Math 110 Assignment 1

(I) Exercises. Not to be handed in, but do them carefully to solidify your understanding and as preparation for exam problems and graded homework problems.

Axler Chapter 1: 2, 3, 5, 8, 10, 11, 12, 14.

(a) Recall from lecture the field $F_2$ with two elements 0, 1 and addition and multiplication given by the tables

\[
\begin{array}{c|cc}
+ & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\quad \begin{array}{c|cc}
\cdot & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]

Verify that $F_2$ satisfies the axioms of a field.

(b) In the field $F_2$, which element is $-1$?

(c) Prove that if $V$ is a vector space over $F_2$, then every vector in $V$ is its own additive inverse. Hint: use Axler, Proposition 1.6 and Exercise (b).

(d) Verify that $F^n$, $F^\infty$ and $P(F)$ satisfy the axioms of a vector space. You should do this using the axioms of the field $F$, so that your verification is not specific only to the fields $F = \mathbb{R}$ or $F = \mathbb{C}$.

(e) Suppose we define a new scalar multiplication operation $\odot$ on $F^n$ by the rule $a \odot v = 0$ for all $a \in F$ and $v \in F^n$. Show that the set $F^n$ with the usual addition operation and this new scalar multiplication satisfies all the axioms of a vector space on page 9 of Axler, except for the multiplicative identity axiom. This demonstrates that the multiplicative identity axiom is not redundant.

(f) Let $S$ be a subset of $\{1, \ldots, n\}$. Let $Z(S) \subseteq F^n$ denote the set of vectors $v = (v_1, \ldots, v_n)$ such that $v_i = 0$ for every index $i \in S$. Show that $Z(S)$ is a subspace of $F^n$. For which subsets $S$ and $T$ do we have $Z(S) \cap Z(T) = 0$? For which $S$ and $T$ do we have $F^n = Z(S) + Z(T)$? For which $S$ and $T$ do we have $F^n = Z(S) \oplus Z(T)$?

(II) Problems. Due Friday, Jan. 27 by 3:00pm at your GSI’s office or mailbox.

Axler, Chapter 1: 9, 15.

Hints for #9: Prove the contrapositive, i.e., prove that if $U_1$ and $U_2$ are subspaces of $V$, $U_1$ is not contained in $U_2$, and $U_2$ is not contained in $U_1$, then $U_1 \cup U_2$ is not a subspace of $V$. How can you use the hypothesis that neither subspace is contained the other? Which axiom in the definition of subspace can you show must be violated by the subset $U_1 \cup U_2$?

Hint for #15: Think about subspaces of $\mathbb{R}^2$. 