Problem Set 8

The final exam is Thursday, Aug. 13. It covers the whole course, with about one half of it on material new since Midterm 2. I am also posting a Review Guide with more details and review exercises.

This assignment is due on Thursday, Aug. 13, at the final exam.

It covers class material for Aug 10–12 and reading from the Notes on Finite Fields, and parts of 9.3–5 and 10.1, 10.3, 10.5–6, as indicated in the 'Guide to Field Theory' notes.

Finite fields:

(a) Verify that $p^2 \equiv 1 \pmod 8$ for every odd integer p.

(b) Use the fact that $F^\times$ is cyclic for every finite field F to show that $\mathbb{F}(p^2)^\times$ contains an element α of order 8 for every odd prime p.

(c) Prove that if $\alpha \in \mathbb{F}(p^2)^\times$ has order 8, then α is a root of $x^4 + 1$.

(d) Deduce that $x^4 + 1$ is reducible in $\mathbb{Z}_p[x]$ for every odd prime p. Hint: let $f(x) \in \mathbb{Z}_p[x]$ be the minimal polynomial of the root α found in (c), and observe that the degree of $f(x)$ is at most 2.

(e) Verify that $x^4 + 1$ is reducible in $\mathbb{Z}_2[x]$.

(f) Factor $x^4 + 1$ as a product of two quadratic polynomials in $\mathbb{R}[x]$. Hint: its complex roots are $e^{\pm\pi i/4}$ and $e^{\pm 3\pi i/4}$.

(g) Use (f) to prove that $x^4 + 1$ is irreducible in $\mathbb{Q}[x]$. This shows that $x^4 + 1$ is an example of a polynomial with integer coefficients that is irreducible over $\mathbb{Q}$, but reducible over $\mathbb{Z}_p$ for every prime p.

Section 9.3: Use the derivative test to show that the polynomial $12x^4 - 52x^3 + 103x^2 - 120x + 63$ has a multiple real root, and to find that root.

Chapter 9: 9.7.1(a). Note that $x^3 + 2x + 1$ has only one real root, so adjoining one root does not give the splitting field. What does this imply about the dimension of the splitting field L?

Section 10.1: 4