Math 110 Spring 2012

HW 9 Solutions

Exercises

7.1 (a) Let $U = \text{span}(x)$, $W = \text{span}(1, x^2)$. Then T is the projection $P_{U,W}$ with respect to the direct sum $\mathcal{P}_2(\mathbb{R}) = U \oplus W$. If T (see Ex. 4) were self-adjoint, this would be an orthogonal projection, but it is not, since W is not orthogonal to U, e.g. $\langle 1, x \rangle = \frac{1}{2} \neq 0$.

b) Not a contradiction because the basis $(1, x, x^2)$ is not orthonormal

7.2 Counterexample: let S, T be the operators on $\mathbb{R}^2$ whose matrices in the standard basis are

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the matrices are symmetric, S and T are self-adjoint (in the Euclidean inner product on $\mathbb{R}^2$). The matrix of ST is the product $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of the two matrices. It is not symmetric, so ST is not self-adjoint.

7.3 In both (a), (b), the set of self-adjoint operators in $\mathcal{L}(V)$ is closed under addition. For (a), since $(\lambda T)^* = \bar{\lambda} T^*$, we have λT self-adjoint if $T = T^*$ and $\lambda \in \mathbb{R}$. So the set of self-adjoint operators is closed under scalar multiplication. For (b) this is not true, e.g. I is self-adjoint but λI is self-adjoint iff λ is real.

7.4 Suppose P is an orthogonal projection P_U. Then $V = U \oplus U^\perp$, so if $\underline{e}$ is an orthonormal basis of U, and $\underline{f}$ is an orthonormal basis of $U^\perp$, then $(\underline{e}, \underline{f})$ is an orthonormal basis of V. The matrix of P in this basis looks like

$$\begin{pmatrix} 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \\ & 0 & & 0 \end{pmatrix},$$

i.e. it is diagonal and real. Hence P_U is self-adjoint.

Conversely, suppose $P^2 = P$ and $P = P^*$. By 6.17, P is a projection $P_{U,W}$ for some $V = U \oplus W$, where $W = N(P)$ and $U = N(P-I)$. Since $P = P^*$, $U^\perp$ is P-invariant. Let $P' = P|_{U^\perp}$. Then $P' = (P')^2$, so P' is a projection. If P' were non-zero, then P' and hence P would have an eigenvector with eigenvalue 1 in $U^\perp$. But $U^\perp \cap U = 0$, so this is not the case. Hence $P' = 0$, i.e. $U^\perp \subseteq N(P) = W$. Both $U^\perp$ and W have dimension $\dim V - \dim U$, so $U^\perp = W$, and $P = P_U$ is an orthogonal projection.

7.5 Each of the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

commutes with its transpose, since $A = A^T$ and $B = -B^T$. Hence they represent normal operators on $\mathbb{F}^2$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). But $A + B$ does not commute with its transpose (as you can check), so it represents a non-normal operator. For $\mathbb{F}^n$ with $n > 2$ we can get similar examples by taking matrices with A, B in the upper left 2×2 block, and all other entries zero.

This shows that the set of normal operators is not closed under addition. (It is closed under scalar multiplication, however).

7.16 The operator in Ex. 7.1 is an example. $W = \operatorname{span}(1, x^2)$ is invariant. Its orthogonal complement is $\operatorname{span}(15x^2 - 16x + 3)$, which is not invariant.

7.21 The operator $S(x_1, x_2) = (x_1 + x_2, 0)$ on $\mathbb{R}^2$ has $Se_1 = Se_2 = e_1$. So it satisfies $\|Se_i\| = 1$ for $i = 1, 2$, but it is not an isometry since it is not invertible.

<u>Problem</u>

Let $T \in \mathcal{L}(\mathbb{F}^n)$ (with Euclidean inner product on $\mathbb{F}^n$) be the operator $T(\vec{x}) = Q\vec{x}$. As we showed in class, Q is unitary ($\mathbb{F} = \mathbb{C}$) or orthogonal ($\mathbb{F} = \mathbb{R}$) iff T is an isometry.

(a) $\Leftrightarrow$ (b) : Q unitary/orthogonal $\Leftrightarrow$ T isometry $\Leftrightarrow$ $TT^* = I$
$\Leftrightarrow QQ^* = I_n$.

(b) $\Rightarrow$ (c) : $QQ^* = I_n \Rightarrow Q^* = Q^{-1} \Rightarrow Q^*Q = I_n$. Then (since $(Q^*)^* = Q$, Q^* is unitary/orthogonal by (b) $\Rightarrow$ (a) applied to Q^*.

(c) $\Rightarrow$ (d) : (c) implies (by definition) that the complex conjugates of the rows of Q form an orthonormal basis of $\mathbb{F}^n$. But if $\bar{u}$ denotes the (entry-by-entry) complex conjugate of a vector u, then $\langle \bar{u}, \bar{v} \rangle = \overline{\langle u, v \rangle}$ by the definition of the Euclidean inner product. So the conjugates of an orthonormal basis form an orthonormal basis.

(d) $\Rightarrow$ (b) By definition, (d) implies that Q^T is unitary/orthogonal. By (a) $\Rightarrow$ (b) we therefore have $Q^T Q^{T*} = I_n$. Transposing both sides gives $Q^*Q = I_n$, which implies $Q^* = Q^{-1}$, so $QQ^* = I_n$

We proved

$$\begin{tikzcd} (a) \arrow[r, Leftrightarrow] & (b) \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\ & (c) \arrow[r, Rightarrow] & (d) \arrow[ul, Rightarrow] \end{tikzcd}$$

, so all conditions are equivalent.