1. (10 points each) For each of the following counting problems, express the answer in terms of factorials, falling powers, binomial, multinomial and multichoose coefficients, Stirling numbers (of either kind) and/or partition numbers. Write answers in an unsimplified form which suggests how you derived them. No further explanation in words is required.

(a) Find the number of compositions of the integer 50 into 4 parts, in which each part is greater than or equal to 5.

$$\left\langle {4 \atop 30} \right\rangle \quad \text{or} \quad \binom{33}{30} \quad \text{or} \quad \binom{33}{3}$$

$$\uparrow$$

$$30 = 50 - 4 \cdot 5$$

(b) Find the number of rearrangements of the letters in the word REARRANGING.

$$\binom{11}{3, 2, 2, 2, 1, 1}$$

(c) Find the number of partitions of the set $\{1, 2, \ldots, 10\}$ into 5 blocks, with the property that 1 and 10 are in different blocks.

$$S(10, 5) - S(9, 5)$$

$$\uparrow$$

If 1 and 10 are in the same block we can treat them as if they were a single element.

2. (20 points) Let f be a function from $X = \{1, 2, \ldots, 10\}$ to $Y = \{1, 2, 3, 4\}$. Prove that there must be three (distinct) elements $x, y, z \in X$ such that $f(x) + f(y) + f(z)$ is divisible by 3.

By generalized pigeonhole principle, since $10 > 2 \cdot 4$, there must be x, y, z such that $f(x) = f(y) = f(z)$.
Then $f(x) + f(y) + f(z) = 3f(x)$.

3. (20 points) Recall that $c(n, k)$ denotes the number of permutations of an n-element set with k cycles (signless Stirling number of the first kind). Prove the identity

$$k\, c(n, k) = \sum_{j=1}^{n} \binom{n}{j} (j-1)!\, c(n-j, k-1).$$

The left hand side counts permutations of $[n]$ with k cycles, and one of the cycles distinguished.
On the right, count the same thing by first selecting the number of elements $j > 0$ for the distinguished cycle. For each j we have $\binom{n}{j}$ ways to choose the elements in the distinguished cycle, $(j-1)!$ ways to arrange them into a cycle, and $c(n-j, k-1)$ ways to make the remaining elements into a permutation with $k-1$ cycles.

4. (15 points each) (a) Let $g(n)$ be the number of compositions of n with parts in the set $\{1, 2, 3\}$. Find a formula for the generating function

$$G(x) = \sum_{n=0}^{\infty} g(n)\, x^n.$$

By the sequence principle,

$$G(x) = \frac{1}{1-(x+x^2+x^3)}$$

(b) Let $h(n)$ be the number of partitions of n with parts in the set $\{1, 2, 3\}$. Find a formula for the generating function

$$H(x) = \sum_{n=0}^{\infty} h(n)\, x^n.$$

We can choose a partition $\lambda = (3^{m_3}, 2^{m_2}, 1^{m_1})$ by choosing m_1, m_2 and m_3 independently. This gives

$$H(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)}$$