Name Solutions Section time & instructor ____________

Student ID____________

**Math 110—Linear Algebra, Spring 2012—Haiman**

**Midterm 1**

**Instructions:**

1. Write your name, ID number and discussion section time and instructor's name at the top of this page. Do not look at the other pages until the signal to start is given.
2. You may use one sheet (written on both sides) of prepared notes. No other notes, books, calculators, computers, cell phones, audio players, or other aids may be used.
3. Use your own scratch paper for preliminary work, then write your solutions on the exam paper. Hand in only the exam paper itself.
4. If a problem has a yes/no or numerical answer, give a **brief justification** of your answer. This need only be a sentence or two, not a complete proof.
5. If a problem asks you to **prove** something, give a complete proof, written out in clear and mathematically correct language. In proofs you may use any theorems which were proved in the textbook, the lectures, or which you were asked to prove on homework assignmnents.
6. There are nine problems worth a total of 100 points.

1. (6 pts each) Which of the following maps $T$ are linear?

(a) $T\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2)$

Not linear.
For instance $T(-(1,1,\ldots,1)) = (1,1,\ldots,1) \neq -T(1,1,\ldots,1)$ shows it does not preserve scalar multiplication. It's also not additive.

(b) $T\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(x_1, \ldots, x_n) = (\overline{x_1}, \ldots, \overline{x_n})$, where $\overline{a+bi} = a - bi$

Not linear. Does not preserve scalar multiplication by $i$. (It is additive, however, as [redacted] well as linear over $\mathbb{R}$.)

(c) $T\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ defined by $T(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2)$

Both elements $x = 0, 1$ of $\mathbb{F}_2$ satisfy $x^2 = x$.
Hence $T$ is actually the identity map, so it <u>is</u> linear.

2. (10 pts) Find a basis of the subspace $W = \{(x_1, \ldots, x_5) : x_1 = x_2 \text{ and } x_3 = x_4 = 2x_5\}$ of $\mathbb{R}^5$.

$(w_1, w_2)$ where $w_1 = (1,1,0,0,0)$, $w_2 = (0,0,2,2,1)$.
Then every vector $(a, a, 2b, 2b, b) \in W$ is uniquely expressed as $aw_1 + bw_2$.

3. (10 pts) Find a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ such that $\operatorname{null} T$ is equal to the subspace $W$ in Problem 2.

$T(x_1, x_2, x_3, x_4, x_5) = (x_1 - x_2, x_3 - x_4, x_4 - 2x_5)$.

This is linear, and the right hand side is zero if and only if $x_1 = x_2$ and $x_3 = x_4 = 2x_5$, i.e. $\underset{\sim}{x} \in W$.

4. (12 pts) Prove that every linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ such that $\operatorname{null} T = W$ is surjective, where $W$ is the subspace in Problem 2.

Problem 2 shows that $\dim(W) = 2$. Hence if $\operatorname{null} T = W$, then $\dim(\operatorname{range} T) = \dim(\mathbb{R}^5) - \dim(W) = 5-2=3$, by the rank and nullity theorem. Therefore $\operatorname{range} T = \mathbb{R}^3$, i.e., $T$ is surjective.

5. (10 pts) Let $\mathbb{F}$ be a field. What is the smallest possible dimension of $U \cap V$ if $U$ and $V$ are subspaces of $\mathbb{F}^{10}$, $\dim U = 7$, and $\dim V = 8$?

$\dim(U\cap V) = 5$ is the smallest possible, because

$\dim(U \cap V) = 7+8-\dim(U+V)$, and $\dim(U+V) \leq 10$,

so this is at least $7+8-10=5$. An example in which the lower bound is realized is

$$U = \operatorname{span}(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$$
$$V = \operatorname{span}(e_1, e_2, e_3, e_4, e_5, e_8, e_9, e_{10})$$

where $(e_1, \ldots, e_{10})$ is the standard basis. Then

$$U \cap V = \operatorname{span}(e_1, e_2, e_3, e_4, e_5).$$

6. (10 pts) What is the largest possible dimension of $U \cap V$ for $U$ and $V$ subject to the same conditions as in Problem 5?

$\dim(U \cap V) = 7$ is the largest possible. It can't be larger since $\dim(U\cap V) \leq \dim(U)$. An example in which the upper bound is realized is when $U \subseteq V$.

7. (10 pts) If $S\colon V \to W$ and $T\colon U \to V$ are surjective linear maps, does it follow that $S \circ T$ is a surjective linear map?

Yes. The composite of linear maps is linear, and the composite of surjective maps is surjective.

8. (10 pts) Let $D\colon P(\mathbb{R}) \to P(\mathbb{R})$ be the differentiation map, $D(p(z)) = p'(z)$. Find a subspace $W \subseteq P(\mathbb{R})$ such that $P(\mathbb{R}) = W \oplus \operatorname{null} D$

The subspace null $D$ consists of the constant polynomials, i.e. $\operatorname{null} D = \operatorname{span}(1)$. Since $(1, z, z^2, \dots)$ is a basis of $P(\mathbb{R})$, it follows that

$$P(\mathbb{R}) = W \oplus \operatorname{null} D$$

if we take $W = \operatorname{span}(z, z^2, \dots)$, i.e. $W$ is the space consisting of polynomials such that $p(z)=0$.

9. (10 pts) Prove that the polynomials $1$, $z$, $z(z+1)$, $z(z+1)(z+2)$ and $z(z+1)(z+2)(z+3)$ in $P(\mathbb{R})$ are linearly independent.

Let's denote them by $p_0 = 1$, $p_1 = z$, $p_2 = z(z+1)$, ...

Then $p_k$ has degree exactly $k$, so $p_0, \dots, p_k \in P_k(\mathbb{R})$, and hence $\operatorname{span}(p_0, \dots, p_k) \subseteq P_k(\mathbb{R})$, for each $k$. Since $p_{k+1} \notin P_k(\mathbb{R})$, $p_{k+1} \notin \operatorname{span}(p_0, \dots, p_k)$. Thus the list $(p_0, p_1, \dots, p_4)$ has the property that none of the vectors is in the span of the previous ones. We proved in a homework exercise that this implies the list is linearly independent.