Math 110 Assignment 7

(I) Exercises.

Axler Chapter 6: 2, 3, 4, 5, 6, 9, 10, 13, 14

(II) Problems. Due Friday, Mar. 16 by 3pm at the location your GSI has specified for turning in homework.

Initially I assigned Axler 6.8 (for $\mathbb{F} = \mathbb{R}$), but this problem seems to be too hard, so I’m changing it.

1. (3/10) Prove the identity

\[ b \|u + av\|^2 + a \|u - bv\|^2 = (a + b)(\|u\|^2 + a b \|v\|^2) \]

for all vectors $u, v$ in an inner product space $V$, and all real scalars $a, b \in \mathbb{R}$. (The parallelogram identity, Axler Proposition 6.14, is the special case of this with $a = b = 1$.)

2. (3/10) Prove the identity

\[ \|u + v + w\|^2 - \|u + v\|^2 - \|u + w\|^2 - \|v + w\|^2 + \|u\|^2 + \|v\|^2 + \|w\|^2 = 0 \]

for all vectors $u, v, w$ in an inner product space $V$. (The parallelogram identity is also a special case of this identity: take $w = -v$.)

3. (4/10) Let $V$ be a vector space over $\mathbb{R}$, and suppose a norm $\| \cdot \| : V \to \mathbb{R}$ is given which satisfies the conditions in Axler, Exercise 6.8, and also the generalized parallelogram identities in Problems 1 and 2, above. Prove that there is an inner product on $V$ such that $\|v\| = \sqrt{\langle v, v \rangle}$ for all $v \in V$.

Hint: find an expression for $\langle u, v \rangle$ in terms of $\|u + v\|$, $\|u\|$ and $\|v\|$ which holds in any inner product space over $\mathbb{R}$, and use this to define the inner product in terms of the given norm. In this version of the problem, you will not need to use the assumption that $\| \cdot \|$ satisfies the triangle inequality.