ and $z = 100 - y \in Y$ s.t. $x = z$, or $x + y = 100$.

$(y \in X)$

4. (8 pts) Prove the identity

$$k\, S(n,k) = \sum_{m=0}^{n-1} \binom{n}{m} S(m, k-1).$$

The left-hand side counts partitions of $[n]$ into k blocks, with one block distinguished. On the right-hand side, we can count the same thing as follows:

- Select a size, $n-m$, for the distinguished block (so we have m elements in the remaining blocks)
- Choose the distinguished block, in $\binom{n}{n-m} = \binom{n}{m}$ ways
- Partition the remaining m elements into $k-1$ blocks, in $S(m, k-1)$ ways.

Note that the distinguished block must not be empty, so the allowed choices for m are 0 to $n-1$. (Actually, m must be at least $k-1$, but this is taken care of by the fact that $S(m, k-1) = 0$ for $m < k-1$.)

5. (8 pts ea.) (a) Let $q_2(n)$ denote the number of partitions of n in which each part is repeated at most twice. Find a formula for the ordinary generating function

$$\sum_{n=0}^{\infty} q_2(n)x^n$$

$$Q_2(x) = \prod_{i=1}^{\infty} (1+x^i+x^{2i})$$

(b) Show that $q_2(n)$ is equal to the number of partitions of n with no parts divisible by 3.

$$\prod_{i=1}^{\infty} (1+x^i+x^{2i}) = \prod_{i=1}^{\infty} \frac{(1-x^{3i})}{(1-x^i)} = \prod_{\substack{i=1 \\ i \text{ not divisible} \\ \text{by } 3}}^{\infty} \frac{1}{(1-x^i)},$$

and the last expression is the generating function for partitions with no parts divisible by 3.

6. (8 pts) Let c_n be the number of permutations σ of $\{1, \ldots, n\}$ such that σ^3 is the identity permutation. Find the exponential generating function

$$\sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$$

We have $\sigma^3 = 1$ iff all cycles of σ are length 1 or 3. There are two 3-cycles on a 3-element set, so the E.G.F. for 1 or 3 cycles is $x + \frac{2x^3}{3!} = x + \frac{x^3}{3}$.

Then the composition principle gives

$$\sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = e^{(x+x^3/3)}.$$

7. (8 pts) Let $f(n,k)$ be the number of labelled rooted forests with vertex set $\{1,\dots,n\}$ and k connected components (i.e., the forest consists of k disjoint rooted trees). Find a closed-form expression (depending on n) for the ordinary generating function

$$\sum_k f(n,k)x^k$$

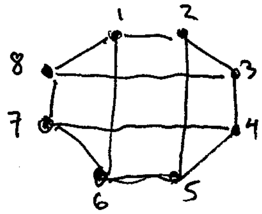
Attaching an extra vertex 0 to the roots, we see that $f(n,k)$ is also the number of spanning trees on $\{0,1,\dots,n\}$ in which vertex 0 has degree k.

In Cayley's formula, $\displaystyle\sum_{\substack{T \text{ on} \\ \{0,\dots,n\}}} x_0^{d_0} x_1^{d_1}\cdots x_n^{d_n} = x_0 x_1 \cdots x_n (x_0+\dots+x_n)^{n-1}$,

Set $x_0 = x$, $x_i = 1$ for $i \neq 0$, to get

$$\sum_k f(n,k)x^k = x(x+n)^{n-1}.$$

8. (8 pts) Let G be the simple graph on vertex set $\{1,\dots,8\}$ in which i and j are connected by an edge if and only if $i-j$ is odd. Express the number of spanning trees in G in terms of the determinant of some matrix. Write out your matrix explicitly, but you need not evaluate the determinant.



Cross out one row and column from $3I_8 - A_G$ to get

$$M = \begin{pmatrix} 3 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 & 3 \end{pmatrix}$$

Then the number of spanning trees in G is $\det(M)$, by the matrix-tree theorem.

9. (7 pts ea.) We define a *binary tree* to be an unordered rooted tree in which every vertex has either 0 or 2 children. Let B_n be the set of labelled binary trees with vertex set $\{1, \ldots, n\}$, and let $b_n = |B_n|$ be the number of them.

(a) Find an equation whose solution is the exponential generating function

$$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

You need not solve for $B(x)$.

The structure of the trees and the product and composition principles give $B(x) = x\left(1 + B(x)^2/2\right)$, since the E.G.F.'s for trivial one-element set and trivial 0 or 2 element set are x, and $1 + x^2/2$.

(b) Let $\widehat{Z}_B(p_1, p_2, \ldots)$ be the reduced cycle generating function for binary trees, i.e.,

$$\widehat{Z}_B(p_1, p_2, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} |B_n^\sigma| p_1^{c_1(\sigma)} \cdots p_k^{c_k(\sigma)},$$

where B_n^σ is the set of labelled binary trees fixed by the permutation σ, and $c_i(\sigma)$ denotes the number of i-cycles in σ.

Show that

$$\widehat{Z}_B(p_1, p_2, \ldots) = p_1 \left(1 + \frac{\widehat{Z}_B(p_1, p_2, \ldots,)^2 + \widehat{Z}_B(p_2, p_4, \ldots)}{2}\right).$$

The reduced cycle G.F.'s are
trivial 1-element set: p_1
trivial 0 or 2-element set: $1 + \frac{1}{2}(p_1^2 + p_2)$.

Therefore $\hat{Z}_B = p_1\left(\left(1 + \frac{p_1^2 + p_2}{2}\right) * \hat{Z}_B\right)$, which is the same as the identity above.

(c) Let u_n be the number of unlabelled binary trees with n vertices. Find an equation whose solution is the ordinary generating function

$$U(x) = \sum_{n=0}^{\infty} u_n x^n.$$

You need not solve for $U(x)$.

We have $U(x) = \hat{Z}_B(x, x^2, \ldots)$. Setting $p_k = x^k$ in (b), this gives $U(x) = x\left(1 + \frac{U(x)^2 + U(x^2)}{2}\right)$.

(Notice that part (a) also comes from (b) with $p_1 = x$, $p_k = 0$ for $k > 0$.)