Exercises

1.2: Use \((a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\) with
\[
a = -\frac{1}{2}, \quad b = \frac{\sqrt{3}}{2}i,
\]
to get
\[
\left(-\frac{1+i\sqrt{3}}{2}\right)^3 = -\frac{1}{8} + \frac{3\sqrt{3}i}{8} + \frac{9}{8} - \frac{3\sqrt{3}i}{8} = 1.
\]

1.3: Let \(w\) denote \(-v\). By Prop 1.2, \(w\) is uniquely determined by the equation \(w + (-v) = 0\). But \(v + (-v) = 0\), so \(w = v\).

1.5: (a) subspace, (b) not (doesn't contain 0), (c) not (isn't closed under +), (d) subspace.

1.8: Let \(\{W_a\}_{a \in F}\) be a collection of subspaces (i.e. we have one subspace \(W_a\) for each \(a\) in some (possibly infinite) index set \(I\). Let \(W = \bigcap_{a \in I} W_a\). We verify:

- \(W\) is closed under +: if \(x, y \in W\), then \(x, y \in W_a\) for all \(a\), so \(x + y \in W_a\) for all \(a\) since each \(W_a\) is a subspace. Hence \(x + y \in W\).
- \(W\) is closed under \(.\): if \(x \in W\), \(a \in F\), then \(ax \in W_a\) for all \(a\) since \(W_a\) is a subspace. Hence \(ax \in W\).
- \(W\) contains 0: true, since 0 \(\in W_a\) for all \(a\).

1.10: \(U + U = U\). More precisely, \(U + U \subseteq U\) since \(U\) is closed under +, and \(U \subseteq U + U\) since \(0 \in U\), so for every \(v \in U\) we have \(v = v + 0 \in U + U\).

1.12: The zero subspace is an identity: \(U + 0 = U\). The only subspace with an inverse is 0. Indeed, if \(U + V = 0\), then \(U\) and \(V\) are both 0, since \(U, V \subseteq U + V\).

1.14: The simplest choice is to take \(W\) to be the subspace consisting of all polynomials \(p(x) \in P(F)\) whose coefficients of \(x^2\) and \(x^3\) are both equal to zero. (But this is not the only solution—see Problem 1.15.)
(a) Since \( \mathbb{F}_2 \) has only two elements, you can verify the axioms case-by-case. E.g., this table verifies that 
\( + \) is associative (\( (x+y)+z = x+(y+z) \)):

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>( (x+y)+z )</th>
<th>( x+(y+z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The last two columns are computed using the table that defines \( + \) in \( \mathbb{F}_2 \). Note that 0 is identity for \( + \) and 1 is identity for \( \cdot \) (because of the + and \( \cdot \) tables, not because I named these elements "0" and "1").

(b) \(-1 = 1\)

(c) By Prop. 1.16, \(-v = (-1)v = av = v\).

(d) There's a lot to check, but it's all straightforward. I won't write down details.

(e) The first four axioms in Axler p. 9 only refer to \( + \), so they also hold for \( \mathbb{F}^m \) with \( + \) and \( \cdot \). For the distributive laws:

\[
(a+b)v = av + bv \quad \text{also because } 0 = 0+0.
\]

(f) \( \mathbb{Z}(S) \cap \mathbb{Z}(T) = 0 \iff S \cup T = \{1, \ldots, m\} \).

\( \mathbb{Z}(S) + \mathbb{Z}(T) = \mathbb{F}^m \iff S \cap T = \emptyset \)

\( \mathbb{F}^m = \mathbb{Z}(S) \oplus \mathbb{Z}(T) \iff S \cup T = \{1, \ldots, m\} \quad \text{and} \quad S \cap T = \emptyset \), i.e. \( \{1, \ldots, m\} \) is the disjoint union of \( S \) and \( T \).
1.9 First let's write out what is to be proved more fully:

**Theorem** Let $U_1, U_2$ be subspaces of $V$. Then $U_1 \cup U_2$ is a subspace if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

*Proof.* $(\Rightarrow)$ Exchanging the two subspaces if necessary, we can assume that $U_1 \subseteq U_2$. Then $U_1 \cup U_2 = U_2$, which is a subspace by hypothesis.

$(\Leftarrow)$ We'll prove the contrapositive: $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$ implies $U_1 \cup U_2$ is not a subspace.

Since $U_1 \not\subseteq U_2$, we can choose a vector $u \in U_1 \setminus U_2$.

Since $U_2 \not\subseteq U_1$, we can choose a vector $u_2 \in U_2 \setminus U_1$.

Then $u_1$ and $u_2$ are both in $U_1 \cup U_2$. We'll show that $u_1 + u_2 \notin U_1 \cup U_2$, so $U_1 \cup U_2$ is not closed under $+$. Suppose for contradiction that $u_1 + u_2 \in U_1 \cup U_2$, i.e., $u_1 + u_2 \in U_1$, or $u_1 + u_2 \in U_2$.

If $u_1 + u_2 \in U_1$ then, since $U_1 \subseteq U_1$, we have $u_2 = (u_1 + u_2) + (-u_1) \in U_1$.

Since $U_1$ is a subspace. But this contradicts our choice of $u_2$ to be in $U_2 \setminus U_1$. A similar argument leads to a contradiction if $u_1 + u_2 \in U_2$.

1.15 Counterexample: Let $U_1, U_2, W$ be three distinct lines through $0$ in $\mathbb{R}^2$ (and $V = \mathbb{R}^2$). Then $V$ is the $\Theta$ of any two $\Theta$ of $U_1, U_2$ and $W$. In particular, $V = U_1 \oplus W$ and $V = U_2 \oplus W$,

but $U_1 \not\subseteq U_2$

Note: You should write solutions to problems requiring a proof in a grammatically and mathematically complete style, similar to the above.