

The first 10 questions are true/false, worth 5 points each. Give brief reasons (one or two sentences) for your answers.

1. **TRUE** or FALSE: The list of polynomials $(z+1, z-1, z^2, z^3-z)$ is a basis of $\mathcal{P}_3(\mathbb{R})$.

Each standard monomial $1, z, z^2, z^3$ can be written as a linear combination of the given polynomials, so they span.
Since $\dim(\mathcal{P}_3(\mathbb{R})) = 4$, the list is a basis.

2. **TRUE** or **FALSE**: If U , V , and W are subspaces of a vector space Z , and we have $\dim(Z) = 10$, $\dim(U) = 2$, $\dim(V) = 3$, $\dim(W) = 5$, then $Z = U \oplus V \oplus W$.

We also need $U + V + W = Z$.

3. **TRUE** or FALSE: If $S: V \rightarrow W$ and $T: U \rightarrow V$ are linear maps, and ST is injective, then T is injective.

$Tx = 0 \Rightarrow STx = 0 \Rightarrow x = 0$, since ST is injective.

4. **TRUE** or FALSE: If S and T are linear operators on V , and $U \subseteq V$ is an invariant subspace for both S and T , then U is invariant for ST .

$ST(u) \subseteq S(u) \subseteq u$.

5. **TRUE** or FALSE: Let $T \in \mathcal{L}(\mathbb{R}^3)$ be the linear operator whose matrix with respect to the standard basis is

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then T is diagonalizable.

The eigenvalues of T are $\lambda=1, \lambda=3$. The eigenspace $\text{null}(T-3I)$ is equal to $\text{span}((1,0,0), (0,1,0))$, so its dimension is 2. Hence the eigenspace dimensions add up to 3.

6. TRUE or FALSE: Let V be a finite-dimensional vector space over \mathbb{R} , and let $T \in \mathcal{L}(V)$. Then for every integer d between 0 and $\dim(V)$, there exists a T -invariant subspace $U \subseteq V$ such that $\dim(U) = d$.

See #7.

7. TRUE or FALSE: Let V be a finite-dimensional vector space over \mathbb{C} , and let $T \in \mathcal{L}(V)$. Then for every integer d between 0 and $\dim(V)$, there exists a T -invariant subspace $U \subseteq V$ such that $\dim(U) = d$.

Over \mathbb{C} it follows from the theorem that there is a basis in which $M(T, \underline{\epsilon})$ is upper-triangular, but this does not hold over \mathbb{R} . In particular, over \mathbb{R} , any T with no non-zero eigenvector is a counterexample.

8. TRUE or FALSE: If V is an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and $\phi: V \rightarrow \mathbb{F}$ is a linear functional, then there is a unique vector $u \in V$ such that $\phi(v) = \langle v, u \rangle$ for all $v \in V$.

This was proved in the text and the lecture — it is the theorem which justifies the definition of adjoints.

9. TRUE or FALSE: Let V and W be inner product spaces. If $S: V \rightarrow W$ is an isometry, then $S^*: W \rightarrow V$ is an isometry.

$$(S^*)^* = S = (S^*)^{-1}, \text{ so } S^* \text{ is an isometry.}$$

↑
since S is an isometry

Or, $S^* = S^{-1}$, and the inverse of an isometry is an isometry.

10. TRUE or FALSE: If T is a positive operator, then so is T^k , for every positive integer k .

$$(T^k)^* = (T^*)^k = T^k, \text{ so } T^k \text{ is self-adjoint.}$$

The eigenvalues of T^k are λ^k , where λ is an eigenvalue of T . These are real and ≥ 0 , so T^k is positive.

11. (6 pts) Let (v_1, v_2, v_3) be a basis of a vector space V over \mathbb{R} . Let S be the linear operator whose matrix with respect to this basis is

$$\begin{pmatrix} 2 & -3 & 5 \\ 1 & 6 & -2 \\ 0 & 4 & -1 \end{pmatrix}.$$

Find the matrix of S with respect to the basis (v_3, v_2, v_1) .

$$\begin{pmatrix} -1 & 4 & 0 \\ -2 & 6 & 1 \\ 5 & -3 & 2 \end{pmatrix} \quad \begin{array}{l} \text{This is clear from the definition of} \\ \text{the matrix of an operator, or it can} \\ \text{be gotten using a change of basis matrix} \\ \text{as } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 & 5 \\ 1 & 6 & -2 \\ 0 & 4 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{array}$$

12. Let T be the operator on the Euclidean inner product space \mathbb{R}^3 whose matrix with respect to the standard basis is

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) (5 pts) Find the polar decomposition of T . Hint: the matrix of T^*T is diagonal.

$$T^*T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \text{ so } \sqrt{T^*T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \text{ So the polar}$$

decomposition, expressed in terms of matrices, is

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) (5 pts) Find the singular value decomposition of T . You may express your answer either as a matrix factorization, or by specifying bases (e_i) , (f_i) and scalars s_i such that $T(v) = \sum_i s_i \langle v, e_i \rangle f_i$.

Since the unit vectors are eigenvectors of $\sqrt{T^*T}$, the SVD in matrix form is

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Equivalently, $T(v) = \sum_{i=1}^3 s_i \langle v, e_i \rangle f_i$; where $s_1 = s_2 = 1$, $s_3 = 2$, (e_1, e_2, e_3) is the standard basis, and

$$f_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

13. (8 pts) Let V be an inner product space and let U be a subspace. Then U is an inner product space with the restriction of the inner product on V . Let $J: U \rightarrow V$ be the inclusion map $J(u) = u$ for all $u \in U$. Let $P: V \rightarrow U$ be the orthogonal projection on U . Prove that $J^* = P$.

By the definition of J^* , we need to prove that $\langle u, Pv \rangle = \langle Ju, v \rangle$ for all $u \in U$, $v \in V$. Let $v = u' + w$, where $u' \in U$, $w \in U^\perp$. Then $Pv = u'$ by definition. Hence $\langle u, Pv \rangle = \langle u, u' \rangle$, while $\langle Ju, v \rangle = \langle u, v \rangle$ $= \langle u, u' + w \rangle = \langle u, u' \rangle + \langle u, w \rangle = \langle u, u' \rangle$, since $\langle u, w \rangle = 0$ (because $w \in U^\perp$).

14. Let T be the linear operator on $P_3(\mathbb{C})$ defined by $T(p(z)) = 2p(z) - p'(z)$.

- (a) (4 pts) Show that the only eigenvalue of T is $\lambda = 2$.

$T - 2I = -D$, where $D p(z) = p'(z)$. Since D is nilpotent, the whole space $P_3(\mathbb{C})$ is the generalized eigenspace of T for $\lambda = 2$. This shows that $\lambda = 2$ is the only eigenvalue.

- (b) (3 pts) Describe the generalized eigenspaces of T .

See above : the whole space $P_3(\mathbb{C})$ is the generalized e-space for $\lambda = 2$.

- (c) (4 pts) Find the matrix of T in Jordan canonical form.

The $\lambda = 2$ eigenspace (not generalized) of T is the nullspace of D , i.e. $\text{span}(1)$. Since its dimension is 1, T has a single Jordan block :

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

15. Let $M_{n \times n}(\mathbb{F})$ denote the space of n by n matrices over a field \mathbb{F} .

(a) (4 pts) Find a linear operator R on $M_{n \times n}(\mathbb{F})$ such that the nullspace of R is the set of symmetric matrices in $M_{n \times n}(\mathbb{F})$.

$R(A) = A - A^T$ is clearly linear with

$$\text{null}(R) = \{ A \in M_{n \times n}(\mathbb{F}) : A = A^T \}.$$

(b) (4 pts) What is the dimension of the range of the operator R in part (a)?

The entries of a symmetric matrix on and above the diagonal may be chosen freely, and they determine the remaining entries. Hence the space of symmetric matrices $\text{null}(R)$ has dimension $n + \frac{n^2-n}{2} = \frac{n^2+n}{2}$. By the rank/nullity theorem,

$$\dim(\text{range}(R)) = n^2 - \dim(\text{null}(R)) = \frac{n^2-n}{2} = \frac{n(n-1)}{2}.$$

16. (7 pts) We can define an inner product on $\mathcal{P}_2(\mathbb{R})$ by

$$\langle p(x), q(x) \rangle = \frac{1}{2} \int_{-1}^1 p(x)q(x) dx.$$

Note: the limits of integration are different than those for the familiar inner product from the text and homework.

Given that the polynomials $p_0(x) = 1$, $p_1(x) = \sqrt{3}x$, $p_2(x) = (\sqrt{5}/2)(3x^2 - 1)$ form an orthonormal basis, find the polynomial $p(x) \in \mathcal{P}_2(\mathbb{R})$ which best approximates the function $f(x) = x^3$ on $[-1, 1]$, in the sense that it minimizes $\|p(x) - f(x)\|$.

$$\langle f, p_0 \rangle = \frac{1}{2} \int_{-1}^1 1 \cdot f(x) dx = \frac{1}{2} \int_{-1}^1 x^3 dx = 0 \quad \text{since } x^3 \text{ is an odd function.}$$

$$\langle f, p_1 \rangle = \frac{1}{2} \int_{-1}^1 \sqrt{3}x \cdot x^3 dx = \frac{\sqrt{3}}{2} \int_{-1}^1 x^4 dx = \frac{\sqrt{3}}{5}$$

$$\langle f, p_2 \rangle = \frac{1}{2} \int_{-1}^1 (\sqrt{5}/2)(3x^2 - 1) x^3 dx = 0 \quad \text{since the integrand is again an odd function.}$$

$$\text{Then } p(x) = \sum_i \langle f, p_i \rangle p_i = \frac{\sqrt{3}}{5} p_1(x) = \frac{3}{5} x.$$