Exercises

7.6 By the spectral theorem, \( V \) has an orthonormal basis consisting of eigenvectors \( e_i \) of \( T \), say \( T e_i = \lambda_i e_i \). Then \( \text{range}(T) = \text{span}(\{ e_i : \lambda_i \neq 0 \}) \). But \( T^* e_i = \overline{\lambda_i} e_i \), so \( \text{range}(T^*) = \text{span}(\{ e_i : \overline{\lambda_i} \neq 0 \}) = \text{range}(T) \).

7.7 We can describe \( \text{range}(T) \) as in 7.6, above. Then \( \text{range}(T^*) = \text{span}(\{ e_i : \lambda_i \neq 0 \}) = \text{range}(T) \). Similarly, \( \text{null}(T^*) = \text{span}(\{ e_i : \lambda_i = 0 \}) = \text{null}(T) \).

7.8 The conditions say that \( (1,2,3) \) and \( (2,5,7) \) are eigenvectors of \( T \) with eigenvalues \( 0, 1 \); in particular, with different eigenvalues. If \( T \) were self-adjoint, this would imply that the two vectors are orthogonal, but \( \langle (1,2,3), (2,5,7) \rangle = 33 \neq 0 \).

[It is understood that we are using the Euclidean inner product on \( \mathbb{R}^3 \).]

7.10 \( T \) is diagonalizable, by the spectral theorem, and its eigenvalues satisfy \( \lambda^8 = \lambda^9 \), i.e. \( \lambda^8 (\lambda - 1) = 0 \). Hence all eigenvalues are \( 0 \) or \( 1 \), so \( T \) is a projection and \( T^2 = T \). It's self-adjoint (in fact, an orthogonal projection) because the eigenvalues are real.

7.11 Diagonalize \( T \) and take for \( S \) any operator whose matrix is \( (\alpha_1, \ldots, \alpha_n) \) where the matrix of \( T \) is \( (\lambda_1, \ldots, \lambda_n) \) and \( \alpha_i^2 = \lambda_i \). This is possible, since every complex number has a square root.

7.13 True, by same argument as in 7.11 and using the fact that every real number has a cube root.

7.15 "If" follows from the spectral theorem. For "only if," let \( e_1, \ldots, e_n \) be a basis consisting of eigenvectors of \( T \) and define \( \langle u, v \rangle = \Sigma a_i \overline{b_i} \) where \( u = \Sigma a_i e_i \), \( v = \Sigma b_i e_i \), so that \( b \) becomes an \( L^2 \)-normal basis.
7.17 If $S, T$ are positive, then $(S + T)^* = S^* + T^* = S + T$, so $S + T$ is self-adjoint, and $\langle (S + T)v, v \rangle = \langle Sv, v \rangle + \langle Tv, v \rangle \geq 0$ for all $v$, so $S + T$ is positive.

7.18 $(T^k)^* = (T^k)^k = T^k$, so $T^k$ is self-adjoint. The eigenvalues $\lambda_i^k$ are $\lambda_i^k$, where $\lambda_i$ are the eigenvalues of $T$, so they are $\geq 0$, hence $T^k$ is positive.

7.19 If $\langle Tv, v \rangle > 0$ for all $v$, then null$(T) = 0$ (since $v \in$ null$(T)$ implies $\langle Tv, v \rangle = \langle 0, v \rangle = 0$), so $T$ is invertible. Since $T$ is positive, let $e_i$ be a basis of $V$ consisting of eigenvectors of $T$, say $Te_i = \lambda_i e_i$, where $\lambda_i > 0$ (by spectral theorem and definition of positivity). If $T$ is invertible, then all $\lambda_i > 0$. If $v = \Sigma a_i e_i$, then $\langle Tv, v \rangle = \Sigma \lambda_i |a_i|^2$. This is $> 0$ unless all $a_i = 0$, i.e., unless $v = 0$.

7.20 True. For every orthonormal basis $(e_1, e_2)$ of $F^2$, the operator $T e_1 = -e_1$, $T e_2 = e_2$ is self-adjoint with $T^2 = I$. This operator determines span$(e_1) = \text{null}(T + I)$. By Gram-Schmidt, span$(e_1)$ can be any 1-dimensional subspace of $F^2$, i.e., every such subspace contains the first member of some orthonormal basis). Since $F^2$ has infinitely many 1-dimensional subspaces (note $F = \mathbb{R}$ or $F = \mathbb{C}$ – $F$ is not a finite field!), there are infinitely many distinct operators of the above form $T$.

7.23 The matrices of $T$, $T^T$, and $S$ in the standard basis of $F^3$ are (it doesn't matter whether $F = \mathbb{R}$ or $F = \mathbb{C}$ since the solution is real):

$$
T = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \quad T^T = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T^T T = \begin{pmatrix} 200 \\ 0 & 30 \\ 0 & 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
$$

because $T^T T$ happens to be diagonal.
Problems

1. I'll prove this assuming either that $T$ is self-adjoint, or, in the case $\mathbb{F} = \mathbb{C}$, normal. In either case we can find an orthonormal basis $e_1, \ldots, e_n$ of $V$ consisting of eigenvectors of $T$, say $Te_i = \lambda_i e_i$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $T$. Let $\alpha = \min |\lambda_i|$. If $\|v\| = 1$, and we express $v$ in terms of the basis $(e_1, \ldots, e_n)$ as

$$v = a_1 e_1 + \cdots + a_n e_n,$$

then $Tv = a_1 \lambda_1 e_1 + \cdots + a_n \lambda_n e_n$.

$$\|Tv\|^2 = \sum |a_i|^2 |\lambda_i|^2 \geq \alpha^2 \sum |a_i|^2 = \alpha^2,$$

since $\sum |a_i|^2 = 1$.

and $|\lambda_i|^2 \geq \alpha^2$ for all $i$. This shows

$$\min_{\|v\| = 1} \|Tv\| \geq \alpha.$$ 

To prove equality, let $i$ be an index for which $\lambda_i = |\lambda_i|$ and take $v = e_i$. Then $\|v\| = 1$, and $\|Tv\| = |\lambda_i| = \alpha$.

2. In the case $\mathbb{F} = \mathbb{R}$, $T - \lambda I$ is self-adjoint. In the case $\mathbb{F} = \mathbb{C}$, $(T - \lambda I)^* = T^* - \lambda I$, and since $T$ is self-adjoint, this is equal to $T - \lambda I$, which commutes both $T - \lambda I$, so $T - \lambda I$ is normal. By hypothesis, $\min_{\|v\| = 1} \|T - \lambda I\| \leq \varepsilon$,

so $T - \lambda I$ has an eigenvalue $\beta$ with $|\beta| \leq \varepsilon$ by Problem 1. Then $\lambda' = \lambda + \beta$ is an eigenvalue of $T$, and $|\lambda' - \lambda| = |\beta| \leq \varepsilon$.

[If you only proved Problem 1 for self-adjoint operators, you will only be able to use it to solve Problem 2 in the case $\mathbb{F} = \mathbb{R}$.]