

Math 110 Spring 2012
HW 2 Solutions

Exercises

2.1 It's enough to show that each $v_i \in \text{Span}(v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n)$.
But $v_i = (v_i - v_{i+1}) + (v_{i+1} - v_{i+2}) + \dots + (v_{n-1} - v_n) + v_n$, so this is true.

2.2 Suppose $a_1(v_1 - v_2) + a_2(v_2 - v_3) + \dots + a_{n-1}(v_{n-1} - v_n) + a_n v_n = 0$

Then $a_1 v_1 + (a_2 - a_1) v_2 + \dots + (a_n - a_{n-1}) v_n = 0$ by rewriting the left-hand side, hence $a_1, a_2 - a_1, \dots, a_n - a_{n-1}$ are all equal to 0, since (v_1, \dots, v_n) are independent. It follows that

$$0 = a_1 = a_2 = \dots = a_n,$$

i.e. all $a_i = 0$, which shows that $(v_1 - v_2, \dots, v_{n-1} - v_n, v_n)$ are independent.

2.5 Let $e_i = (0, \dots, 0, 1, 0, \dots) \in F^\infty$, where the 1 is in the i th place. Then $a_1 e_1 + \dots + a_n e_n = (a_1, a_2, \dots, a_n, 0, 0, \dots)$, which is 0 only if all $a_i = 0$. This shows (e_1, \dots, e_n) is linearly independent, for every n . If F^∞ were spanned by m vectors, then no independent list could be longer than m . So F^∞ is infinite-dimensional. [In the terminology of Exercises (b)-(e), the set $\{e_1, e_2, \dots\}$ is independent.] ~~the set $\{e_1, e_2, \dots\}$ is independent~~

2.6 The space $P(\mathbb{R})$ is a subspace of the space $C([0, 1])$, (if we identify a polynomial with the function it defines on $[0, 1]$). If $C([0, 1])$ were finite-dimensional, then so would $P(\mathbb{R})$ be.

2.8 The vectors $v_1 = (3, 1, 0, 0, 0)$, $v_2 = (0, 0, 7, 1, 0)$, $v_3 = (0, 0, 0, 0, 1)$ belong to U . Any vector in U can be written $v = (3x_2, x_2, 7x_4, x_4, x_5) = x_2 v_1 + x_4 v_2 + x_5 v_3$. This is clearly the unique expression for v as a linear combination of v_1, v_2, v_3 , so (v_1, v_2, v_3) is a basis.

2.9 True. For example, $B = \langle 1, z, z^3 + z^2, z^3 \rangle$ is a basis of $P_3(\mathbb{F})$. To prove that B is a basis, note that $1, z, z^3$ and $z^2 = (z^3 + z^2) - z^3$ are in $\text{span}(B)$, so B spans, and $\dim P_3(\mathbb{F}) = 4$, so B is a basis by Prop. 2.16.

2.11 Let (v_1, \dots, v_d) be a basis of U (since V is finite-dimensional, so is U , so it has a finite basis). Then $d = \dim U = \dim V$, and (v_1, \dots, v_d) is independent. Hence (v_1, \dots, v_d) is a basis of V , by Prop. 2.17. In particular, $U = \text{span}(v_1, \dots, v_d) = V$.

2.13 $\dim(U \cap W) = \dim(U + W) - \dim(U) - \dim(W) = 8 - 5 - 3 = 0$.

2.16 For $m=1$, the desired inequality is $\dim(U_1) \leq \dim(U_1)$, which is trivially true. For $m > 1$ we can assume that we already know it holds in the case of $m-1$ subspaces (this is the principle of mathematical induction). So we have $\dim(U_1 + \dots + U_{m-1}) \leq \dim U_1 + \dots + \dim U_{m-1}$. Let $W = U_1 + \dots + U_{m-1}$. Then $\dim(U_1 + \dots + U_m) = \dim(W + U_m) = \dim(W) + \dim(U_m) - \dim(W \cap U_m) \leq \dim(W) + \dim(U_m) \stackrel{\text{since } \dim(W \cap U_m) \geq 0}{\leq} \dim(U_1 + \dots + \dim U_{m-1} + \dim U_m)$.
 \uparrow since $\dim(W) \leq \dim U_1 + \dots + \dim U_{m-1}$.

Or, without using mathematical induction, let $d_i = \dim U_i$ and let $b_{i,1}, \dots, b_{i,d_i}$ be a basis of U_i , for each i . Then $(b_{1,1}, \dots, b_{1,d_1}, \dots, b_{m,1}, \dots, b_{m,d_m})$ is a list of $d_1 + \dots + d_m$ vectors which spans $U_1 + \dots + U_m$. Therefore $\dim(U_1 + \dots + U_m) \leq d_1 + \dots + d_m = \dim U_1 + \dots + \dim U_m$.

2.17 Take $b_{i,1}, \dots, b_{i,d_i}$ as in 2.16. In this case the sum being direct implies that $(b_{1,1}, \dots, b_{1,d_1}, \dots, b_{m,1}, \dots, b_{m,d_m})$ is a basis of V , so $\dim V = \dim U_1 + \dots + \dim U_m$.

a) (i) \Rightarrow (ii). Suppose $a_1v_1 + \dots + a_nv_n = 0$ with not all $a_i = 0$. Let j be the last index such that $a_j \neq 0$, so $a_{j+1} = \dots = a_n = 0$. Then

$$a_1v_1 + \dots + a_jv_j = 0$$

$$\Rightarrow v_j = -\frac{1}{a_j}(a_1v_1 + \dots + a_{j-1}v_{j-1}) \in \text{Span}(v_1, \dots, v_{j-1})$$

[this also works if $j=1$, in which case it shows that $v_1 = 0 \in \text{Span}(\emptyset)$].

(ii) \Rightarrow (iii) is trivial, since ~~the span of the~~
 $\text{Span}(v_1, \dots, v_{j-1}) \subseteq \text{Span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n)$

(iii) \Rightarrow (i)

Suppose $v_j \in \text{Span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n)$, say

$$v_j = a_1v_1 + \dots + a_{j-1}v_{j-1} + a_{j+1}v_{j+1} + \dots + a_nv_n.$$

Then $0 = a_1v_1 + \dots + a_{j-1}v_{j-1} + (-1)v_j + a_{j+1}v_{j+1} + \dots + a_nv_n$ is a non-trivial linear dependence, since the coefficient of v_j is $-1 \neq 0$.

b) Both sets are equal to $\{a_1v_1 + \dots + a_nv_n : a_i \in \mathbb{F}\}$.

c) By the definition, if $\{v_1, \dots, v_n\}$ is independent then (v_1, \dots, v_n) is independent. For the converse, if (v_1, \dots, v_n) we need to show that any list of distinct vectors from $\{v_1, \dots, v_n\}$ is independent. But such a list is a rearrangement of a sub-list of (v_1, \dots, v_n) , so it is independent.

d) If $X \subseteq W$, where W is a subspace, then every linear combination of vectors in X belongs to W . So $\text{Span}(X) \subseteq W$. $\text{Span}(X)$ is itself a subspace because a sum or scalar multiple of finite linear combinations of elements of X is itself a linear combination of elements of X .

c) This is just a restatement of the definition of X being linearly independent, with the condition that finite lists of distinct vectors in X are independent spelled out.

Problems

2.7 "if": suppose we have $v_1, v_2, \dots \in V$ with (v_1, \dots, v_n) independent for all n . If V had a spanning set of size m , this could not occur for $n > m$. So V must be infinite-dimensional.

"only if": Suppose V is infinite-dimensional. Then no finite list spans V , so we can construct a sequence v_1, v_2, \dots with the property that $v_n \notin \text{span}(v_1, \dots, v_{n-1})$ for all n . By Exercise (a), this implies that (v_1, \dots, v_n) is independent for all n . (If you use Exercise (a) as part of your solution you should also include your proof of Exercise (a).)

$$\begin{aligned} 2.14 \quad \dim(U \cap W) &= \dim(U) + \dim(W) - \dim(U + W) \\ &= 5 + 5 - \dim(U + W) \\ &= 10 - \dim(U + W). \end{aligned}$$

But $U + W \subseteq \mathbb{R}^5$, so $\dim(U + W) \leq 5$, hence $\dim(U \cap W) \geq 1$, and therefore $U \cap W \neq 0$.