Name ______________________ Section time & instructor ______________________

Student ID______________________

Math 110—Linear Algebra, Spring 2012—Haiman

Midterm 2

Instructions:

1. Write your name, ID number and discussion section time and instructor's name at the top of this page. Do not look at the other pages until the signal to start is given.
2. You may use one sheet (written on both sides) of prepared notes. No other notes, books, calculators, computers, cell phones, audio players, or other aids may be used.
3. Use your own scratch paper for preliminary work, then write your solutions on the exam paper. Hand in only the exam paper itself.
4. If a problem has a yes/no or numerical answer, give a **brief justification** of your answer. This need only be a sentence or two, not a complete proof.
5. If a problem asks you to **prove** something, give a complete proof, written out in clear and mathematically correct language. In proofs you may use any theorems which were proved in the textbook, the lectures, or which you were asked to prove on homework assignmnents.
6. There are six problems, worth a total of 100 points.

For grading use only			
1		2	
3		4	
5		6	
Total:			

1. (15 pts) Find a basis of $\mathbb{R}^3$ which is orthonormal in the Euclidean inner product, and which has the property that a subset of this basis spans the subspace $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$.

Start with a (non-orthonormal) basis in which the first two elements span W, such as $v_1 = (1,-1,0)$, $v_2 = (0,1,-1)$, $v_3 = (0,0,1)$.

Apply Gram-Schmidt:

$u_1 = v_1/\|v_1\| = \boxed{\frac{1}{\sqrt{2}}(1,-1,0)}$

$w_2 = v_2 - \langle v_2, u_1\rangle u_1 = (0,1,-1) + (\frac{1}{2}, -\frac{1}{2}, 0) = (\frac{1}{2}, \frac{1}{2}, -1)$

$u_2 = w_2/\|w_2\| = \boxed{\frac{1}{\sqrt{6}}(1,1,-2)}$

$w_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_2, u_2\rangle u_2$

$= (0,0,1) - (0,0,0) + (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

$u_3 = w_3/\|w_3\| = \boxed{\frac{1}{\sqrt{3}}(1,1,1).}$

Your solution may differ if you started with a different basis v, but you should always get $\pm\frac{1}{\sqrt{3}}(1,1,1)$ as one of the vectors, and two orthonormal vectors in W as the other two.

2. (15 pts) Let $T: V \to W$ be a linear map between finite-dimensional vector spaces. Let M be the matrix of T with respect to some bases of V and W. Suppose the columns of M are linearly independent. What can you conclude from this about T being injective, surjective, or invertible?

T is injective. To see why, let $\underset{\sim}{v}$, $\underset{\sim}{w}$ be the bases such that $M = \mathcal{M}(T, \underset{\sim}{v}, \underset{\sim}{w})$. By definition, the columns of M are the matrices of the vectors $T(v_i)$ in the basis $\underline{w}$. Since $u \mapsto \mathcal{M}(u, \underline{w})$ is an isomorphism $W \to \mathbb{F}^m$ ($m = \dim W$), this means the vectors $T(v_i)$ are linearly independent.

The nullspace of T consists of those vectors $a_1 v + \dots + a_n v_n$ ($n = \dim V$) such that $a_1 T(v_1) + \dots + a_n T(v_n) = 0$. So $T(v_i)$ being linearly independent $\Rightarrow N(T) = 0 \Rightarrow T$ is injective.

3. (15 pts) Let $T\colon V \to W$ be an invertible linear map between finite-dimensional vector spaces, and let $\mathbf{v} = (v_1, \ldots, v_n)$ be a basis of V. Since T is invertible, $\mathbf{w} = (Tv_1, \ldots, Tv_n)$ is a basis of W. Describe the matrix $M(T, \mathbf{v}, \mathbf{w})$ of T with respect to the basis $\mathbf{v}$ of V and the basis $\mathbf{w}$ of W.

By definition, the columns of $M(T, \underline{v}, \underline{w})$ are the matrices of the vectors $T(v_i)$ in the basis $\underline{w}$. Since $Tv_i = w_i$, M is the identity matrix.

4. (15 pts) Let $T \in \mathcal{L}(V)$ be a linear operator. Prove that the range of T^2 is an invariant subspace for T.

Let $v \in R(T^2)$. Then $v = T^2 u$ for some $u \in V$.
Hence $Tv = T^3 u = T^2(Tu) \in R(T^2)$. This shows $R(T^2)$ is T-invariant.

5. Let E be the linear operator on $P_3(\mathbb{R})$ defined by $Ep(x) = (x-1)p'(x)$.

(a) (8 pts) Find a basis of $P_3(\mathbb{R})$ such that the matrix of E with respect to this basis is upper triangular.

Each subspace $P_d(\mathbb{R})$ is invariant for E.
Therefore, E has upper triangular matrix in the standard basis $(1, x, x^2, x^3)$.

(b) (9 pts) Find all eigenvalues of E (you are not required to find the eigenvectors).

The matrix of E in the standard basis is

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Its diagonal entries are the eigenvalues : $\lambda = 0, 1, 2, 3$.

(c) (8 pts) Determine whether or not E is diagonalizable (you are not required to diagonalize E).

E is an operator on a space of dimension 4, with 4 distinct eigenvalues. Therefore E is diagonalizable.

6. (15 pts) Let T be a linear operator on a finite-dimensional inner product space V. Suppose that $\langle Tu, Tv\rangle = \langle u, v\rangle$ for all $u, v \in V$. Prove that T is invertible.

Since V is finite-dimensional, it is enough to prove that T is injective. If $v \in N(T)$ then
$\|v\|^2 = \langle v, v\rangle = \langle Tv, Tv\rangle = \langle 0, 0\rangle = 0$, so $v = 0$.
This shows that $N(T) = 0$, and hence T is injective.