Exercises

3.2 Many examples are possible. Here is one:

\[ f(x,y) = \begin{cases} x & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases} \]

You can check that \( f(a(x,y)) = af(x,y) \). It's not linear because, for instance, \( f(1,0) + f(0,1) = 0 + 0 = 0 \) but \( f(1,1) = 1 \).

3.3 Let \((u_1,\ldots,u_m)\) be a basis of \(U\), and extend it to a basis \((u_1,\ldots,u_m,u_{m+1},\ldots,u_n)\) of \(V\). Using the theorem that a linear map can be defined by specifying it on a basis, we have \( T:V \to W \) such that \( T(u_i) = S(u_i) \) for \( i = 1,\ldots,m \) and \( T(u_i) \) can be chosen at will, let's say \( T(u_i) = 0 \) for \( i > m \). Then since \((u_1,\ldots,u_m)\) is a basis of \(U\) and \( T(u_i) = S(u_i) \) for all \( i = 1,\ldots,m \), we have \( T(u) = S(u) \) for all \( u \in U \).

3.6 [There is a typo in the book: it should say \( S_1\cdots S_n \) rather than \( S_1\ldots Sn \), since what he means is composition.] It's enough to show that the composition of two injective maps is injective, since this implies the case of any number of maps by applying it repeatedly. So suppose

\[ V \xrightarrow{S_2} W \xrightarrow{S_1} Z \]

are both injective. If \( S_1 \circ S_2(x) = S_1 \circ S_2(y) \), then \( S_2(x) = S_2(y) \) since \( S_1 \) is injective. Then \( x = y \) since \( S_2 \) is injective. This shows \( S_1 \circ S_2 \) is injective.

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\[ HW 3 \ Solutions \]
3.10 The space in the problem, call it $\mathbb{F}^5$, consists of all vectors $(3a, a, b, b, b)$. A basis is $\{(3,1,0,0,0), (0,0,1,1,1)\}$, so $\dim U = 2$. If $T: \mathbb{F}^5 \to \mathbb{F}^2$ is a linear map, then

$$\dim N(T) = 5 - \dim R(T)$$

and $\dim R(T) \leq 2$, so $\dim N(T) \geq 3$. So $N(T) = U$ is impossible.

3.12 The "only if" part is Corollary 3.6. For "if", let $\dim W = m \leq n = \dim V$. Let $(w_1, \ldots, w_m)$ be a basis of $W$ and $(v_1, \ldots, v_n)$ a basis of $V$. We can define a linear map $T: V \to W$ by

$$T(v_i) = w_i \quad i = 1, \ldots, m$$

$$T(v_i) = 0 \quad i = m+1, \ldots, n.$$  

(this makes sense since $m \leq n$). Then all the basis elements $w_i$ belong to $R(T)$, so $R(T) = W$, i.e., $T$ is surjective.

3.13 "If": let $\dim V = n$, $\dim W = m$, $\dim U = k$, so we are assuming $n \leq m+k$. Choose a basis $(u_1, \ldots, u_k)$ of $U$, and extend it to a basis $(u_1, \ldots, u_k, v_1, \ldots, v_n)$ of $V$. Note that $l + n = m$, so $l \leq m$. Define $T: V \to W$ by

$$T(u_i) = 0$$

$$T(v_i) = w_i \quad i = 1, \ldots, l$$

(this makes sense since $l \leq m$). Then $U \subseteq N(T)$ since $T(u_i) = 0$ for all $i$, and $\dim R(T) = l$, since $R(T) = \text{span}(w_1, \ldots, w_l)$ and the $w_i$'s are independent. Hence

$$\dim N(T) = \dim V - \dim R(T) = n - l = k = \dim U,$$

so $N(T) = U$.  

(a), (b): The finite-dimensional version is the special case where $B = \{v_1, \ldots, v_m\}$ and $f(v_i) = w_i$. So it's enough to prove the general version.

Every $v \in V$ can be expressed as

$$v = a_1 u_1 + \ldots + a_k u_k$$

for some $u_1, \ldots, u_k \in B$. This expression is unique apart from adding or removing terms with coefficient $a_i = 0$. So we can define a map $T : V \rightarrow W$ by

$$T(v) = a_1 f(u_1) + \ldots + a_k f(u_k),$$

where $v$ is expressed as above. This is well-defined since adding or removing terms with coefficient 0 doesn't change the right-hand side in the formula for $T(v)$.

To show $T$ is linear, suppose we have another vector $w = b_1 u_1 + \ldots + b_k u_k$.

Note that by adding terms to each expression with coefficient 0, we can assume that both $v$ and $w$ are written in terms of the same elements $u_i \in B$. Then

$$T(v+w) = T((a_1+b_1) u_1 + \ldots + (a_k+b_k) u_k)$$

$$= (a_1+b_1) f(u_1) + \ldots + (a_k+b_k) f(u_k) \quad \text{(by definition)}$$

$$= a_1 f(u_1) + \ldots + a_k f(u_k) + b_1 f(u_1) + \ldots + b_k f(u_k)$$

$$= T(v) + T(w)$$

Similarly, for any scalar $c \in F$,

$$T(cv) = T(c a_1 u_1 + \ldots + c a_k u_k)$$

$$= c a_1 f(u_1) + \ldots + c a_k f(u_k)$$

$$= a_1 (c f(u_1) + \ldots + c a_k f(u_k))$$

$$= c T(v).$$

Finally $T$ is unique because for any linear map $T' : V \rightarrow W$ such that $T'(v) = f(v)$ for $v \in B$, we have
\[ T'(a_1 u_1 + \ldots + a_k u_k) = \]
\[ a_1 T'(u_1) + \ldots + a_k T(u_k) \quad \text{(by linearity)} \]
\[ = a_1 f(u_1) + \ldots + a_k f(u_k) \quad \text{(since } u_i \in B) \]
\[ = T(a_1 u_1 + \ldots + a_k u_k). \]

(c) \[ T((a+bi) + (a'+b'i)) = T((a+a') + (b+b'i)) \]
\[ = (a+a') + (b+b'i) \]
\[ = (a+0i) + (a'+0i) \]
\[ = T(a+0i) + T(a'+0i). \]

So \( T \) is additive (in fact it's a linear map of vector spaces over \( \mathbb{R} \)). But, for example, \( T \)
\[ T(1) = 1 \]
\[ T(i,1) = T(i) = 0 \neq i T(1), \]
so \( T \) is not linear (over \( \mathbb{C} \)).

(a) In \( \mathbb{R} \) considered as a vector space over \( \mathbb{Q} \),
\[ \text{span}(1) = \mathbb{Q} \cdot 1 = \mathbb{Q}. \]
Since \( \sqrt{2} \notin \mathbb{Q} \), \( 1 \) and \( \sqrt{2} \)
are linearly independent over \( \mathbb{Q} \) (using Homework 2, Exercise (a)). Let \( B \) be a basis of \( \mathbb{R} \) over \( \mathbb{Q} \) such that \( 1, \sqrt{2} \in B \). (Recall from the lecture that
any independent set can be extended to a basis, even in an infinite-dimensional vector space.) Define
\( T : \mathbb{R} \to \mathbb{R} \) to be the \( \mathbb{Q} \)-linear map such that
\[ T(1) = 1 \]
\[ T(\sqrt{2}) = 0 \]
\[ T(b) = 0 \quad \text{for all } b \in B, \ b \neq 1, \sqrt{2} \]

Then \( T \) is additive, but not linear over \( \mathbb{R} \), since
\[ T(\sqrt{2} \cdot 1) = 0 \neq \sqrt{2} T(1) = \sqrt{2}. \]
e) Let \((u_1, \ldots, u_k)\) be a basis of \(U_1\) and \((v_1, \ldots, v_d)\) a basis of \(U_2\). Since \(V = U_1 \oplus U_2\), \((u_1, \ldots, u_k, v_1, \ldots, v_d)\) is a basis of \(V\). Define \(T: V \to W\) by \(T(u_i) = T_1(u_i)\) for all \(u_i \in U_1\), and \(T(v) = T_2(v)\) for all \(v \in U_2\). Then \(T(u_i) = T_1(u_i)\) for all \(u_i \in U_1\) since the \(u_i\) span \(U_1\), and \(T(v) = T_2(v)\) for all \(v \in U_2\) since the \(v_i\) span \(U_2\). Another way to do this problem is to use the fact that every \(v \in V\) has a unique expression
\[
v = u_1 + u_2, \quad u_1 \in U_1, \quad u_2 \in U_2.
\]

Then define \(T(v) = T_1(u_1) + T_2(u_2)\). If \(v \in U_1\), then \(u_2 = 0\), so \(u_1 = v\), so \(T(v) = T_1(v)\), and similarly if \(v \in U_2\) then \(T(v) = T_2(v)\). One then must verify that \(T\), defined this way, is linear.

Either way, \(T\) is unique since any linear map \(T'\) satisfying the conditions must have \(T'(u_1 + u_2) = T_1(u_1) + T_2(u_2)\).

Problems

3.4 You may assume \(V\) finite-dimensional. Then since \(T\) can't be the 0 map, \(R(T) = IF\), so \(\dim R(T) = 1\), hence \(\dim N(T) = \dim V - 1\). Also, \(\dim IF \cdot u = 1\) since \(IF \cdot u = \text{Span}(u)\). Now if \(u \in N(T)\) then \(0 = T(au) = aT(u) = a\cdot 0\) since \(T(u) \neq 0\). This shows \(N(T) \cap IF \cdot u = 0\). By the dimension formula,
\[
\dim (N(T) + IF \cdot u) = \dim N(T) + \dim IF \cdot u - \dim (N(T) \cap IF \cdot u) = \dim V - 1 + 1 - 0 = \dim V.
\]

So \(N(T) + IF \cdot u = V\), hence \(V = N(T) \oplus IF \cdot u\).
If you don't want to assume $U$ finite-dimensional, you can prove $N(T) + \text{Im } T = 0$ just as before, and prove $N(T) + \text{Im } T = V$ as follows: given $v \in V$, let

$$a = T(u)v, T(v)$$

(note that $T(u), T(v)$ are elements of $F$, and $T(u) \neq 0$, so this makes sense). Then $T(au) = a T(u) = T(v)$, so $T(v-au) = 0$, i.e., $v-au \in N(T)$. Hence

$$v = (v-au) + au \in N(T) + \text{Im } T.$$

3.5 Suppose

$$a_1Tv_1 + \ldots + a_nTv_n = 0$$

we have to prove all $a_i = 0$. Now

$$T(a_1v_1 + \ldots + a_nv_n) = a_1Tv_1 + \ldots + a_nTv_n = 0,$$

and $N(T) = 0$ since $T$ is injective, so

$$a_1v_1 + \ldots + a_nv_n = 0.$$

Since $(v_1, \ldots, v_n)$ was assumed independent, all $a_i = 0$.

3.7 Given $w \in W$, since $T$ is surjective there is some $v \in V$ such that $T(v) = w$. Since $(v_1, \ldots, v_n)$ spans, we can write

$$v = a_1v_1 + \ldots + a_nv_n \quad (a_i \in F).$$

Then

$$w = T(v) = T(a_1v_1 + \ldots + a_nv_n) = a_1T(v_1) + \ldots + a_nT(v_n) \in \text{Span } (Tv_1, \ldots, Tv_n).$$