Math 110 - Haiman - Fall '09
Problem Set 6 Solutions

(1) If AB is invertible, then $L_{AB} = L_A L_B$ is invertible. In particular, L_A is onto, since $R(L_A L_B) \subseteq R(L_A)$ and $L_A L_B$ is onto. Now L_A maps $\mathbb{F}^n$ to $\mathbb{F}^n$, so L_A is 1-to-1 by Thm 2.5. Thus L_A is invertible, hence A is invertible. It follows that B is invertible with inverse $(AB)^{-1}A$, since the inverse of $(AB)^{-1}A$ is $A^{-1}(AB) = B$.

For a counterexample with non-square matrices we can take $A = (1\ 0)$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $AB = (1) = I_1$ which is invertible.

(2) Φ is linear, since matrix multiplication is linear in each variable. We'll construct the inverse map: define $\Psi(A) = BAB^{-1}$. Then

$$\Psi\Phi(A) = BB^{-1}ABB^{-1} = I_n A I_n = A$$
$$\Phi\Psi(A) = B^{-1}BAB^{-1}B = I_n A I_n = A.$$

This shows Ψ is inverse to Φ, hence Φ is 1-1 and onto.

(3) This one is a "gimme." The statement is false in general. As a counterexample take any non-zero vector spaces V, W over a field $\mathbb{F}$; then the zero transformation T_0 is not invertible (T_0 is 1-1 and onto only if $V = \{0_V\}$ and $W = \{0_W\}$), but any subspace of $\mathcal{L}(V,W)$ must contain T_0.

(4) a) $R(e_1) = e_1$, $R(e_2) = \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$, $R(e_3) = \begin{pmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$, so

$$[R] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.$$

$S(e_1) = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$, $S(e_2) = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$, $S(e_3) = e_3$, so

$$[S] = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiplying gives

$$[RS] = [R][S] = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 1/2 & 1/2 & -\sqrt{2}/2 \\ 1/2 & 1/2 & \sqrt{2}/2 \end{pmatrix}.$$

b) Now $([RS]-I)v = 0$ is a system of 3 linear equations in the coefficients of $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, namely

$$\left(\tfrac{\sqrt{2}}{2}-1\right)a - \tfrac{\sqrt{2}}{2}b = 0$$
$$\tfrac{1}{2}a - \tfrac{1}{2}b - \tfrac{\sqrt{2}}{2}c = 0$$
$$\tfrac{1}{2}a + \tfrac{1}{2}b + \left(\tfrac{\sqrt{2}}{2}-1\right)c = 0,$$

whose solutions are the scalar multiples of $v = \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 1 \end{pmatrix}$.

(5) Observe that S acts the same way on the ordered basis $\gamma = \{e_3, e_1, e_2\}$ as R does on the standard one $\beta = \{e_1, e_2, e_3\}$, i.e. $[S]_\gamma = [R]_\beta$. Then $B = [S]_\beta = [I]_\gamma^\beta [S]_\gamma [I]_\beta^\gamma = [I]_\gamma^\beta [R]_\beta [I]_\beta^\gamma$ $= [I]_\gamma^\beta A [I]_\beta^\gamma$. This tells us that the required matrix is $Q = [I]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. You can then check by hand that $A = [R]$ and $B = [S]$ above satisfy $B = Q^{-1}AQ$. Note that $Q^{-1} = [I]_\gamma^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

(6) a) No. For example if $V = \mathbb{R}^n$, $\alpha = \beta =$ standard basis, and $\gamma = \delta =$ any other basis, we'll have $[I]_\alpha^\beta = I_n = [I]_\gamma^\delta$, but $\alpha \neq \gamma$ and $\beta \neq \delta$.

b) Yes. $[I]_\alpha^\beta$ and $[I]_\alpha^\gamma$ are invertible, so taking inverses, $[I]_\alpha^\beta = [I]_\alpha^\gamma \Rightarrow [I]_\beta^\alpha = [I]_\gamma^\alpha$. By definition, the columns of $[I]_\beta^\alpha$ are the coordinates of the basis vectors v_j in β with respect to α, i.e., the jth column is $[v_j]_\alpha$. Similarly the jth column of $[I]_\gamma^\alpha$ is $[w_j]_\alpha$, where $\gamma = \{w_1, \dots, w_n\}$. So equality of the two matrices implies $[v_j]_\alpha = [w_j]_\alpha$ for all j, hence $v_j = w_j$, i.e., $\beta = \gamma$.