Math 110—Linear Algebra
Fall 2009, Haiman
Problem Set 7

Due Monday, Oct. 19 at the beginning of lecture.

Reminder: Midterm 2 is Friday, Oct. 23, covering material from Problem Sets 1 through 7. The emphasis will be on Problem Sets 4 through 7, but you are responsible for knowing the earlier material as well.

1. For the matrices A and D in Section 3.2, Example 3, find invertible matrices G and F such that $D = GAF$.

2. Let $T\colon P_4(\mathbb{R}) \to P_4(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f(x) + f(1-x)$.

(a) Find the matrix of T with respect to the basis of monomials $\{1, x, x^2, x^3, x^4\}$, and calculate $\operatorname{rank}(T)$.

(b) Find a basis of the nullspace $N(T)$.

3. Invert the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

and show your method (*i.e.*, don't just plug it into a computer algebra program).

4. Prove that every 2×2 invertible matrix over $\mathbb{R}$ can be expressed as a product of elementary matrices using only the following three types:

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad D(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ where } a, b \neq 0.$$

Hint: first express each 2×2 elementary matrix as a product of the above types of matrices.

5. In this problem we'll work out a more conceptual proof, using dual spaces, of the fact that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$ for any $A \in M_{m \times n}(\mathbb{F})$. Recall that a linear transformation $\lambda\colon V \to \mathbb{F}$ is called a *linear functional* on V. The vector space $\mathcal{L}(V, \mathbb{F})$ of all linear functionals on V is called the *dual space* of V, and denoted V^*.

(a) Given a linear transformation $T\colon V \to W$, we define $T^*\colon W^* \to V^*$ by $T^*(\lambda) = \lambda T$. Use Theorem 2.10 to show that T^* is linear.

(b) Let V be finite-dimensional, with ordered basis $\beta = \{v_1, \ldots, v_n\}$. Using Theorem 2.9, we can define for each $i = 1, \ldots, n$ a unique linear functional λ_i such that

$$\lambda_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Using the basis $\{1\}$ in $\mathbb{F}$, show that the coordinate vector $[\lambda_i]_\beta^{\{1\}}$ is the i-th unit vector e_i. Deduce that $\{\lambda_1, \ldots, \lambda_n\}$ is a basis of V^*, and in particular, that $\dim(V^*) = n = \dim(V)$. The basis $\{\lambda_1, \ldots, \lambda_n\}$ is called the *dual basis* to the basis β of V.

(c) Let V and W be finite-dimensional, with ordered bases β and γ. Let β' and γ' be their dual bases in V^* and W^*. Show that if $T\colon V \to W$ is a linear transformation, then $[T^*]_{\gamma'}^{\beta'} = \left([T]_\beta^\gamma\right)^T$.

(d) If $S \subseteq V$ is a subspace, we define $S^\perp = \{\lambda \in V^* : \lambda(v) = 0 \text{ for all } v \in S\}$. Show that $S^\perp$ is a subspace of V^*. When V is finite dimensional, with basis $\{v_1, \ldots, v_n\}$, and dual basis $\{\lambda_1, \ldots, \lambda_n\}$ in V^*, show that if $S = \operatorname{Span}(v_1, \ldots, v_k)$, then $S^\perp = \operatorname{Span}(\lambda_{k+1}, \ldots, \lambda_n)$.

(e) Use part (d) to prove that if V is finite-dimensional and $S \subseteq V$ is any subspace, then $\dim(S^\perp) = \dim(V) - \dim(S)$.

(f) Prove that $N(T^*) = R(T)^\perp$.

(g) Use parts (e) and (f) and the dimension theorem to prove that $\operatorname{rank}(T) = \operatorname{rank}(T^*)$.

(h) Use parts (c) and (g) to prove that the rank of a matrix is equal to the rank of its transpose.