1. (7 pts) Express the multinomial coefficient \( \binom{10}{4,3,2,1} \) as a product of binomial coefficients.

\[
\binom{10}{4} \binom{6}{3} \binom{3}{2} . \quad \text{Other factorizations are possible.}
\]

2. (8 pts ea.) (a) Show that the number of \((n+1)\)-element multisets with elements in \(\{1, \ldots, k+1\}\), and containing each element with multiplicity at least one, is equal to \(\binom{n}{k}\).

It's the same as the number of \((n-k)\)-element multisets from \([1, \ldots, k+1]\), given by \(\binom{k+1}{n-k} = \binom{n-k+k+1-1}{n-k} = \binom{n-k}{n-k} = \binom{n}{k}\).

(b) Find a closed-form expression (depending on \(k\)) for the ordinary generating function

\[
\sum_{n=0}^{\infty} \binom{n}{k} x^n .
\]

Part (a) implies that

\[
\sum_{n=0}^{\infty} \binom{n}{k} x^{n+1} = \frac{x^{k+1}}{(1-x)^{k+1}} .\]

Hence

\[
\sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}} .
\]
3. (8 pts) Prove that in every set $X$ of 50 integers between 1 and 99, there are two elements $x, y \in X$ such that $x + y = 100$.

Let $Y = \{100-x : x \in X\}$. Then $Y$ is also a set of 50 elements from $[99]$. Hence $X \cap Y \neq \emptyset$, i.e. we can find $x \in X$ and $y = 100 - x \in Y$ s.t. $x + y = 100$.

4. (8 pts) Prove the identity

$$k S(n, k) = \sum_{m=0}^{n-1} \binom{n}{m} S(m, k-1).$$

The left-hand side counts partitions of $[n]$ into $k$ blocks, with one block distinguished. On the right-hand side, we can count the same thing as follows:

- Select a size, $n-m$, for the distinguished block (so we have $m$ elements in the remaining blocks)
- Choose the distinguished block, in $\binom{n}{m}$ ways
- Partition the remaining $m$ elements into $k-1$ blocks,
  in $S(m, k-1)$ ways.

Note that the distinguished block must not be empty, so the allowed choices for $m$ are 0 to $n-1$. (Actually, $m$ must be at least $k-1$, but this is taken care of by the fact that $S(m, k-1) = 0$ for $m < k-1$.)
5. (8 pts ea.) (a) Let $q_2(n)$ denote the number of partitions of $n$ in which each part is repeated at most twice. Find a formula for the ordinary generating function

$$\sum_{n=0}^{\infty} q_2(n)x^n$$

$$Q_2(x) = \prod_{i=1}^{\infty} \frac{1}{(1+x^i+x^{2i})}$$

(b) Show that $q_2(n)$ is equal to the number of partitions of $n$ with no parts divisible by 3.

3. 

$$\prod_{i=1}^{\infty} \frac{1}{(1+x^i+x^{2i})} = \prod_{i=1}^{\infty} \frac{1}{(1-x^{3i})} = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)} \quad \text{if not divisible by 3}$$

and the best expression is the generating function for partitions with no parts divisible by 3.

6. (8 pts) Let $c_n$ be the number of permutations $\sigma$ of $\{1, \ldots, n\}$ such that $\sigma^3$ is the identity permutation. Find the exponential generating function

$$\sum_{n=0}^{\infty} \frac{c_n x^n}{n!}$$

We have $\sigma^3 = 1$ iff all cycles of $\sigma$ are length 1 or 3. There are two 3-cycles on a 3-element set, so the E.G.F. for 1 or 3 cycles is

$$x + \frac{2x^3}{3!} = x + \frac{x^3}{3}.$$ 

Then the composition principle gives

$$\sum_{n=0}^{\infty} \frac{c_n x^n}{n!} = e^{(x+x^3/3)}.$$
7. (8 pts) Let \( f(n, k) \) be the number of labelled rooted forests with vertex set \( \{1, \ldots, n\} \) and \( k \) connected components (i.e., the forest consists of \( k \) disjoint rooted trees). Find a closed-form expression (depending on \( n \)) for the ordinary generating function

\[
\sum_k f(n, k) x^k
\]

Attaching an extra vertex 0 to the root, we see that

\( f(n, k) \) is also the number of spanning trees on \( \{0, 1, \ldots, n\} \) in which vertex 0 has degree \( k \).

In Cayley's formula, \( \sum_{T \in \text{trees}} x_0 x_1 \cdots x_n = x_0 x_1 \cdots x_n (x_0 + \cdots + x_n)^{n-1} \),

set \( x_0 = x \), \( x_i = 1 \) for \( i \neq 0 \), to get

\[
\sum_k f(n, k) x^k = x (x+n)^{n-1}.
\]

8. (8 pts) Let \( G \) be the simple graph on vertex set \( \{1, \ldots, 8\} \) in which \( i \) and \( j \) are connected by an edge if and only if \( i - j \) is odd. Express the number of spanning trees in \( G \) in terms of the determinant of some matrix. Write out your matrix explicitly, but you need not evaluate the determinant.

![Graph](image)

Cross out one row and column from

\[3I_8 - A_G\]

to get

\[
M = \begin{pmatrix}
3 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\
-1 & 3 & -1 & 0 & -1 & 0 & 1 & 0 \\
0 & -1 & 3 & -1 & 0 & -1 & 0 & 1 \\
-1 & 0 & -1 & 3 & -1 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 & 3 & -1 & 0 & 1 \\
-1 & 0 & -1 & 0 & -1 & 3 & -1 & 0 \\
0 & -1 & 0 & -1 & 0 & -1 & 3 & -1 \\
0 & -1 & 0 & -1 & 0 & -1 & 3 & -1 \\
\end{pmatrix}
\]

Then the number of spanning trees is \( \det(M) \), by the matrix-tree theorem.
9. (7 pts ea.) We define a binary tree to be an unordered rooted tree in which every vertex has either 0 or 2 children. Let \( B_n \) be the set of labelled binary trees with vertex set \( \{1, \ldots, n\} \), and let \( b_n = |B_n| \) be the number of them.

(a) Find an equation whose solution is the exponential generating function

\[
B(x) = \sum_{n=0}^{\infty} \frac{b_n x^n}{n!}
\]

You need not solve for \( B(x) \).

The structure of the trees and the composition principles give

\[
B(x) = x \left( 1 + B(x^2)/2 \right),
\]

since the E.C.F.'s for trivial one-element set and trivial 0 or 2 element set are \( x \) and \( 1 + x^2/2 \).

(b) Let \( \hat{Z}_B(p_1, p_2, \ldots) \) be the reduced cycle generating function for binary trees, i.e.,

\[
\hat{Z}_B(p_1, p_2, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} |B_n^\sigma| p_1^{c_1(\sigma)} \cdots p_k^{c_k(\sigma)},
\]

where \( B_n^\sigma \) is the set of labelled binary trees fixed by the permutation \( \sigma \), and \( c_i(\sigma) \) denotes the number of \( i \)-cycles in \( \sigma \).

Show that

\[
\hat{Z}_B(p_1, p_2, \ldots) = p_1 \left( 1 + \frac{\hat{Z}_B(p_1, p_2, \ldots) + \hat{Z}_B(p_2, p_4, \ldots)}{2} \right).
\]

The reduced cycle G.F.'s are

trivial 1-element set: \( p_1 \)

trivial 2-element set: \( 1 + \frac{1}{2} (p_1^2 + p_2^2) \).

Therefore \( \hat{Z}_B = p_1 \left( \frac{1 + p_1^2 + p_2^2}{2} \right) \hat{Z}_B \), which is the same as the identity above.

(c) Let \( u_n \) be the number of unlabelled binary trees with \( n \) vertices. Find an equation whose solution is the ordinary generating function

\[
U(x) = \sum_{n=0}^{\infty} u_n x^n.
\]

You need not solve for \( U(x) \).

We have \( U(x) = \hat{Z}_B(x, x^2, \ldots) \). Setting \( p_k = x^k \) in (a), this gives

\[
U(x) = x \left( 1 + \frac{UX^2 + U(x^2)}{2} \right).
\]

(Notice that part (a) also comes from (c) with \( p_1 = x \), \( p_k = 0 \) for \( k > 0 \)).