Math 110 Spring 2012
HW 4 Solutions

<u>Exercises</u>

3.17 $A(B+C)$ makes sense (and so does $AB+AC$) when A is $m\times n$ and B, C are both $n\times p$, for some m, n, p. Then there are linear maps

$$\mathbb{F}^p \xrightarrow[T_B, T_C]{} \mathbb{F}^n \xrightarrow[T_A]{} \mathbb{F}^m$$

whose matrices with respect to the standard bases are $\mathcal{M}(T_A) = A$, $\mathcal{M}(T_B) = B$, $\mathcal{M}(T_C) = C$. We have

$$\begin{aligned} A(B+C) &= \mathcal{M}(T_A)(\mathcal{M}(T_B)+\mathcal{M}(T_C)) = \mathcal{M}(T_A)\mathcal{M}(T_B+T_C) \\ &= \mathcal{M}(T_A\circ(T_B+T_C)) \\ &= \mathcal{M}(T_A\circ T_B + T_A\circ T_C) \quad \leftarrow [\text{Ch. 3, p. 41 : distributive property of composition of linear maps}] \\ &= \mathcal{M}(T_A T_B) + \mathcal{M}(T_A T_C) \\ &= \mathcal{M}(T_A)\mathcal{M}(T_B) + \mathcal{M}(T_A)\mathcal{M}(T_C) \\ &= AB + AC. \end{aligned}$$

3.21 Let $T: M_{n\times 1}(\mathbb{F}) \to M_{m\times 1}(\mathbb{F})$ be a linear map and let $A = \mathcal{M}(T)$ with respect to the standard basis of unit column vectors. Then for any $v \in M_{n\times 1}(\mathbb{F})$ we have $\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v) = A\mathcal{M}(v)$ [Prop. 3.14]. But using the standard bases, we have $\mathcal{M}(v) = v$ in $M_{n\times 1}(\mathbb{F})$ and $\mathcal{M}(w) = w$ in $M_{m\times 1}(\mathbb{F})$. So

$$Tv = \mathcal{M}(Tv) = A\mathcal{M}(v) = Av.$$

3.22 If S and T are invertible then ST is invertible with inverse $T^{-1}S^{-1}$. For the converse, if $Tv = 0$ then $STv = 0$, so ST invertible $\Rightarrow N(ST) = 0 \Rightarrow N(T) = 0 \Rightarrow T$ injective. By Thm. 3.21, T is invertible. Then $S = (ST)\circ T^{-1}$ is also invertible.

3.23 If $ST = I$, then ST is invertible, so S and T are invertible by Ex. 3.23, and then $T = S^{-1}ST = S^{-1}I = S^{-1}$. Hence $TS = S^{-1}S = I$.

3.25 We have $\mathcal{M} : \mathcal{L}(V) \to M_{n\times n}(\mathbb{F})$ an invertible linear map, and $\mathcal{M}(\{\text{noninvertible operators}\}) = \{\text{non-invertible matrices } A \in M_{n\times n}(\mathbb{F})\}$. Under an invertible map, subspaces correspond to subspaces, so it suffices to show that the set of non-invertible $n\times n$ matrices is not a subspace ~~when~~ of $M_{n\times n}(\mathbb{F})$, for $n > 1$. We'll show that it's not additive, with a counterexample: let

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

The linear maps $T_A, T_B \in \mathcal{L}_V$ with matrices A, B are non-invertible, because if $(v_1, \ldots, v_n)$ is the basis of V we are using for $\mathcal{M}$, then $T_A v_2 = 0$, since col 2 is zero in A, and $T_B v_1 = 0$, since col 1 is zero in B. So T_A and T_B have non-zero nullspaces. But $A + B = I_n$, so $T_A + T_B = I_V$, which is invertible. (This doesn't work if $n = 1$ since then there is no v_2. In fact, if $\dim V = 1$ then $\mathcal{L}(V)$ consists of scalars, and the set of non-invertible ~~ones~~ ~~are~~ is $\{0\}$, which is a subspace.)

3.26 Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be $Tv = Av$ where A is the matrix with entries $a_{i,j}$. Then (a) holds $\Leftrightarrow$ T has nullspace $= 0$ $\Leftrightarrow$ T injective, and (b) holds $\Leftrightarrow$ T is surjective. Hence (a) $\Leftrightarrow$ (b) by Thm. 3.21

Exercises not from book

(a) The operators $D\,p(z) = p'(z)$ and $Z\,p(z) = z\,p(z)$ are clearly linear, and $T = D^2 - 2ZD + Z$, so T is linear.

If $p(z) \in P_d(\mathbb{R})$, the first two terms ~~exercise~~ of $Tp(z)$ have degree $\leq d$ and the last has degree $\leq d+1$, so $Tp(z) \in P_{d+1}(\mathbb{R})$.

(b)
$$\begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 1 & -2 & 0 & 6 & 0 \\ 0 & 1 & -4 & 0 & 12 \\ 0 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) S is injective, since a polynomial is determined by its coefficients. S is not surjective, since elements in $\mathbb{F}^\infty$ which have infinitely many nonzero entries, such as $(1, 1, 1, \ldots)$ are not in $R(S)$. Since S is not surjective, it is not invertible.

Problems

(a) $L_j((a_1u_1+\dots+a_nu_n)+(b_1u_1+\dots+b_nu_n))$
$= L_j((a_1+b_1)u_1+\dots+(a_n+b_n)u_n) = a_j+b_j =$
$L_j(a_1u_1+\dots+a_nu_n) + L_j(b_1u_1+\dots+b_nu_n)$, so
L_j is additive. $L_j(c(a_1u_1+\dots+a_nu_n)) =$
$L_j(ca_1u_1+\dots+ca_nu_n) = ca_j = cL_j(a_1u_1+\dots+a_nu_n)$, so
L_j preserves scalar multiplication.

$K_i(a+b) = (a+b)v_i = av_i+bv_i$
$K_i(ca) = cav_i = c(av_i)$
shows K_i is linear.

(b) $L_j(u_i) = L_j(0u_1+\dots+1u_i+\dots 0u_n) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$,

so $\mathcal{M}(L_j) = (0,0,\dots,\underset{\uparrow \text{ position } j}{1},0,\dots 0) \in M_{1\times n}(\mathbb{F})$
(with respect to the bases $(u_1,\dots,u_n)$ in $\mathcal{U}$ and (1) in $\mathbb{F}$).
$K_i(1) = v_i = 0v_1+\dots+1v_i+\dots+0v_m$, so

$$\mathcal{M}(K_i) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{position } i \qquad \in M_{m\times 1}(\mathbb{F})$$

with respect to (1) in $\mathbb{F}$ and $(v_1,\dots,v_m)$ in V.

Then $\mathcal{M}(K_iL_j) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}(0\dots 1\dots 0) = \begin{array}{c} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & & 0 \end{pmatrix} \leftarrow \text{row } i \\ \uparrow \text{col } j \end{array}$

$\in M_{m\times n}(\mathbb{F})$.

(c) $\mathcal{M}: \mathcal{L}(\mathcal{U},V) \to M_{m\times n}(\mathbb{F})$ is invertible, and the matrices $\mathcal{M}(K_iL_j)$ for $i=1,\dots,m$, $j=1,\dots,n$ form the basis of unit vectors in $M_{m\times n}(\mathbb{F})$. Hence the K_iL_j form a basis of $\mathcal{L}(\mathcal{U},V)$.