Math 110 Spring 2012
HW 3 Solutions

Exercises

3.2 Many examples are possible. Here is one:

$$f(x,y) = \begin{cases} x & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 . \end{cases}$$

You can check that $f(a(x,y)) = a f(x,y)$. It's not linear because, for instance, $f(1,0) + f(0,1) = 0 + 0 = 0$ but $f(1,1) = 1$.

3.3 Let $(u_1, \dots, u_m)$ be a basis of U, and extend it to a basis $(u_1, \dots, u_m, u_{m+1}, \dots, u_n)$ of V. Using the theorem that a linear map can be defined by specifying it on a basis, we have $T : V \to W$ such that $T(u_i) = S(u_i)$ for $i = 1, \dots, m$ and $T(u_i)$ can be chosen at will, let's say $T(u_i) = 0$, for $i > m$. Then since $(u_1, \dots, u_m)$ is a basis of U and $T(u_i) = S(u_i)$ for all $i = 1, \dots, m$, we have $T(u) = S(u)$ for all $u \in U$.

3.6 [There is a typo in the book: it should say $S_1 \cdots S_n$ rather than $S_1 \dots S_n$, since what he means is composition.] It's enough to show that the composition of two injective maps is injective, since this implies the case of any number of maps by applying it repeatedly. So suppose

$$V \xrightarrow[S_2]{} W \xrightarrow[S_1]{} Z$$

are both injective. If $S_1 \cdot S_2(x) = S_1 \cdot S_2(y)$, then $S_2(x) = S_2(y)$ since S_1 is injective. Then $x = y$ since S_2 is injective. This shows $S_1 \cdot S_2$ is injective.

3.10 The space in the problem, call it $U \subseteq \mathbb{F}^5$, consists of all vectors $(3a, a, b, b, b)$. A basis is $\{(3,1,0,0,0), (0,0,1,1,1)\}$, so $\dim U = 2$. If $T: \mathbb{F}^5 \to \mathbb{F}^2$ is a linear map, then

$$\dim N(T) = 5 - \dim R(T)$$

and $\dim R(T) \leq 2$, so $\dim N(T) \geq 3$. So $N(T) = U$ is impossible.

3.12 The "only if" part is Corollary 3.6. For "if", let $\dim W = m \leq n = \dim V$. Let $(w_1, \ldots, w_m)$ be a basis of W and $(v_1, \ldots, v_n)$ a basis of V. We can define a linear map $T: V \to W$ by

$$T(v_i) = w_i \qquad i = 1, \ldots, m$$
$$T(v_i) = 0 \qquad i = m+1, \ldots, n.$$

(this makes sense since $m \leq n$). Then all the basis elements w_i belong to $R(T)$, so $R(T) = W$, i.e., T is surjective.

3.13 "If": Let $\dim V = n$, $\dim W = m$, $\dim U = k$, so we are assuming $n \leq m + k$. Choose a basis $(u_1, \ldots, u_k)$ of U, and extend it to a basis $(u_1, \ldots, u_k, v_1, \ldots, v_\ell)$ of V. Note that $k + \ell = n$, so $\ell \leq m$. Let $(w_1, \ldots, w_m)$ be a basis of W, and define $T: V \to W$ by

$$T(u_i) = 0$$
$$T(v_i) = w_i \qquad i = 1, \ldots, \ell$$

(this makes sense since $\ell \leq m$). Then $U \subseteq N(T)$ since $T(u_i) = 0$ for all i, and $\dim R(T) = \ell$, since $R(T) = \operatorname{span}(w_1, \ldots, w_\ell)$ and the w_i's are independent. Hence $\dim N(T) = \dim V - \dim R(T) = n - \ell = k = \dim U$, so $N(T) = U$.

(a), (b): The finite-dimensional version is the special case where $B = \{v_1, \dots, v_n\}$ and $f(v_i) = w_i$. So it's enough to prove the general version.

Every $v \in V$ can be expressed as

$$v = a_1 u_1 + \dots + a_k u_k$$

for some $u_1, \dots, u_k \in B$. This expression is unique apart from adding or removing terms with coefficient $a_i = 0$. So we can define a map $T: V \to W$ by

$$T(v) = a_1 f(u_1) + \dots + a_k f(u_k),$$

where v is expressed as above. This is well-defined since adding or removing terms with coefficient 0 doesn't change the right-hand side in the formula for $T(v)$.

To show T is linear, suppose we have another vector

$$w = b_1 u_1 + \dots + b_k u_k .$$

Note that by adding terms to each expression with coefficient 0, we can assume that both v and w are written in terms of the same elements $u_i \in B$. Then

$$\begin{aligned} T(v+w) &= T\left((a_1+b_1)u_1 + \dots + (a_k+b_k)u_k\right) \\ &= (a_1+b_1) f(u_1) + \dots + (a_k+b_k) f(u_k) \quad \text{(by definition)} \\ &= a_1 f(u_1) + \dots + a_k f(u_k) + b_1 f(u_1) + \dots + b_k f(u_k) \\ &= T(v) + T(w) \end{aligned}$$

Similarly, for any scalar $c \in \mathbb{F}$,

$$\begin{aligned} T(cv) &= T(ca_1 u_1 + \dots + ca_k u_k) \\ &= ca_1 f(u_1) + \dots + ca_k f(u_k) \\ &= c\,(a_1 f(u_1) + \dots + a_k f(u_k)) \\ &= c\,T(v). \end{aligned}$$

Finally T is unique because for any linear map $T': V \to W$ such that $T'(v) = f(v)$ for $v \in B$, we have

$$T'(a_1u_1+\dots+a_ku_k) =$$
$$a_1T'(u_1)+\dots+a_kT'(u_k) \qquad \text{(by linearity)}$$
$$= a_1 f(u_1)+\dots+a_k f(u_k) \qquad (\text{since } u_i \in B)$$
$$= T(a_1u_1+\dots+a_ku_k).$$

(c) $$T((a+bi)+(a'+b'i)) = T((a+a')+(b+b')i)$$
$$= (a+a')+0i$$
$$= (a+0i)+(a'+0i)$$
$$= T(a+bi)+T(a'+b'i).$$

So T is additive [in fact it's a linear map of vector spaces over $\underline{\mathbb{R}}$.] But, for example,
$$T(1) = 1$$
$$T(i\cdot 1) = T(i) = 0 \neq iT(1),$$
so T is not linear (over $\mathbb{C}$).

(d) In $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$, $\operatorname{span}(1) = \mathbb{Q}\cdot 1 = \mathbb{Q}$. Since $\sqrt{2} \notin \mathbb{Q}$, 1 and $\sqrt{2}$ are linearly independent over $\mathbb{Q}$ (using Homework 2, Exercise (a)). Let B be a basis of $\mathbb{R}$ over $\mathbb{Q}$ such that $1, \sqrt{2} \in B$. (Recall from the lecture that any independent set can be extended to a basis, even in an infinite-dimensional vector space.) Define $T : \mathbb{R} \to \mathbb{R}$ to be the $\mathbb{Q}$-linear map such that
$$T(1) = 1$$
$$T(\sqrt{2}) = 0$$
$$T(b) = 0 \quad \text{for all } b \in B,\ b \neq 1, \sqrt{2}$$
Then T is additive, but not linear over $\mathbb{R}$, since
$$T(\sqrt{2}\cdot 1) = 0 \neq \sqrt{2}\,T(1) = \sqrt{2}$$

e) Let $(u_1, \dots, u_k)$ be a basis of U_1 and $(v_1, \dots, v_\ell)$ a basis of U_2. Since $V = U_1 \oplus U_2$, $(u_1, \dots, u_k, v_1, \dots, v_\ell)$ is a basis of V. Define $T : V \to W$ by $T(u_i) = T_1(u_i)$, $T(v_i) = T_2(v_i)$. Then $T(v) = T_1(v)$ for all $v \in U_1$ since the u_i span U_1, and $T(v) = T_2(v)$ for all $v \in U_2$ since the v_i span U_2.

Another way to do this problem is to use the fact that every $v \in V$ has a unique expression

$$v = u_1 + u_2 \qquad u_1 \in U_1, \; u_2 \in U_2,$$

Then define $T(v) = T_1(u_1) + T_2(u_2)$. If $v \in U_1$, then $u_2 = 0$, $u_1 = v$, so $T(v) = T_1(v)$, and similarly if $v \in U_2$ then $T(v) = T_2(v)$. One then must verify that T, defined this way, is linear.

Either way, T is unique since any linear map T' satisfying the conditions must have $T'(u_1 + u_2) = T_1(u_1) + T_2(u_2)$.

<u>Problems</u>

3.4 You may assume V finite-dimensional. Then since T can't be the 0 map, $R(T) = \mathbb{F}$, so $\dim R(T) = 1$, hence $\dim N(T) = \dim V - 1$. Also, $\dim \mathbb{F} \cdot u = 1$ since $\mathbb{F} \cdot u = \operatorname{span}(u)$. Now if $au \in N(T)$ then $0 = T(au) = a\,T(u) \Rightarrow a = 0$ since $T(u) \neq 0$. This shows $N(T) \cap \mathbb{F}u = 0$. By the dimension formula,

$$\begin{aligned} \dim(N(T) + \mathbb{F}u) &= \dim N(T) + \dim \mathbb{F}u - \dim(N(T) \cap \mathbb{F}u) \\ &= \dim V - 1 + 1 - 0 = \dim V. \end{aligned}$$

So $N(T) + \mathbb{F}u = V$, hence $V = N(T) \oplus \mathbb{F}u$.

If you don't want to assume V finite-dimensional, you can prove $N(T) \cap \mathbb{F}u = 0$ just as before, and prove $N(T) + \mathbb{F}u = V$ as follows: given $v \in V$, let $a = T(u)^{-1} T(v)$ (note that $T(u), T(v)$ are element ~~of~~ $\mathbb{F}$, and $T(u) \neq 0$, so this makes sense). Then $T(au) = aT(u) = T(v)$, so $T(v - au) = 0$, i.e. $v - au \in N(T)$. Hence

$$v = (v - au) + au \in N(T) + \mathbb{F}u.$$

3.5 Suppose

$$a_1 Tv_1 + \dots + a_n Tv_n = 0$$

we have to prove all $a_i = 0$. Now

$$T(a_1 v_1 + \dots + a_n v_n) = a_1 Tv_1 + \dots + a_n Tv_n = 0,$$

and $N(T) = 0$ since T is injective, so

$$a_1 v_1 + \dots + a_n v_n = 0.$$

Since $(v_1, \dots, v_n)$ was assumed independent, all $a_i = 0$.

3.7 Given $w \in W$, since T is surjective there is some $v \in V$ such that $T(v) = w$. Since $(v_1, \dots, v_n)$ spans, we can write

$$v = a_1 v_1 + \dots + a_n v_n \qquad (a_i \in \mathbb{F}).$$

Then

$$w = T(v) = T(a_1 v_1 + \dots + a_n v_n)$$
$$= a_1 T(v_1) + \dots + a_n T(v_n) \in \operatorname{Span}(Tv_1, \dots, Tv_n).$$