Math 110—Linear Algebra
Fall 2009, Haiman
Problem Set 13

Due Friday, Dec. 4, at the beginning of lecture.

1. In the textbook and lecture we saw a quick but "tricky" proof of the Cauchy-Schwarz inequality, Theorem 6.2(c). In this problem we will work out a slightly longer but easier to motivate proof using Gram-Schmidt. Note that the proof of Theorem 6.4, which justifies the Gram-Schmidt process, does not use the Cauchy-Schwarz or triangle inequalities, so it is not circular reasoning to use Gram-Schmidt to prove Cauchy-Schwarz.

(a) If x and y are linearly dependent, then they are both scalar multiples of some vector u. Verify that Cauchy-Schwarz holds with equality in this case.

(b) If x and y are linearly independent, deduce from the Gram-Schmidt process that there is an orthonormal basis $\{u, v\}$ of $\mathrm{Span}(\{x, y\})$ such that x is a scalar multiple of u. Letting $x = tu$, $y = au + bv$, express $\langle x, y\rangle$, $\|x\|$, and $\|y\|$ in terms of t, a, and b, and use this to verify that the Cauchy-Schwarz inequality holds, with strict inequality in this case.

2. Prove that if $f\colon [a, b] \to \mathbb{C}$ is a continuous function, then

$$\left|\int_a^b f(t)\,dt\right| \leq \sqrt{(b-a)\int_a^b |f(t)|^2\,dt}$$

Hint: find a way to apply Cauchy-Schwarz.

3. Section 6.1, Exercises 28 and 29.

4. Section 6.2, Exercise 2(g).

5. Section 6.2, Exercise 21.

6. Use Theorem 6.6 to prove that if W is a finite-dimensional subspace of an inner product space V, then $(W^\perp)^\perp = W$.

7. Section 6.3, Exercise 20(b).