

Exercises

6.13 Let $U = \text{span}(e_1, \dots, e_n)$. If $v \in U$, then Theorem 6.17 applied to U gives the desired identity. If $v \notin U$, let $v = u + w$ where $u \in U$ and $w \in U^\perp$. Then $\|v\|^2 = \|u\|^2 + \|w\|^2$, and $\langle u, e_i \rangle = \langle v, e_i \rangle$ for all i , so $\|u\|^2 = \sum |\langle v, e_i \rangle|^2$. If this is also equal to $\|v\|^2$, then $\|w\|^2 = 0$, contrary to the assumption that $v \notin U$.

6.14 Since $P_0(\mathbb{R})$, $P_1(\mathbb{R})$ are invariant subspaces for D , the matrix of D is upper triangular in any basis (e_1, e_2, e_3) such that $e_1 \in P_0(\mathbb{R})$, $e_2 \in P_1(\mathbb{R})$. We can get such a basis by applying Gram-Schmidt to a (non-orthonormal) basis with the same property, such as the monomial basis $(1, z, z^2)$.

In lecture, we did this example, and found

$$e_1 = 1$$

$$e_2 = \sqrt{3}(2x - 1)$$

$$e_3 = \sqrt{5}(6x^2 - 6x + 1)$$

i.e., the solution to Exercise 6.10.

6.15 Follows immediately from Ex. 2.17 and Theorem 6.29.

6.16 Follows from Ex. 6.15.

6.17 By Hw6 Problem 2, $P^2 = P$ implies P is the projection $P_{U,W}$ on U for some direct sum decomposition $V = U \oplus W$. Moreover, $R(P) = U$ and $N(P) = W$. So the hypothesis that $R(P)$ is orthogonal to $N(P)$ implies $W = U^\perp$, and therefore P is the orthogonal projection P_U .

6.20 Suppose U and U^\perp are T -invariant. Given any $v \in V$, let $v = u + w$ with $u \in U$, $w \in U^\perp$, so $P_U v = u$. Then $Tv = Tu + Tw$, and $Tu \in U$, $Tw \in U^\perp$ by invariance. Hence $P_U T v = Tu = T P_U v$. This shows $P_U T = T P_U$.

Conversely, suppose $P_U T = T P_U$. Since $U = R(P_U)$, we have $T(U) = T P_U(V) = P_U T(V) \subseteq U$, i.e. U is T -invariant. Since $U^\perp = N(P_U)$, if $w \in U^\perp$ then $P_U T w = T P_U w = 0$, hence $Tw \in N(P_U) = U^\perp$. This shows U^\perp is T -invariant.

6.21 It is understood that we are using the Euclidean inner product on \mathbb{R}^4 . We start by using Gram-Schmidt to find an orthonormal basis of U :

$$w_1 = (1, 1, 0, 0)$$

$$e_1 = w_1 / \|w_1\| = \frac{1}{\sqrt{2}} (1, 1, 0, 0)$$

$$\begin{aligned} w_2 &= (1, 1, 1, 2) - \langle (1, 1, 1, 2), e_1 \rangle e_1 \\ &= (1, 1, 1, 2) - (1, 1, 0, 0) = (0, 0, 1, 2) \end{aligned}$$

$$e_2 = w_2 / \|w_2\| = \frac{1}{\sqrt{5}} (0, 0, 1, 2)$$

$$\text{Now } u = P_U (1, 2, 3, 4)$$

~~$$\langle (1, 2, 3, 4), e_1 \rangle e_1 + \langle (1, 2, 3, 4), e_2 \rangle e_2$$~~

~~$$= \left(\frac{3}{2}, \frac{3}{2}, 0, 0 \right) + \left(0, 0, \frac{11}{5}, \frac{22}{5} \right)$$~~

~~$$= \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5} \right)$$~~

$$\begin{aligned}
 6.22 \quad \text{Let } U &= \{p(x) \in P_3(\mathbb{R}) : p(0) = p'(0) = 0\} \\
 &= \{ax^3 + bx^2 : a, b \in \mathbb{R}\} \\
 &= \text{span}(x^2, x^3).
 \end{aligned}$$

Using the inner product $\langle p, q \rangle = \int_0^1 p(x)q(x) dx$, we want the orthogonal projection ~~$p(x)$~~ $p(x) = P_U(2+3x)$. First find an orthonormal basis for U :

$$w_1 = x^2$$

$$e_1 = w_1 / \|w_1\| = \sqrt{5}x^2 \cancel{\#}$$

$$w_2 = x^3 - \langle x^3, e_1 \rangle e_1 = x^3 - \frac{5}{6}x^2$$

$$e_2 = w_2 / \|w_2\| = \sqrt{7}(6x^3 - 5x^2)$$

$$\begin{aligned}
 \text{Now } P_U(2+3x) &= \langle 2+3x, e_1 \rangle e_1 + \langle 2+3x, e_2 \rangle e_2 \\
 &= \frac{17\sqrt{5}}{12} e_1 - \frac{29\sqrt{7}}{60} e_2 \\
 &= -\frac{1}{10}(203x^3 - 240x^2)
 \end{aligned}$$

6.24 We have a linear function $L: P_2(\mathbb{R}) \rightarrow \mathbb{R}$ given by $L(p(x)) = p(\frac{1}{2})$, and we want the element $q \in P_2(\mathbb{R})$ such that $\langle p, q \rangle = Lp$ for all p , using the same inner product on $P_2(\mathbb{R})$ as before. To find it, we'll let $q = ax^2 + bx + c$, impose the condition $\langle p, q \rangle = Lp$ for p in the basis $(1, x, x^2)$, and solve for a, b, c . This gives:

$$\langle 1, ax^2 + bx + c \rangle = \frac{a}{3} + \frac{b}{2} + c = L(1) = 1$$

$$\langle x, ax^2 + bx + c \rangle = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} = L(x) = \frac{1}{2}$$

$$\langle x^2, ax^2 + bx + c \rangle = \frac{a}{5} + \frac{b}{4} + \frac{c}{3} = L(x^2) = \frac{1}{4}$$

$$a = -15, \quad b = 15, \quad c = -\frac{3}{2}, \quad q(x) = -15x^2 + 15x - \frac{3}{2}$$

[This problem can also be solved using the orthonormal basis found in Ex. 6.14: in terms of that basis, $q(x) = \overline{L(e_1)}e_1 + \overline{L(e_2)}e_2 + \overline{L(e_3)}e_3.$]

6.25 The desired $q(x)$ is the orthogonal projection of $\cos \pi x$ on $P_2(\mathbb{R})$, using the inner product $\langle P, q \rangle = \int_0^1 P(x)q(x) dx$ on continuous functions of x with \mathbb{R} values, namely

$$q(x) = \langle \cos \pi x, e_1 \rangle e_1 + \langle \cos \pi x, e_2 \rangle e_2 + \langle \cos \pi x, e_3 \rangle e_3$$

with e_1, e_2, e_3 as in 6.14. After doing the various integrals we get

$$q(x) = \frac{-24x + 12}{\pi^2}$$

6.27 Here we are using the Euclidean inner product, in which the standard basis of \mathbb{F}^n is orthonormal. The matrix of T in the standard basis is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & & \vdots \\ 0 & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Its conjugate transpose $M^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is

the matrix of T^* , so $T^*(z_1, \dots, z_n) = (z_2, z_3, \dots, z_n, 0)$.

6.30 a) By Prop. 6.46, T injective $\Leftrightarrow N(T) = 0 \Leftrightarrow R(T^*) = V \Leftrightarrow T^*$ surjective

b) Similarly, T surjective $\Leftrightarrow R(T) = W \Leftrightarrow N(T^*) = 0 \Leftrightarrow T^*$ injective.

Or, deduce (b) by applying (a) to T^* and using $T^{**} = T$.

6.31 Using Prop. 6.46 and the rank-nullity theorem:

$$\dim N(T^*) = \dim R(T)^\perp = \dim W - \dim R(T) = \dim W - (\dim V - \dim N(T)) \\ = \dim N(T) + \dim W - \dim V.$$

Rewriting this as

$$\dim V - \dim N(T) = \dim W - \dim N(T^*)$$

and using rank-nullity, we get

$$\dim R(T) = \dim R(T^*).$$

Problems

6.18 By HW 6 Problem 2, P is the projection $P_{U,W}$ for some $V = U \oplus W$. As in Ex. 6.17, if we prove that U and W are orthogonal, then P is the orthogonal projection on U .

Suppose to the contrary that there are vectors $u \in U$, $w \in W$ such that $\langle u, w \rangle \neq 0$. Then we can find a scalar multiple $u' = au$ such that $\langle u', w \rangle = a \langle u, w \rangle$ is real and $\langle u', w \rangle < -\|w\|^2/2$.

Let $v = u' + w$. Then $Pv = u'$, and we have

$$\begin{aligned}\|v\|^2 &= \|u'\|^2 + \|w\|^2 + 2 \operatorname{Re} \langle u', w \rangle && \text{(as in proof of } \\ &&& \text{Theorem 6.9)} \\ &= \|u'\|^2 + \|w\|^2 + 2 \langle u', w \rangle && \text{(since } \langle u', w \rangle \text{ is real)} \\ &< \|u'\|^2 && \text{(since } 2 \langle u', w \rangle < -\|w\|^2\text{).}\end{aligned}$$

In other words, $\|v\|^2 < \|Pv\|^2$, contradicting the hypothesis that $\|Pv\| \leq \|v\|$ for all v .

6.28 The properties of adjoints on p. 119 of Axler imply that $(T - \lambda I)^* = T^* - \bar{\lambda} I$. By exercise 6.31, $T - \lambda I$ invertible $\Leftrightarrow (T - \lambda I)^*$ invertible. But $T - \lambda I$ is not invertible iff λ is an eigenvalue of T , and $(T - \lambda I)^* = T^* - \bar{\lambda} I$ is not invertible iff $\bar{\lambda}$ is an eigenvalue of T^* . Hence λ eigenvalue of $T \Leftrightarrow \bar{\lambda}$ eigenvalue of T^*