PS 10

(i) For "if," suppose $\det(A) = \pm 1$. By Corollary to Cramer's Rule, the $(i,j)$ entry of $A^{-1}$ is given by $(-1)^{i+j} \frac{\det(A_{ij})}{\det(A)}$, which is an integer since $A$ has integer entries and we are dividing by $\pm 1$.

For "only if," suppose $A$ and $A^{-1}$ both have integer entries. Then $\det(A)$ and $\det(A^{-1}) = \frac{1}{\det(A)}$ are both integers. The only integers $m$ s.t. $\det(A) = m \pm 1$.

(ii) For $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$, we have $\det(A - \lambda I) =$

$$\det \begin{pmatrix} 2-\lambda & 0 & -1 \\ 4 & 1-\lambda & -4 \\ 2 & 0 & -1-\lambda \end{pmatrix} = -(1-\lambda)^2 \lambda.$$ Hence the eigenvalues are $\lambda = 0, 1$.

ii) $\lambda = 0$: $\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \iff x_3 = 2x_1$, so eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $c \neq 0$.

$\lambda = 1$: $\begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \iff x_1 = x_3$, so eigenvectors are $\begin{pmatrix} a \\ b \\ a \end{pmatrix}$, for $a, b$ not both 0.

iii) A basis is $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

iv) Take $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $Q^{-1}Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.
(3) a) We know that $T$ is invertible if its nullspace is $\mathbb{R}^3$, which is equivalent to $0$ not being an eigenvalue.

Or, the value of the characteristic polynomial $p(\lambda) = \det (T - \lambda I)$ at $\lambda = 0$ is $\det(A)$, so $\det(A) \neq 0$ if $\lambda = 0$ is not a root of $p(\lambda)$.

b) Let $\lambda$ be an eigenvalue of $T$, $v$ a corresponding eigenvector, so $T(v) = \lambda v$. Applying $T^{-1}$ to both sides, we get $v = T^{-1}(\lambda v) = \lambda T^{-1}(v)$, hence $T^{-1}(v) = \lambda^{-1} v$ (note $\lambda \neq 0$ by (a), so we can divide by $\lambda$). This shows $\lambda^{-1}$ is an eigenvalue of $T^{-1}$.

The same argument with $T$ replaced by $T^{-1}$ shows that conversely, if $\lambda$ is an eigenvalue of $T^{-1}$, then $\lambda$ is an eigenvalue of $T$.

c) (a) An $n \times n$ matrix $A$ is invertible $\iff 0$ is not an eigenvalue of $A$.

(b) If $A$ is invertible, then $\lambda$ is an eigenvalue of $A$ iff $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

Proof. Follows from (a) and (b) above by taking $T = A^{-1}$.

(4) a) If $Q^*AQ = \lambda I$ then $A = Q \lambda I Q^* = \lambda \lambda Q^* Q = \lambda I$.

b) If $Q^*AQ = \lambda I$ is diagonal, the diagonal entries of $D$ are the eigenvalues of $A$. So if $A$ has only one eigenvalue, then all the diagonal entries of $D$ are equal to $\lambda$, i.e., $D = \lambda I$.

c) The char. poly. of $(d'1')$ is $(1-\lambda)^2$, so its only eigenvalue is $1$. Since it isn't a scalar matrix, it's not diagonalizable, by (b).

(5) a) The characteristic polynomial $f(x)$ of $T$ is a non-constant polynomial over $\mathbb{C}$. Every such polynomial has at least one root in $\mathbb{C}$, so $T$ has at least one eigenvalue and hence at least one eigenvector.

b) If $f(x) \not\equiv \text{zero}$ is not the zero polynomial, let $d = \text{deg}(f)$. Then $\text{deg}(xf(x)) = d+1$, hence $xf(x)$ is not a scalar multiple of $f(x)$, i.e. $f(x)$ is not an eigenvector of $T$. 
(1) a) Suppose \( p(\lambda) = c(\lambda-a_1) \cdots (\lambda-a_n) \). We know \( c = (-1)^n \).

Then \( \det(A) = p(0) = (-1)^n (-a_1) \cdots (-a_n) = a_1 \cdots a_n \) is the product of the roots, repeated according to their multiplicities.

b) The coefficient of \( \lambda^{n-1} \) in \( p(\lambda) \) is \( (-1)^{n-1} (a_1 + \cdots + a_n) \) if \( p(\lambda) \) splits as above. We claim that, on the other hand, the coefficient of \( \lambda^{n-1} \) in \( \det(A-\lambda I) \) is \( (-1)^{n-1} \text{tr}(A) \), showing \( \text{tr}(A) = a_1 + \cdots + a_n \).

To prove the claim, observe that in the cofactor expansion on first row of \( A-\lambda I \), only the term \( (a_{11}-\lambda) \det(A_{11}-\lambda I_{n-1}) \) contributes to the coefficient of \( \lambda^{n-1} \) (all the other terms are polynomials of degree \( \leq n-1 \) in \( \lambda \)). The contribution from \( (a_{11}-\lambda) \det(A_{11}-\lambda I_{n-1}) \) has two terms: \( a_{11} \) (coefficient of \( \lambda^{n-1} \) in \( \det(A_{11}-\lambda I_{n-1}) \)) = \( (-1)^{n-1} a_{11} \), plus \( -1 \) (coefficient of \( \lambda^{n-2} \) in \( \det(A_{11}-\lambda I_{n-1}) \), which is \( -(-1)^{n-2} \text{tr}(A_{11}) \) by induction. So the total is \( (-1)^{n-1} (a_{11} + \text{tr}(A_{11})) = (-1)^{n-1} \text{tr}(A) \).

(2) Diagonalize the matrix \( A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \), to get

[After some work finding eigenvalues \( \lambda = 1 \) (mult. 2), \( \lambda = 2 \) (mult. 1) and corresponding eigenvectors,]

\( Q^T A Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \) where \( Q = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \).

Therefore, if we set \( \vec{x}(t) = Q \vec{y}(t) \), the system is equivalent to

\( \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \), with solution \( \vec{y}(t) = \begin{pmatrix} a_1 e^t \\ a_2 e^t \\ a_3 e^{2t} \end{pmatrix} \)

for initial condition \( \vec{y}(0) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \). Then \( \vec{x}(t) = Q \vec{y}(t) \) is given by

\( \vec{x}(t) = \begin{pmatrix} a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ a_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} e^t + a_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \)
Diagonalize by finding eigenvalues \( \lambda = -1, \lambda = 5 \) and corresponding eigenvectors:

\[ Q^{-1} A Q = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \]

where

\[ Q = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \]

So \( A = Q D Q^{-1} \).

Then

\[ A^n = Q D^n Q^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \]

\[ = \begin{pmatrix} (-1)^n \frac{1}{3} + 5^n \frac{1}{3} & (-1)^n \frac{1}{3} - 5^n \frac{1}{3} \\ -2(-1)^n + 2 \cdot 5^n \frac{1}{3} & (-1)^n + 2 \cdot 5^n \frac{1}{3} \end{pmatrix} \]

[As a check, you can see that this correctly gives \( A^2 \) for \( n=0 \) and \( A \) for \( n=1 \).]

Likewise,

\[ e^{At} = Q e^{D t} Q^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \]

\[ = \begin{pmatrix} \frac{2e^{-t} + e^{5t}}{3} & \frac{-e^{-t} + e^{5t}}{3} \\ \frac{-2e^{-t} + 2e^{5t}}{3} & \frac{e^{-t} + 2e^{5t}}{3} \end{pmatrix} \].