Prof. Haiman | Spring 2012

Math 110 Assignment 1

(I) *Exercises.* Not to be handed in in, but do them carefully to solidify your understanding and as preparation for exam problems and graded homework problems.

Axler Chapter 1: 2, 3, 5, 8, 10, 11, 12, 14.

(a) Recall from lecture the field $\mathbb{F}_2$ with two elements 0, 1 and addition and multiplication given by the tables

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}.$$

Verify that $\mathbb{F}_2$ satisfies the axioms of a field.

(b) In the field $\mathbb{F}_2$, which element is -1?

(c) Prove that if V is a vector space over $\mathbb{F}_2$, then every vector in V is its own additive inverse. Hint: use Axler, Proposition 1.6 and Exercise (b).

(d) Verify that $\mathbb{F}^n$, $\mathbb{F}^\infty$ and $\mathcal{P}(\mathbb{F})$ satisfy the axioms of a vector space. You should do this using the axioms of the field $\mathbb{F}$, so that your verification is not specific only to the fields $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

(e) Suppose we define a new scalar multiplication operation $\odot$ on $\mathbb{F}^n$ by the rule $a \odot v = 0$ for all $a \in \mathbb{F}$ and $v \in \mathbb{F}^n$. Show that the set $\mathbb{F}^n$ with the usual addition operation and this new scalar multiplication satisfies all the axioms of a vector space on page 9 of Axler, except for the multiplicative identity axiom. This demonstrates that the multiplicative identity axiom is not redundant.

(f) Let S be a subset of $\{1, \ldots, n\}$. Let $Z(S) \subseteq \mathbb{F}^n$ denote the set of vectors $v = (v_1, \ldots, v_n)$ such that $v_i = 0$ for every index $i \in S$. Show that $Z(S)$ is a subspace of $\mathbb{F}^n$. For which subsets S and T do we have $Z(S) \cap Z(T) = 0$? For which S and T do we have $\mathbb{F}^n = Z(S) + Z(T)$? For which S and T do we have $\mathbb{F}^n = Z(S) \oplus Z(T)$?

(II) *Problems.* Due Friday, Jan. 27 by 3:00pm at your GSI's office or mailbox.

Axler, Chapter 1: 9, 15.

Hints for #9: Prove the contrapositive, *i.e.*, prove that if U_1 and U_2 are subspaces of V, U_1 is not contained in U_2, and U_2 is not contained in U_1, then $U_1 \cup U_2$ is not a subspace of V. How can you use the hypothesis that neither subspace is contained the other? Which axiom in the definition of subspace can you show must be violated by the subset $U_1 \cup U_2$?

Hint for #15: Think about subspaces of $\mathbb{R}^2$.