Math 110 – Fall 2009 – Haiman
Problem Set 10 + 11 Solutions

PS 10

(1) For "if," suppose $\det(A) = \pm 1$. By Corollary to Cramer's Rule, the (i,j) entry of A^{-1} is given by $(-1)^{i+j}\det(A_{ji})/\det(A)$, which is an integer since A has integer entries and we are dividing by ± 1.

For "only if," suppose A and A^{-1} both have integer entries. Then $\det(A)$ and $\det(A^{-1}) = 1/\det(A)$ are both integers. The only integers m s.t. $1/m$ is an integer are $m = \pm 1$.

(2) i) For $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$, we have $\det(A - \lambda I) =$

$$\det\begin{pmatrix} 2-\lambda & 0 & -1 \\ 4 & 1-\lambda & -4 \\ 2 & 0 & -1-\lambda \end{pmatrix} = -(1-\lambda)^2\lambda.$$ Hence the eigenvalues are $\lambda = 0, 1$

ii) $\lambda = 0$: $\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \iff \begin{matrix} x_3 = 2x_1, \\ x_2 = 4x_3 - 4x_1 \end{matrix}$, so eigenvectors are $c \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ $(c \neq 0)$.

$\lambda = 1$: $\begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \iff x_1 = x_3$, so eigenvectors are $\begin{pmatrix} a \\ b \\ a \end{pmatrix}$, for a, b not both $= 0$.

iii) A basis is $\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

iv) Take $Q = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. Then $Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(3) a) We know that T is invertible iff its nullspace is $\{0\}$, which is equivalent to 0 not being an eigenvalue. Or, the value of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ at $\lambda = 0$ is $\det(A)$, so $\det(A) \neq 0$ iff $\lambda = 0$ is not a root of $p(\lambda)$.

b) Let λ be an eigenvalue of T, v a corresponding eigenvector, so $T(v) = \lambda v$. Applying T^{-1} to both sides, we get $v = T^{-1}(\lambda v) = \lambda T^{-1}(v)$, hence $T^{-1}(v) = \lambda^{-1} v$ (note $\lambda \neq 0$ by (a), so we can divide by it). This shows λ^{-1} is eigenvalue of T^{-1}. The same argument with T replaced by T^{-1} shows that conversely, if λ^{-1} is an eigenvalue of T^{-1}, then λ is an eigenvalue of T.

c) (a) An $n \times n$ matrix A is invertible $\Leftrightarrow$ 0 is not an eigenvalue of A.

(b) If A is invertible, then λ is an eigenvalue of A iff λ^{-1} is an eigenvalue of A^{-1}.

<u>Proof</u>. Follows from (a) and (b) above by taking $T = L_A$.

(4) a) If $Q^{-1}AQ = \lambda I$ then $A = Q\lambda I Q^{-1} = \lambda Q I Q^{-1} = \lambda Q Q^{-1} = \lambda I$.

b) If $Q^{-1}AQ = D$ is diagonal, the diagonal entries of D are the eigenvalues of A. So if A has only one eigenvalue λ then all the diagonal entries of D are equal to λ, i.e., $D = \lambda I$.

c) The char. poly. of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $(1-\lambda)^2$, so its only eigenvalue is 1. Since it isn't a scalar matrix, it's not diagonalizable, by (b).

(5) a) The characteristic polynomial $p(\lambda)$ of T is a non-constant polynomial over $\mathbb{C}$. Every such polynomial has at least one root in $\mathbb{C}$, so T has at least one eigenvalue and hence at least one eigenvector.

b) If $f(x) \in P(\mathbb{C})$ is not the zero polynomial, let $d = \deg(f)$. Then $\deg(xf(x)) = d+1$, hence $xf(x)$ is not a scalar multiple of $f(x)$, i.e. $f(x)$ is not an eigenvector of T.

PS 11

(1) a) Suppose $p(\lambda) = c(\lambda - a_1)\cdots(\lambda - a_n)$. We know $c = (-1)^n$.
Then $\det(A) = p(0) = (-1)^n(-a_1)\cdots(-a_n) = a_1 \cdots a_n$ is the product of the roots, repeated according to their multiplicities.

b) The coefficient of λ^{n-1} in $p(\lambda)$ is $(-1)^{n-1}(a_1 + \dots + a_n)$ if $p(\lambda)$ splits as above. We claim that, on the other hand, the coefficient of λ^{n-1} in $\det(A - \lambda I)$ is $(-1)^{n-1}\operatorname{tr}(A)$, showing $\operatorname{tr}(A) = a_1 + \dots + a_n$.
To prove the claim, observe that in the cofactor expansion on first row of $A - \lambda I$, only the term $(a_{11} - \lambda)\det(A_{11} - \lambda I_{n-1})$ contributes to the coefficient of λ^{n-1} (all the other terms are polynomials of degree $< n-1$ in λ). The contribution from $(a_{11} - \lambda)\det(A_{11} - \lambda I_{n-1})$ has two terms: $a_{11} \cdot$ (coefficient of λ^{n-1} in $\det(A_{11} - \lambda I_{n-1})$) $= (-1)^{n-1} a_{11}$, plus $-1 \cdot$ (coefficient of λ^{n-2} in $\det(A_{11} - \lambda I_{n-1})$), which is $-(-1)^{n-2}\operatorname{tr}(A_{11})$ by induction. So the total is $(-1)^{n-1}(a_{11} + \operatorname{tr}(A_{11})) = (-1)^{n-1}\operatorname{tr}(A)$.

(2) Diagonalize the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, to get
[after some work finding eigenvalues $\lambda = 1$ (mult. 2), $\lambda = 2$ (mult. 1) and corresponding eigenvectors]

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{where} \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Therefore, if we set $\vec{x}(t) = Q\vec{y}(t)$, the system is equivalent to $\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, with solution $\vec{y}(t) = \begin{pmatrix} a_1 e^t \\ a_2 e^t \\ a_3 e^{2t} \end{pmatrix}$

for initial condition $\vec{y}(0) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. Then $\vec{x}(t) = Q\vec{y}(t)$ is given by

$$\vec{x}(t) = \left(a_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right)e^t + a_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{2t}$$

(3) Diagonalize by finding eigenvalues $\lambda = -1$, $\lambda = 5$ and corresponding eigenvectors:

$$Q^{-1}AQ = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} = D \quad \text{where} \quad Q = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix},$$

So $A = QDQ^{-1}$.

Then $A^n = QD^nQ^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n \, 2/3 & (-1)^n(-1/3) \\ 5^n \cdot 1/3 & 5^n \cdot 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2\cdot(-1)^n + 5^n}{3} & \frac{-(-1)^n + 5^n}{3} \\ \frac{-2\cdot(-1)^n + 2\cdot 5^n}{3} & \frac{(-1)^n + 2\cdot 5^n}{3} \end{pmatrix}$$

[As a check, you can see that this correctly gives I_2 for $n=0$ and A for $n=1$]

Likewise,

$$e^{At} = Qe^{Dt}Q^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2e^{-t} + e^{5t}}{3} & \frac{-e^{-t} + e^{5t}}{3} \\ \frac{-2e^{-t} + 2e^{5t}}{3} & \frac{e^{-t} + 2e^{5t}}{3} \end{pmatrix}.$$