

Exercises

3.14 "If": suppose $ST = I_v$. If $T(v_1) = T(v_2)$ then $ST(v_1) = ST(v_2)$, so $v_1 = v_2$. This shows T is injective.

"Only if": suppose $T: V \rightarrow W$ is injective. Let B be a basis of V . Then ~~the~~ the vectors $T(b)$ for $b \in B$ are linearly independent by Ex. 3.5. We can complete $T(B)$ to a basis $T(B) \cup C$ of W and define $S: W \rightarrow V$ by specifying its values on this basis to be $S(T(b)) = b$, $S(c) = 0$ for $c \in C$. Then ST agrees with I_v on every $b \in B$, so $ST = I_v$. (The assumption that ~~was~~ $\dim W < \infty$ is not necessary.)

3.15 "If": Suppose $TS = I_w$. Clearly $R(TS) = T(R(S)) \subseteq R(T)$.

But $TS = I_w \Rightarrow R(TS) = W$, so $R(T) = W$, i.e. T is Surjective.

"Only if": ~~Let B be a basis of V . For each $b \in B$ we have the fact that T is surjective. By Ex. 3.7, $T(B)$ spans W so $T(B) \cup C$ is a basis of W . Define $S: W \rightarrow V$ on~~

~~Let B be a basis of W . For each $b \in B$, pick some $v_b \in V$ such that $T(v_b) = b$. Define $S: W \rightarrow V$ on the basis B by $S(b) = v_b$. Then $TS(b) = b = I_w(b)$ for all $b \in B$, hence $TS = I_w$. (Again, finite dimension is not needed.)~~

10.1 If T is invertible, say $S = T^{-1}$, then $m(S)m(T) = m(T)m(S) = m(I_v) = I_n$, so $m(S)$ is inverse to $m(T)$ (all matrices with respect to basis v). The converse also holds since $m: L(v) \rightarrow M_{n \times n}(\mathbb{F})$ is an isomorphism.

10.2 Let $T_A, T_B: \mathbb{F}^n \rightarrow \mathbb{F}^n$ be given by $T_A v = Av$, $T_B v = Bv$ (identifying \mathbb{F}^n with $M_{n \times 1}(\mathbb{F})$). Then $AB = I \Rightarrow T_A T_B = I$. In particular, by 3.14, T_B is injective, hence invertible, and T_A is its inverse. Therefore $T_B T_A = I$ which implies $BA = I$.

10.5 Let B be $n \times n$, $T_B : \mathbb{F}^n \rightarrow \mathbb{F}^n$ defined by $T_B v = Bv$.

By Theorem 5.13, there is a basis (\underline{v}) of \mathbb{F}^n such that $M(T_B, \underline{v})$ is upper-triangular. But

$$\begin{aligned} M(T_B, \underline{v}) &= M(I, \underline{e}, \underline{v}) \circ M(I, \underline{v}, \underline{e}) \\ &= A^{-1}BA, \end{aligned}$$

where \underline{e} is the standard basis of \mathbb{F}^n and $A = M(I, \underline{v}, \underline{e})$ is the matrix whose columns are the vectors v_i .

S.1 Let $v = u_1 + \dots + u_m$ where $u_i \in U_i$ (this is the general form of a vector $v \in U_1 + \dots + U_m$). Then

$$Tv = Tu_1 + \dots + Tu_m \in U_1 + \dots + U_m$$

since each U_i is T -invariant.

S.2 Let $\{U_\alpha\}$ be a collection of invariant subspaces.

If $v \in \bigcap U_\alpha$ then $v \in U_\alpha$ for all α , hence $Tv \in U_\alpha$ for all α by T -invariance of U_α . This shows $Tv \in \bigcap U_\alpha$.

S.3 True. Proof Suppose $U \subseteq V$, $U \neq 0$, $U \neq V$. Let

~~choose~~ B be a basis of U . Since $U \neq 0$, $B \neq \emptyset$.

let $v \in V$ be a vector not in U (which exists since $U \neq V$). Extend B to a basis $B \cup C$ of V

and define $T: V \rightarrow V$ by taking $T(b) = v$ for every $b \in B \cup C$. In particular, there is at least one $b \in B$, and for this b we have $b \in U$, $T(b) = v \notin U$, so U is not T -invariant.

S.5 If $T(w, z) = \lambda(w, z)$, we have $(z, w) = \lambda(w, z)$, i.e.

$$z = \lambda w, \quad w = \lambda z. \quad \text{Then } z = \lambda w = \lambda(\lambda z) = \lambda^2 z, \text{ and } z \neq 0$$

Since $z = 0$ would imply $w = 0$ and $(w, z) = (0, 0)$. Hence

$$\lambda^2 = 1, \quad \lambda = \pm 1.$$

For $\lambda = 1$ we have $z = w$: so the eigenvectors with eigenvalue 1 are ~~all~~ $C(1, 1)$

$0 \neq c \in \mathbb{F}$. For $\lambda = -1$ we have $z = -w$: so the eigenvectors

with eigenvalue -1 are $c \cdot (1, -1)$ for $0 \neq c \in \mathbb{F}$.

Note that this answer is correct for every field \mathbb{F} , but its meaning is a little different if \mathbb{F} is a field such as \mathbb{F}_2 in which $1 = -1$. In this case, both solutions are the same, the only eigenvalue is $1 = -1$, and the eigenvectors are $c \cdot (1, 1)$ ($c \neq 0$). So T is diagonalizable if $1 \neq -1$ in \mathbb{F} , but not if $1 = -1$ in \mathbb{F} .

Problems

S.12 We'll prove that every vector ($\neq 0$) in V is an eigenvector with the same eigenvalue λ . Then $Tv = \lambda v$ for every $v \in V$ (including $v=0$, trivially), so $T = \lambda I$. Suppose to the contrary that we had two eigenvectors $v, w \in V$ with different eigenvalues λ, μ . In particular, by Theorem S.6, v and w are linearly independent, so $v+w \neq 0$. By assumption, $v+w$ is also an eigenvector, say with eigenvalue ρ . Since $(v, w, v+w)$ are linearly dependent, and $\lambda \neq \mu$, Theorem S.6 implies that $\rho = \lambda$ or $\rho = \mu$. If $\rho = \lambda$, then both v and $v+w$ belong to the eigenspace $N(T-\lambda I)$, hence so does $w = (v+w)-v$. But then $\mu = \lambda$. By symmetry, we also conclude that $\mu = \lambda$ if $\rho = \mu$. So T could not have had more than one eigenvalue, and the result follows.

Problem 2. A is the transpose of the matrix of T with respect to the basis $(1, z, z^2, \dots, z^d)$ of $P_d(\mathbb{F})$ and the standard basis of \mathbb{F}^{d+1} . Since T is invertible, A^t is invertible, and then so is A , with inverse ~~$(A^t)^{-1}$~~ B^t where $B = (A^t)^{-1}$, since $AB^t = (BA^t)^t = I^t = I$.

Exercises not from book

(a) If $p(a) = 0$ then $p(z) = (z-a)q(z)$, so
 $p'(z) = q(z) + (z-a)q'(z)$. Hence $p'(a) = q(a)$, so
if $p'(a) = 0$ then $(z-a)$ is a factor of $q(z)$, hence
 $(z-a)^2$ is a factor of $p(z)$. Conversely if
 $p(z) = (z-a)^2r(z)$ then $p'(z) = 2(z-a)r(z) + (z-a)^2r'(z)$,
so $p(a) = 0$ and $p'(a) = 0$.

This implies that $N(S)$ ~~exists~~ is the set of polynomials
 $p(z) \in P_3(\mathbb{R})$ which have $z^2(z-1)^2$ as a factor. But
this implies $p(z) = 0$, since the factor has degree 4.
Thus $N(S) = 0$ and S is injective. (It's clear that
 S is linear.)

(b) Since the domain and target of S both have dimension
4, Ex. (a) implies that S is invertible, i.e. for
every $(a, b, c, d) \in \mathbb{R}^4$ there is a unique $p(z) \in P_3(\mathbb{R})$
such that $S p(z) = (a, b, c, d)$, which means
 $p(0) = 0$, $p'(0) = b$, $p''(0) = c$, $p'''(1) = d$.