Prof. Haiman Spring 2012

Math 110 Assignment 3

(0) *A supplement to the reading.*

Theorem Let V, W be vector spaces over $\mathbb{F}$, and let $v_1, \ldots, v_n$ be a basis of V. Given vectors $w_1, \ldots, w_n \in W$, there is a unique linear map $T : V \to W$ such that $T(v_i) = w_i$ for all i.

This theorem and a sketch of its proof are hidden in the paragraph beginning near the top of page 40 in your textbook. It is actually the most important fact about linear maps! For this reason I have stated it above in full, and will ask you to prove it in full as one of the exercises below.

(I) *Exercises.*

Axler Chapter 3: 2, 3, 6, 10, 12, 13

(a) Give a full proof of the theorem stated at the top of this assignment.

(b) Prove that the above theorem also holds if V is not assumed to be finite-dimensional, provided we rephrase the theorem as follows: given a basis B of V and any map $f : B \to W$, there is a unique linear map $T : V \to W$ such that $T(v) = f(v)$ for all $v \in B$ (the set B is not a vector space, so f is just a map of sets—it would not make sense to say a linear map here).

In the theorem as we stated it for V finite-dimensional, what are B and f?

(c) Consider the map $T : \mathbb{C}^1 \to \mathbb{C}^1$ defined by $T((a + bi)) = (a + 0i)$. Verify that T is additive but not linear.

(d) Use Exercise (b) with V and W both equal to $\mathbb{R}$, considered as a vector space over the field of rational numbers $\mathbb{Q}$, to show that there exists an additive map $T : \mathbb{R} \to \mathbb{R}$ such that $T(1) = 1$ and $T(\sqrt{2}) = 0$. You may take as known the fact that $\sqrt{2}$ is irrational. In particular, T is an example of an additive map from $\mathbb{R}^1$ to itself which is not linear as a map of vector spaces over $\mathbb{R}$. Hint: first show that 1 and $\sqrt{2}$ are linearly independent elements of $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$.

(Axler says in the margin for Exercise 3.2 that advanced tools are needed to construct additive maps which are not linear. Over $\mathbb{C}$, as you can see from Exercise (c), no such tools are needed. Over $\mathbb{R}$, the "advanced tool" which we need in Exercise (d) is the existence of bases for infinite-dimensional vector spaces.)

(e) Prove that if $V = U_1 \oplus U_2$, and we are given linear maps $T_1 : U_1 \to W$ and $T_2 : U_2 \to W$, then there is a unique linear map $T : V \to W$ such that $T(v) = T_1(v)$ for all $v \in U_1$ and $T(v) = T_2(v)$ for all $v \in U_2$.

(II) *Problems.* Due Friday, Feb. 10 by 3pm at the location your GSI has specified for turning in homework.

Axler Chapter 3: 4, 5, 7