

Math 110 Homework 11 Solutions
Spring 2012

Exercises

Akler 7.22 : By Theorem 5.26, S has an eigenvalue, so let x be a non-zero eigenvector, say with eigenvalue λ , so $Sx = \lambda x$. Since S is an isometry, $\| \lambda x \| = \| x \|$, hence $|\lambda| = 1$, since $\| \lambda x \| = |\lambda| \cdot \| x \|$ and $\| x \| \neq 0$. Of course λ is real, so $\lambda = \pm 1$. Then $S^2 x = \lambda^2 x = x$.

7.26 Since T is self-adjoint, it is diagonalizable, with matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \lambda_n \end{pmatrix}$ in some orthonormal basis of V , whose λ_i are the eigenvalues (which are real). The matrix of $T^*T = T^2$ in the same basis is $\begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \lambda_n^2 \end{pmatrix}$, hence that of $\sqrt{T^*T}$ is $\begin{pmatrix} |\lambda_1| & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & |\lambda_n| \end{pmatrix}$. The eigenvalues of this last operator are the singular values of T .

7.27 False. Consider the operator $T \in L(\mathbb{R}^2)$ (using Euclidean inner product on \mathbb{R}^2) whose matrix in the standard basis is $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Its polar decomposition is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, since the operator S with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an isometry, and the operator P with matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ is positive. Hence the singular values of T are 2 and 1. But $T^2 = 2I$, which has singular values 2, 2, not 4, 1.

7.30 In the polar decomposition $T = SP$, the singular values are all = 1 iff $P = I$ iff $T = S$, i.e. T is an isometry.

7.31 Suppose T_1 and T_2 have the same singular values.

Then there are orthonormal bases $\underline{u}, \underline{v}, \underline{y}, \underline{z}$ s.t.

$$m(T_1, \underline{u}, \underline{v}) = m(T_2, \underline{y}, \underline{z}) = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}.$$

Let S_1 be defined by $S_1(\underline{z}_i) = \underline{v}_i$, and S_2 by $S_2(\underline{u}_i) = \underline{y}_i$.

Since all these bases are orthonormal, S_1 and S_2 are isometries.

We have

~~m($T_1 S_2 S_1^*, \underline{u}, \underline{v}$)~~

$$\begin{aligned} m(S_1 T_2 S_2^*, \underline{u}, \underline{v}) &= m(S_1, \underline{z}, \underline{v}) m(T_2, \underline{y}, \underline{z}) m(S_2, \underline{u}, \underline{y}) \\ &= I_n \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix} \cdot I_n \\ &= \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix} = m(T_1, \underline{u}, \underline{v}) \end{aligned}$$

This shows $T_1 = S_1 T_2 S_2$.

Conversely, if $T_1 = S_1 T_2 S_2$ then $T_1^* T_1 = S_2^* T_2^* S_1^* S_1 T_2 S_2$

$$= S_2^* T_2^* T_2 S_2 = S_2^* T_2^* T_2 S_2. \text{ This implies that}$$

$T_1^* T_1$ and $T_2^* T_2$ have the same eigenvalues. The singular values are their positive square roots, so T_1 and T_2 have the same singular values.

7.33 Let $\underline{e}, \underline{f}$ be orthonormal basis such that

$$m(T, \underline{e}, \underline{f}) = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}. \text{ If } v = a_1 \underline{e}_1 + \dots + a_n \underline{e}_n,$$

$$\text{then } \|v\|^2 = \sum |a_i|^2, \text{ and } \|Tv\|^2 = \sum s_i^2 |a_i|^2.$$

$$\text{Since all } s_i \leq s, \|Tv\|^2 \leq s^2 \sum |a_i|^2 = s^2 \|v\|^2, \text{ so } \|Tv\| \leq s \|v\|.$$

$$\text{Since all } s_i \geq \hat{s}, \|Tv\|^2 \geq \hat{s}^2 \sum |a_i|^2 = \hat{s}^2 \|v\|^2, \text{ so } \|Tv\| \geq \hat{s} \|v\|.$$

[Note that all inequalities here involve only non-negative real numbers, so we are free to take square roots.]

8.1 $T^2(w,z) = T(z,0) = (0,0)$ shows $T^2=0$. So every vector in \mathbb{C}^2 is a generalized eigenvector of T with eigenvalue 0. (But the only actual eigenvectors are scalar multiples of $(1,0)$).

8.2 T has eigenvalues $\lambda = \pm i$, with eigenspaces

$$\lambda = i : \text{span}((1,-i))$$

$$\lambda = -i : \text{span}((1,i)) .$$

Since T is diagonalizable, its generalized eigenspaces are the same as its eigenspaces.

8.3 Suppose $a_0 v + a_1 T v + \dots + a_{m-1} T^{m-1} v = 0$. If the a_i are not all 0, let j be the smallest index s.t. $a_j \neq 0$. Then

$$a_j T^j v + a_{j+1} T^{j+1} v + \dots + a_{m-1} T^{m-1} v = 0.$$

Apply T^{m-j-1} to both sides and notice that this kills all but the first term on the left hand side:

$$a_j T^{m-1} v = 0.$$

But $T^{m-1}v \neq 0$ by hypothesis, so this contradicts $a_j \neq 0$. Thus we must have all $a_i = 0$, so $(v, T v, \dots, T^{m-1} v)$ is linearly independent.

8.5 Suppose $(ST)^d = 0$. Then $(TS)^{d+1} = T(ST)^d S = 0$.

8.7 If N is nilpotent, then all of V is a generalized eigenspace for eigenvalue $\lambda=0$. Since N is self-adjoint, it's diagonalizable, so V is an ordinary eigenspace, i.e. it's the nullspace of N . Hence $N=0$.

8.10 A counterexample is the operator T in 8.1. Its range and its nullspace are both equal to $\text{span}((1,0))$.

8.11 In the generalized eigenspace decomposition, Thm. 8.23(a), $\text{null } T^n$ is the subspace U_0 corresponding to $\lambda=0$ (or 0 if T is invertible), and $\text{range } T^n$ is the sum of the other U_j 's.

8.12 If 0 is the only eigenvalue, then its generalized eigenspace (for $\text{IF} = \mathbb{C}$) is all of V , i.e. N is nilpotent. A counterexample for $\text{IF} = \mathbb{R}$ is the operator $T \in \mathcal{L}(\mathbb{R}^3)$ with matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Its complex eigenvalues (i.e., the e-vals of the operator in $\mathcal{L}(\mathbb{C}^3)$ with the same matrix) are $0, \pm i$. So its only real e-val is 0, but it's not nilpotent, since that would imply that its complex e-vals would all be 0.

8.14 The operator with matrix

$$\begin{pmatrix} 7 & & 0 \\ & 7 & 0 \\ 0 & & 8 \end{pmatrix}$$

(or any upper triangular matrix with two 7's and two 8's on the diagonal).

8.15 Since neither 5 nor 6 is the only eigenvalue, their multiplicities are $< n$. So the characteristic polynomial is $(z-5)^k(z-6)^l$ for some $k, l \leq n-1$. By Cayley-Hamilton, $(T-5I)^k(T-6I)^l = 0$, hence $(T-5I)^{n-1}(T-6I)^{n-1} = 0$.

8.31 It is block-diagonal with transposed Jordan blocks, of the form $\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & \cdots & \lambda \end{pmatrix}$.

Problems

1. (7.32) a) We'll check that the alleged formula for T^*v satisfies the property $\langle u, T^*v \rangle = \langle Tu, v \rangle$ for all u , which defines T^*v .

$$\begin{aligned}
 & \langle u, s_1 \langle v, f_1 \rangle e_1 + \dots + s_n \langle v, f_n \rangle e_n \rangle \\
 &= \bar{s}_1 \overline{\langle v, f_1 \rangle} \langle u, e_1 \rangle + \dots + \bar{s}_n \overline{\langle v, f_n \rangle} \langle u, e_n \rangle \\
 &= s_1 \langle f_1, v \rangle \langle u, e_1 \rangle + \dots + s_n \langle f_n, v \rangle \langle u, e_n \rangle \quad [\text{since } s_i \in \mathbb{R}] \\
 &= \langle s_1 \langle u, e_1 \rangle f_1 + \dots + s_n \langle u, e_n \rangle f_n, v \rangle \\
 &= \langle Tu, v \rangle.
 \end{aligned}$$

b) We'll apply T to the alleged formula for T^*v and see if we get v .

$$\begin{aligned}
 & T \left(\frac{\langle v, f_1 \rangle e_1}{s_1} + \dots + \frac{\langle v, f_n \rangle e_n}{s_n} \right) \\
 &= \frac{\langle v, f_1 \rangle T(e_1)}{s_1} + \dots + \frac{\langle v, f_n \rangle T(e_n)}{s_n}.
 \end{aligned}$$

Now since \underline{e} is orthonormal, $T(e_i) = s_i f_i$, so the above is equal to

$$\langle v, f_1 \rangle f_1 + \dots + \langle v, f_n \rangle f_n,$$

and since \underline{f} is orthonormal, this is v .

2. Let $g(z) = (z - \lambda_1) \cdots (z - \lambda_n)$ be the characteristic polynomial of T . Since T is invertible, the constant term $\lambda_1 \cdots \lambda_n$ of $g(z)$ is non-zero: $g(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$

where $a_0 \neq 0$.

By Cayley-Hamilton, $T^n + a_{n-1} T^{n-1} + \dots + a_0 I = 0$, hence

$$T(T^{n-1} + a_{n-1} T^{n-2} + \dots + a_1 I) = -a_0 I, \text{ which implies}$$

$$T^{-1} = -\frac{1}{a_0} (T^{n-1} + a_{n-1} T^{n-2} + \dots + a_1 I) = p(T),$$

$$\text{where } p(z) = -\frac{1}{a_0} (z^{n-1} + a_{n-1} z^{n-2} + \dots + a_1).$$

Strictly speaking, the argument we have just given only works if $\mathbb{F} = \mathbb{C}$. If $T \in \mathcal{L}(V)$, where V is a vector space of dimension n over \mathbb{R} , we can choose a basis of V , and apply the above to the operator T_C on \mathbb{C}^n with the same (real) matrix as T . For this we need to know that $q(z)$ has real coefficients if T_C has a real matrix in the standard basis of \mathbb{C}^n . This follows because the map $(z_1, \dots, z_n) \mapsto (\bar{z}_1, \dots, \bar{z}_n)$ will then send elements of $\text{null}((T - \lambda I)^m)$ into $\text{null}((T - \bar{\lambda} I)^m)$ and vice versa.

Hence every complex eigenvalue λ has the same multiplicity as $\bar{\lambda}$, which implies that the polynomial $q(z)$, whose roots are the eigenvalues with these multiplicities, has real coefficients.

[Full credit for this problem if you only solve it for $\mathbb{F} = \mathbb{C}$.]