

Math 110 Spring 2012
HW 7 Solutions

Exercises

6.2 If $\langle u, v \rangle = 0$ then $\|u + av\|^2 = \|u\|^2 + |a|^2 \|v\|^2 \geq \|u\|^2$, hence $\|u + av\| \geq \|u\|$. ~~Proving~~ For the converse, if $\langle u, v \rangle \neq 0$ then $v \neq 0$, and the projection of u along v is given by λv where $\lambda = \frac{\langle u, v \rangle}{\|v\|^2}$, i.e. if we set $w = u - \lambda v$ then $u = \lambda v + w$ and $\langle \lambda v, w \rangle = 0$. Then $\|u\|^2 = |\lambda|^2 \|v\|^2 + \|w\|^2$, and since ~~both~~ $v \neq 0$ and $\lambda \neq 0$, we have $\|u\|^2 > \|w\|^2$, hence $\|u\| > \|w\|$. But $w = u - \lambda v$, so for $a = -\lambda$ we have $\|u\| > \|u + av\|$.

6.3 ~~Definition~~

let $u = (a_1, \sqrt{2}a_2, \sqrt{3}a_3, \dots, \sqrt{n}a_n) \in \mathbb{R}^n$

$v = (b_1, b_1/\sqrt{2}, b_3/\sqrt{3}, \dots, b_n/\sqrt{n}) \in \mathbb{R}^n$.

In the Euclidean inner product on \mathbb{R}^n , we have

$$\langle u, v \rangle = \sum_{j=1}^n a_j b_j$$

$$\|u\|^2 = \sum_{j=1}^n j a_j^2 \quad \|v\|^2 = \sum_{j=1}^n \frac{b_j^2}{j}$$

The desired inequality is now $\langle u, v \rangle^2 \leq \|u\|^2 \|v\|^2$, which follows from Cauchy-Schwarz.

6.4 By parallelogram identity, $6^2 + 4^2 = 2 \cdot 3^2 + 2 \cdot \|v\|^2$. Hence $\|v\| = \sqrt{17}$.

6.5 There is no such inner product. If there were, then we ~~would~~ would have

$$\|e_1 + e_2\| = 2$$

$$\|e_1 - e_2\| = 2$$

$$\|e_1\| = 1$$

$$\|e_2\| = 1,$$

where $e_1 = (1, 0)$, $e_2 = (0, 1)$. But $2^2 + 2^2 \neq 2(1^2 + 1^2)$, so this violates the parallelogram identity.

$$\begin{aligned}
 6.6 \quad & \|u+v\|^2 - \|u-v\|^2 \\
 &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\
 &\quad - (\langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle) \\
 &= 2\langle u, v \rangle + 2\langle v, u \rangle.
 \end{aligned}$$

Since $\mathbb{R} = \mathbb{R}$, $\langle u, v \rangle = \langle v, u \rangle$, so this is equal to $4\langle u, v \rangle$.

6.9 Using the formulas

$$\sin kx \sin lx = \frac{1}{2} (\cos(k-l)x - \cos(k+l)x)$$

$$\sin kx \cos lx = \frac{1}{2} (\sin(k-l)x + \sin(k+l)x)$$

$$\cos kx \cos lx = \frac{1}{2} (\cos(k-l)x + \cos(k+l)x),$$

$$\text{and } \int_{-\pi}^{\pi} \sin mx dx = 0 = \int_{-\pi}^{\pi} \cos mx dx \text{ for integers } m \neq 0,$$

we deduce that for $k \neq l$ we have

$$\int_{-\pi}^{\pi} \sin kx \sin lx dx = \int_{-\pi}^{\pi} \sin kx \cos lx dx = \int_{-\pi}^{\pi} \cos kx \cos lx dx = 0.$$

Hence all the functions on the list are orthogonal (including $1/\sqrt{2\pi} = (\cos 0x)/\sqrt{2\pi}$).

We have

$$\left\| \frac{1}{\sqrt{2\pi}} \right\|^2 = \int_{-\pi}^{\pi} \frac{1}{2\pi} dx = 1$$

$$\left\| \frac{\sin kx}{\sqrt{2\pi}} \right\|^2 = \int_{-\pi}^{\pi} \frac{\sin^2 kx}{\pi} dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2kx}{2\pi} dx = 1 \quad \text{for } k \neq 0$$

$$\left\| \frac{\cos kx}{\sqrt{2\pi}} \right\|^2 = \int_{-\pi}^{\pi} \frac{\cos^2 kx}{\pi} dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2kx}{2\pi} dx = 1 \quad \text{for } k \neq 0.$$

So the list is orthonormal.

6.10 We did this in class and got the answer

$$u_1 = 1$$

$$u_2 = \sqrt{3}(2x-1)$$

$$u_3 = \sqrt{5}(6x^2 - 6x + 1).$$

6.13 If $v \in \text{span}(e_1, \dots, e_m)$, say

$$v = a_1 e_1 + \dots + a_m e_m,$$

then $\|v\|^2 = |a_1|^2 + \dots + |a_m|^2$ since the $a_i e_i$ are orthogonal and $\|a_i e_i\|^2 = |a_i|^2$. But we also have $a_i = \langle v, e_i \rangle$, so $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_m \rangle|^2$.

For the converse, if v is not in $\text{span}(e_1, \dots, e_m)$, then we can apply Gram-Schmidt to extend (e_1, \dots, e_m) to an orthonormal basis (e_1, \dots, e_{m+1}) of $\text{span}(e_1, \dots, e_m, v)$. Then we do have $v \in \text{span}(e_1, \dots, e_{m+1})$, so by the first part,

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_m \rangle|^2 + |\langle v, e_{m+1} \rangle|^2.$$

Furthermore, writing $v = a_1 e_1 + \dots + a_{m+1} e_{m+1}$, we have $\langle v, e_{m+1} \rangle = a_{m+1} \neq 0$ since $v \notin \text{span}(e_1, \dots, e_m)$.

Hence $\|v\|^2 \neq |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_m \rangle|^2$.

6.14 The basis in Exercise 6.10 works for this, since

$\text{span}(u_1) = P_0(\mathbb{R})$, $\text{span}(u_2, u_3) = P_1(\mathbb{R})$, and these are invariant subspaces for the operator $D_p = p'$.

Problems

$$1. b\|u+av\|^2 + a\|u-bv\|^2 =$$

$$b(\langle u, u \rangle + a(\langle u, v \rangle + \langle v, u \rangle) + a^2 \langle v, v \rangle) + \cancel{a^2 \langle v, v \rangle} \quad \left. \begin{array}{l} \text{we used} \\ \bar{a}=a, \\ \bar{b}=b \text{ here.} \end{array} \right\}$$

$$+ a(\langle u, u \rangle - b(\langle u, v \rangle + \langle v, u \rangle) + b^2 \langle v, v \rangle)$$

$$= (a+b)\langle u, u \rangle + (a^2 b + b^2 a) \langle v, v \rangle = (a+b)(\|u\|^2 + ab\|v\|^2).$$

$$2. \|u+v+w\|^2 = \|u+w\|^2 - \|u+v\|^2 - \|u+v\|^2 + \|v+w\|^2 + \|u\|^2 + \|v\|^2 + \|w\|^2$$

$$= \langle u, u \rangle + \langle v, v \rangle + \langle w, w \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle u, w \rangle + \langle w, u \rangle + \langle v, w \rangle + \langle w, v \rangle$$

$$- (\langle u, u \rangle + \langle u, v \rangle + \langle w, u \rangle + \langle v, v \rangle) - (\langle u, u \rangle + \langle u, w \rangle + \langle w, u \rangle + \langle w, w \rangle)$$

$$- (\langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle) + \langle u, u \rangle + \langle v, v \rangle + \langle w, w \rangle = 0.$$

$$3. \text{ Define } \langle u, v \rangle = \frac{\|u+v\|^2 - \|u\|^2 - \|v\|^2}{2}.$$

$$\text{Then } \langle v, v \rangle = \frac{\|2v\|^2 - 2\|v\|^2}{2} = 4 \frac{\|v\|^2 - 2\|v\|^2}{2} = \|v\|^2.$$

(using $\|av\| = |a|\|v\|$). This shows that the given norm is $\|v\| = \sqrt{\langle v, v \rangle}$ and also shows that \langle , \rangle is positive and definite (since we are given $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$).

For additivity,

$$\begin{aligned} \langle u_1 + u_2, v \rangle &= \frac{1}{2} (\|u_1 + u_2 + v\|^2 - \|u_1 + u_2\|^2 - \|v\|^2) \\ &= \frac{1}{2} (\|u_1 + v\|^2 + \|u_2 + v\|^2 - \|u_1\|^2 - \|u_2\|^2 - 2\|v\|^2) \\ &\quad \uparrow \text{by the identity} \\ &\quad \text{in Problem 2.} \\ &= \langle u_1, v \rangle + \langle u_2, v \rangle \end{aligned}$$

For homogeneity,

$$\begin{aligned} \langle au, v \rangle &= \|au + v\|^2 - a^2\|u\|^2 - \|v\|^2 \\ a\langle u, v \rangle &= a\|u + v\|^2 - a\|u\|^2 - a\|v\|^2, \end{aligned}$$

$$\text{so } \langle au, v \rangle - a\langle u, v \rangle =$$

$$\|au + v\|^2 - a\|u + v\|^2 + a(1-a)(a\|u\|^2 - \|v\|^2)$$

$$\begin{aligned} &= 0 \text{ by the identity in Problem 1, with } b = -1 \\ &\text{and } u, v \text{ switched.} \end{aligned}$$

Symmetry, $\langle u, v \rangle = \langle v, u \rangle$, is obvious from the definition.

So we have proved that \langle , \rangle is an inner product and that $\|v\| = \sqrt{\langle v, v \rangle}$.