

Exercises

5.4 Let  $v \in N(T - \lambda I)$ . Then  $(T - \lambda I)Sv = (TS - \lambda S)v = (ST - \lambda S)v = S(T - \lambda I)v = 0$ . This shows  $Sv \in N(T - \lambda I)$ .  
 Since  $TS = ST$   
 Hence  $N(T - \lambda I)$  is  $S$ -invariant.

5.6 The matrix of  $T$  in the standard basis is

$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ . Since it is upper triangular, the

e-vals of  $T$  are 0 and 5. The eigenvectors for  $\lambda=5$  are the vectors  $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$  such that  
 (written)

$(2z_2, 0, 5z_3) = (5z_1, 5z_2, 5z_3)$ , i.e.  $5z_2 = 0$  and  $5z_1 = 2z_2$ , so  $z_1 = z_2 = 0$ . I.e. the eigenspace  $N(T - 5I)$  is spanned by  $(0, 0, 1)$ .

The eigenspace  $N(T - 0I) = N(T)$  consists of the vectors such that  $z_2 = z_3 = 0$ , i.e. it is spanned by  $(0, 1, 0)$ .

[Note that the dimensions of the eigenspaces add up to 2, not 3, so  $T$  is not diagonalizable.]

5.7 The vector  $(1, 1, \dots, 1)$  is an eigenvector with eigenvalue  $n$ . The vectors  $(z_1, \dots, z_n)$  such that  $z_1 + \dots + z_n = 0$  form the nullspace of  $T$ , i.e., the 0 eigenspace. In particular, since the vectors  $(1, -1, 0, \dots, 0)$ ,  $(0, 1, -1, 0, \dots, 0)$ ,  $\dots$ ,  $(0, \dots, 0, 1, -1)$  are independent,  $N(T)$  has dimension  $n-1$ . Hence  $T$  is diagonalizable and its only eigenvalues are 0 and  $n$ .

5.8 The condition  $Tz = \lambda z$  is equivalent to

$$z_2 = \lambda z_1, z_3 = \lambda z_2, \dots$$

For every  $\lambda$ , this has a 1-dimensional space of solutions, spanned by  $(1, \lambda, \lambda^2, \dots)$ . So every  $\lambda \in \mathbb{F}$  is an e-val, and its eigenspace is  $\mathbb{F} \cdot (1, \lambda, \lambda^2, \dots)$ .

5.11 By symmetry it's enough to show that if  $\lambda$  is an eigenvalue of  $ST$ , then it is also an eigenvalue of  $TS$ . Suppose  $v$  is an eigenvector with  $STv = \lambda v$  ( $v \neq 0$ ). Then  $TSTv = T(\lambda v) = \lambda Tv$ , which shows that  $Tv$  is an eigenvector of  $TS$  with eigenvalue  $\lambda$ , provided  $Tv \neq 0$ . But if  $Tv = 0$  then  $\lambda = 0$ , since  $STv = \lambda v$  and  $Tv = 0$ . In this case  $Tv = 0$  also implies that  $T$  is not invertible, hence not surjective, so  $TS$  is also not surjective, since  $R(TS) \subseteq R(T)$ . Hence  $TS$  is not invertible, so  $0$  is also an eigenvalue of  $TS$ .

5.17 By Thm 5.13,  $V$  has a basis  $(v_1, \dots, v_n)$  such that  $m(T, \vee)$  is upper triangular. By Prop. 5.12, each subspace  $\text{span}(v_1, \dots, v_j)$  is  $T$ -invariant. This subspace has dimension  $j$ .

5.18  $T: \mathbb{F}^2 \rightarrow \mathbb{F}^2$  defined by  $T(z_1, z_2) = (z_2, z_1)$  has matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .  $T$  is invertible, since  $T^2 = I$ , i.e.  $T$  is its own inverse.

5.19 The matrix of the operator in Ex. 5.7 has 1's in every position. For  $n > 1$  we have seen that its nullspace is non-zero, so it is not invertible.

5.20 Let  $\dim V = n$ , and let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues, with eigenvectors  $v_1, \dots, v_n$ . By Thm 5.6,  $(v_1, \dots, v_n)$  is linearly independent, hence it is a basis of  $V$ . By definition,  $Tv_i = \lambda_i v_i$ , and by assumption  $Sv_i = \mu_i v_i$  for some scalars  $\mu_i$ . Then  $STv_i = \lambda_i \mu_i v_i = TSv_i$ . Since the linear operators  $ST$  and  $TS$  agree on the basis  $\vee$ , they are equal.

$$5.23 \quad T(z_1, z_2, z_3, z_4) = (z_2, -z_1, z_4, -z_3).$$

If  $T\vec{z} = \lambda\vec{z}$  then  $z_2 = \lambda z_1$ ,  $-z_1 = \lambda z_2$   
 $z_3 = \lambda z_4$ ,  $z_4 = -\lambda z_3$ .

Then  $z_1 = -\lambda^2 z_1$ ,  $z_2 = -\lambda^2 z_2$ ,  $z_3 = -\lambda^2 z_3$ ,  $z_4 = -\lambda^2 z_4$ .

If  $\vec{z} \neq 0$  then one of these equations implies  $\lambda^2 = -1$ ,  
but this has no real solution.

5.24 Suppose  $U \subseteq V$  is  $T$ -invariant and  $\dim U$  is odd.

By Thm 5.26,  $T|_U$  has an eigenvector in  $U$ . But then  
this is also an eigenvector of  $T$  in  $V$ , contrary to assumption.

(a) Is a typo - I used Hw 5 as a template for Hw 6 and  
forgot to delete this exercise.

### Problems

① 5.21. Let  $v \in V$ . We have  $v = Pv + (I-P)v$ .  
Now  $Pv \in R(P)$ , and  $P(I-P)v = (P-P^2)v = 0$  since  $P=P^2$ .  
So  $(I-P)v \in N(P)$ . This shows  $V = R(P) + N(P)$ . We also  
need to show that  $R(P) \cap N(P) = 0$ , so suppose  
 $v \in R(P) \cap N(P)$ . Then  $\exists w$  s.t.  $v = Pw$ , and we have  
 $0 = Pv = P^2w$ . But  $P^2 = P$ , so  $Pw = 0$ , i.e.  $v = 0$ .

② (a)  $\Rightarrow$  (b). Using ①, we have  $V = R(P) \oplus N(P)$ .  
For all  $v \in R(P)$ , say  $v = Pw$ , we have  $Pv = P^2w = Pw = v$ ,  
i.e.  $R(P)$  is an eigenspace with eigenvalue  $\lambda = 1$ .  
By definition,  $N(P)$  is the eigenspace with eigenvalue  $\lambda = 0$ .  
Since  $V = R(P) \oplus N(P)$ , Prop. 5.21 implies that  $P$  is  
diagonalizable and 0,1 are its only eigenvalues.

(b)  $\Rightarrow$  (c) By Prop. 5.21,  $V = U \oplus W$  where  $U = N(P-I)$   
and  $W = R(P)$ . Given  $v \in V$ , let  $v = u+w$  where  
 $u \in U$ ,  $w \in W$ . Then  $Pv = Pu+Pw = 1 \cdot u + 0 \cdot w = u$ . This  
shows  $P = P_{U,W}$ .

(c)  $\Rightarrow$  (a). Let  $V = U \oplus W$  and let  $P = P_{U,W}$ . If  $v = u+w$ ,  
then  $Pv = u$ , and since  $u = u+0$  ( $u \in U$ ,  $0 \in W$ ),  $Pu = u$ .  
So  $P^2v = P_u = u = Pv$ . This holds for all  $v$ , so  $P^2 = P$ .