Math 110 HW 1 solutions

Exercises

1.2 : Use $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = -\frac{1}{2}$, $b = \frac{\sqrt{3}}{2}i$, to get

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = -\frac{1}{8} + \frac{3\sqrt{3}i}{8} + \frac{9}{8} - \frac{3\sqrt{3}i}{8} = 1.$$

1.3: Let w denote $-(-v)$. By Prop 1.2, w is uniquely determined by the equation $w + (-v) = 0$. But $v + (-v) = 0$, so $w = v$.

1.5 : (a) subspace, (b) not (doesn't contain 0), (c) ~~not~~ (isn't closed under +), (d) subspace.

1.8 : Let $\{W_\alpha\}_{\alpha \in I}$ be a collection of subspaces (i.e. we have one subspace W_α for each α in some (possibly infinite) index set I. Let $W = \bigcap_{\alpha \in I} W_\alpha$. We verify:

- W is closed under + : if $x, y \in W$, then $x, y \in W_\alpha$ for all α, so $x+y \in W_\alpha$ for all α since each W_α is a subspace. Hence $x+y \in W$.
- W is closed under $\cdot$: if $x \in W$, $a \in \mathbb{F}$, then $ax \in W_\alpha$ for all α since W_α is a subspace. Hence $ax \in W$.
- W contains 0: true, since $0 \in W_\alpha$ for all α.

1.10 : $U + U = U$. More precisely, $U + U \subseteq U$ since U is closed under +, and $U \subseteq U + U$ since $0 \in U$, so for every $v \in U$ we have $v = v + 0 \in U + U$.

1.12 : The zero subspace is an identity : $U + 0 = U$. The only subspace with an inverse is 0. Indeed, if $U + V = 0$, then U and V are both 0, since $U, V \subseteq U + V$.

1.14 : The simplest choice is to take W to be the subspace consisting of all polynomials $p(z) \in \mathcal{P}(\mathbb{F})$ whose coefficients of z^2 and z^3 are both equal to zero. (But this is not the only solution — see Problem 1.15)

(a) Since $\mathbb{F}_2$ has only two elements, you can verify the axioms case-by-case. E.g. this table verifies that $+$ is associative ($(x+y)+z = x+(y+z)$):

x	y	z	$(x+y)+z$	$x+(y+z)$
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
1	0	0	1	1
0	1	1	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

The last two colums are computed using the table that defines $+$ in $\mathbb{F}_2$. Note that 0 is identity for $+$ and 1 is identity for $\cdot$ (because of the $+$ and $\cdot$ tables, not because I named these elements "0" and "1").

(b) $-1 = 1$

(c) By Prop. 1.6, $-v = (-1)v = 1v = v$.

(d) There's a lot to check, but it's all straightforward. I won't write down details.

(e) The first four axioms on Axler p. 9 only refer to $+$, so they also hold for $\mathbb{F}^n$ with $+$ and $\odot$. For the distributive laws:

$$a \odot (v+u) = a \odot u + a \odot v \quad \text{because } 0 = 0+0.$$

$$(a+b) \odot v = a \odot v + b \odot v \quad \text{also because } 0 = 0+0.$$

(f) $Z(S) \cap Z(T) = 0 \iff S \cup T = \{1, \dots, n\}$.

$Z(S) + Z(T) = \mathbb{F}^n \iff S \cap T = \emptyset$

$\mathbb{F}^n = Z(S) \oplus Z(T) \iff S \cup T = \{1, \dots, n\}$ and $S \cap T = \emptyset$, i.e. $\{1, \dots, n\}$ is the disjoint union of S and T.

Problems

1.9 First let's write out what is to be proved more fully:

<u>Theorem</u> Let U_1, U_2 be subspaces of V. Then $U_1 \cup U_2$ is a subspace if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

<u>Proof.</u> ($\Leftarrow$) Exchanging the two subspaces if necessary, we can assume that $U_1 \subseteq U_2$. Then $U_1 \cup U_2 = U_2$, which is a subspace by hypothesis.

($\Rightarrow$) We'll prove the contrapositive: $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$ implies $U_1 \cup U_2$ is not a subspace.

Since $U_1 \not\subseteq U_2$, we can choose a vector $u_1 \in U_1 \setminus U_2$.
Since $U_2 \not\subseteq U_1$, we can choose a vector $u_2 \in U_2 \setminus U_1$.

Then u_1 and u_2 are both in $U_1 \cup U_2$. We'll show that $u_1 + u_2 \notin U_1 \cup U_2$, so $U_1 \cup U_2$ is not closed under $+$. Suppose for contradiction that $u_1 + u_2 \in U_1 \cup U_2$, i.e. $u_1 + u_2 \in U_1$ or $u_1 + u_2 \in U_2$.

If $u_1 + u_2 \in U_1$ then, since $u_1 \in U_1$, we have

$$u_2 = \underset{\in U_1}{(u_1+u_2)} + \underset{\in U_1}{(-u_1)} \in U_1,$$

Since U_1 is a subspace. But this contradicts our choice of u_2 to be in $U_2 \setminus U_1$. A similar argument leads to a contradiction if $u_1 + u_2 \in U_2$.

1.15 <u>Counterexample</u>: Let U_1, U_2, W be three distinct lines through 0 in $\mathbb{R}^2$ (and let $V = \mathbb{R}^2$). Then V is the $\oplus$ of any two of U_1, U_2 and W. In particular,

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

but $U_1 \neq U_2$

<u>Note</u>: You should write solutions to problems requiring a proof in a grammatically and mathematically complete style, similar to the above.