Exercises

3.17 \( A(B+C) \) makes sense when \( A \) is \( m \times n \) and \( B, C \) are both \( n \times p \), for some \( m, n, p \). Then there are linear maps

\[
\begin{align*}
&F^n \longrightarrow F^m \\
&T_A, T_B, T_C \\
&F^p \longrightarrow F^n \\
&TA
\end{align*}
\]

whose matrices with respect to the standard bases are

\[ M(T_A) = A, \quad M(T_B) = B, \quad M(T_C) = C. \]

We have

\[ A(B+C) = M(T_A)(M(T_B) + M(T_C)) = M(T_A)M(T_B + T_C) \]

\[ = M(T_A \circ (T_B + T_C)) \]

\[ = M(T_A \circ T_B + T_A \circ T_C) \left[ \text{Ch. 3, p. 41: distributive property of composition of linear maps} \right] \]

\[ = M(T_A T_B) + M(T_A T_C) \]

\[ = M(T_A)M(T_B) + M(T_A)M(T_C) \]

\[ = AB + AC. \]

3.21 Let \( T : M_{n \times 1}(F) \rightarrow M_{m \times 1}(F) \) be a linear map and let \( A = M(T) \) with respect to the standard bases of unit column vectors. Then for any \( v \in M_{n \times 1}(F) \) we have \( M(Tv) = M(T)M(v) = AM(v) \) [Prop. 3.14].

But using the standard bases, we have \( M(v) = v \) in \( M_{n \times 1}(F) \) and \( M(w) = w \) in \( M_{m \times 1}(F) \). So

\[ Tv = M(Tv) = AM(v) = Av. \]

3.22 If \( S \) and \( T \) are invertible then \( ST \) is invertible with inverse \( T^{-1}S^{-1} \). For the converse, if \( Tv = 0 \) then \( STv = 0 \), so \( ST \) invertible \( \Rightarrow N(ST) = 0 \Rightarrow N(T) = 0 \Rightarrow T \) injective.

By Thm. 3.21, \( T \) is invertible. Then \( S = (ST) T^{-1} \) is also invertible.
3.23 If \( ST = I \), then \( ST \) is invertible, so \( S \) and \( T \) are invertible by Ex. 3.23, and then \( T = S^{-1}ST = S^{-1}I = S^{-1} \).

Hence \( TS = S^{-1}S = I \).

3.25 We have \( M : \mathbb{L}(V) \to \mathbb{M}_{n \times n}(\mathbb{F}) \) an invertible linear map, and \( M(\{\text{non-invertible operators}\}) = \{\text{non-invertible matrices} A \in \mathbb{M}_{n \times n}(\mathbb{F})\} \). Under an invertible map, subspaces correspond to subspaces, so it suffices to show that the set of non-invertible \( n \times n \) matrices is not a subspace of \( \mathbb{M}_{n \times n}(\mathbb{F}) \), for \( n > 1 \). We'll show that it's not additive, with a counterexample: let

\[
A = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}.
\]

The linear maps \( T_A, T_B \in \mathbb{L}(V) \) with matrices \( A, B \) are non-invertible, because if \( (v_1, \ldots, v_n) \) is the basis of \( V \) we are using for \( M \), then \( T_A v_2 = 0 \), since col 2 is zero in \( A \), and \( T_B v_1 = 0 \), since col 1 is zero in \( B \). So \( T_A \) and \( T_B \) have non-zero nullspaces. But \( A + B = I_n \), so \( T_A + T_B = I_V \), which is invertible. (This doesn't work if \( n = 1 \) since then there is no \( v_2 \). In fact, if \( \dim V = 1 \) then \( \mathbb{L}(V) \) consists of scalars, and the non-invertible ones are \( \{0\} \), which is a subspace.)

3.26 Let \( T : \mathbb{F}^n \to \mathbb{F}^n \) be \( Tv = Av \) where \( A \) is the matrix with entries \( a_{i,j} \). Then (a) holds \( \iff T \) has nullspace \( = 0 \) \( \iff \)

\( T \) injective, and (b) holds \( \iff T \) is surjective. Hence

(a) \( \iff \) (b) by Thm. 3.21
Exercises not from book

(a) The operators $Dp(z) = p'(z)$ and $zp(z) = zp(z)$ are clearly linear, and $T = D^2 - 2zD + z$, so $T$ is linear.

If $p(z) \in P_d(\mathbb{R})$, the first two terms of $T \cdot p(z)$ have degree $\leq d$ and the last has degree $\leq d + 1$, so $T \cdot p(z) \in P_{d+1}(\mathbb{R})$.

(b) \[
\begin{pmatrix}
0 & 0 & 2 & 0 & 0 \\
1 & -2 & 0 & 6 & 0 \\
0 & 1 & -4 & 0 & 12 \\
0 & 0 & 1 & -6 & 0 \\
0 & 0 & 0 & 1 & -8 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(c) $S$ is injective, since a polynomial is determined by its coefficients. $S$ is not surjective, since elements in $\mathbb{R}^\infty$ which have infinitely many non-zero entries, such as $(1, 1, 1, \ldots)$ are not in $R(S)$. Since $S$ is not surjective, it is not invertible.
Problems

(a) \[ L_j (a_1 u_1 + \ldots + a_n u_n) + (b_1 u_1 + \ldots + b_n u_n) = a_j + b_j = \]
\[ L_j (a_1 u_1 + \ldots + a_n u_n) + L_j (b_1 u_1 + \ldots + b_n u_n), \quad \text{so} \]
\[ L_j \] is additive.
\[ L_j (c (a_1 u_1 + \ldots + a_n u_n)) = \]
\[ L_j (c a_1 u_1 + \ldots + c a_n u_n) = c a_j = c L_j (a_1 u_1 + \ldots + a_n u_n), \quad \text{so} \]
\[ L_j \] preserves scalar multiplication.

\[ k_i (a+b) = (a+b) v_i = a v_i + b v_i; \]
\[ k_i (c a) = c a v_i = c (a v_i); \]
so \( k_i \) is linear.

(b) \[ L_j (v_i) = L_j (0 u_1 + \ldots + 1 u_i + \ldots + 0 u_n) = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i \neq j. \end{cases} \]

So \( M(L_j) = (0, 0, \ldots, 1, 0, \ldots, 0) \in M_{\text{mxn}}(F) \)

with respect to the bases \((u_1, \ldots, u_n)\) in \( U \) and \((1)\) in \( F \).

\[ k_i(1) = v_i = 0 u_1 + \ldots + 1 u_i + \ldots + 0 u_n, \quad \text{so} \]
\[ M(k_i) = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_{\text{mxn}}(F) \]

with respect to \((1)\) in \( F \) and \((u_1, \ldots, u_n)\) in \( V \).

Then \[ M(k_i L_j) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_{\text{mxn}}(F). \]

(c) \( M : L(U, V) \to M_{\text{mxn}}(F) \) is invertible, and the matrices \( M(k_i L_j) \) for \( i=1, \ldots, m, \ j=1, \ldots, n \) form the basis of \( \text{null vectors in } M_{\text{mxn}}(F) \). Hence the \( k_i L_j \) form a basis of \( L(U, V) \).