Math 110—Linear Algebra
Fall 2009, Haiman
Problem Set 9

Due Monday, Nov. 2 at the beginning of lecture.

1. Prove that if A and Q are $n \times n$ matrices over $\mathbb{F}$, with Q invertible, then $\det(Q^{-1}AQ) = \det(A)$. Deduce that if V is a finite-dimensional vector space and $T\colon V \to V$ is a linear transformation, then $\det([T]_\beta)$ does not depend on the choice of the ordered basis β of V.

2. A matrix of the form

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}$$

is called a Vandermonde matrix.

(a) Show that the determinant $\det(A)$ is a polynomial in the variables $x_1, x_2, \dots, x_n$ in which every term has degree $n(n-1)/2$. (The *degree* of a monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is defined to be $a_1 + \cdots + a_n$.)

(b) Show that $\det(A)$ becomes zero if $x_i = x_j$ for any i and j. This implies that $\det(A)$ is divisible as a polynomial in the x_i's by the product

$$\prod_{1 \le i < j \le n} (x_j - x_i).$$

(c) Show that the coefficient of the monomial $x_1^0 x_2^1 \cdots x_n^{n-1}$ in $\det(A)$ is equal to 1.

(d) Deduce from the above that $\det(A)$ is equal to the product in part (b).

3. Suppose M is an $n \times n$ matrix of the form

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A and C are square. Express $\det(M)$ in terms of $\det(A)$ and $\det(C)$. Give reasoning to justify your answer.

4. Prove that an upper triangular matrix (that is, a square matrix A such that $a_{ij} = 0$ for $j < i$) is invertible if and only if all its diagonal entries are non-zero.

5. Suppose $f\colon M_{m\times n}(\mathbb{F}) \to \mathbb{F}$ is an m-multilinear function of the rows of $A \in M_{m\times n}$ (recall that this means f is linear as a function of each row separately when the other rows are held constant). Suppose f also has the property that $f(A) = 0$ whenever A has two

equal rows. Prove that $f(B) = -f(A)$ whenever B is obtained from A by switching two rows.

6. A *permutation of order* n is a bijective function $\pi\colon \{1,\dots,n\} \to \{1,\dots,n\}$. If π is a permutation of order n, we define the *permutation matrix* $P(\pi)$ to be the $n \times n$ matrix with $(\pi(j), j)$-th entry equal to 1 for all $j = 1,\dots,n$, and all other entries equal to zero.

(a) Show that the linear transformation $L_{P(\pi)}$ sends e_j to $e_{\pi(j)}$.

(b) Show that $L_{P(\pi)}$ sends $(x_{\pi(1)},\dots,x_{\pi(n)})^T$ to $(x_1,\dots,x_n)^T$.

(c) The *inversion number* $i(\pi)$ is defined to be the number of pairs of integers $1 \le j < k \le n$ such that $\pi(j) > \pi(k)$. Prove that $\det(P(\pi)) = (-1)^{i(\pi)}$.