If \( AB \) is invertible, then \( L_{AB} = L_A L_B \) is invertible. In particular, 
\( L_A \) is onto, since \( R(L_{AB}) \subseteq R(L_A) \) and \( L_A L_B \) is onto. Now \( L_A \) maps \( \mathbb{F}^n \) to \( \mathbb{F}^n \), so \( L_A \) is 1-to-1 by Thm 2.5. Thus \( L_A \) is invertible, hence \( A \) is invertible. It follows that \( B \) is invertible with inverse \( (AB)^{-1} A \), since the inverse of \( (AB)^{-1} A \) is \( A^{-1} (AB) = B \). 

For a counterexample with non-square matrices we can take 
\( A = (1 \ 0) \), \( B = (0) \), \( AB = (1) = I_1 \), which is invertible.

\( \Phi \) is linear, since matrix multiplication is linear in each variable.

We'll construct the inverse map: define \( \Phi(A) = B A B^{-1} \). Then
\[
\Phi \Phi(A) = B B^{-1} A B B^{-1} = I_n A I_n = A \\
\Phi \Phi(A) = B^{-1} B A B^{-1} B = I_n A I_n = A.
\]
This shows \( \Phi \) is inverse to \( \Phi \), hence \( \Phi \) is 1-1 and onto.

This one is a "gimme." The statement is false in general, as a counterexample take any non-zero vector spaces \( V, W \) over a field \( \mathbb{F} \); then the zero transformation \( T_0 \) is not invertible (\( T_0 \) is 1-to-1 and onto only if \( V = \mathbb{F} V \) and \( W = \mathbb{F} W \)), but any subspace \( U \subseteq \mathbb{L}(V,W) \) must contain \( T_0 \).

a) \( R(e_1) = e_1 \), \( R(e_2) = (0 \ 0) \), \( R(e_3) = (0 \ 0) \), so
\[
[R] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.
\]
\( S(e_1) = (\sqrt{2}/2 \ 0 \ \sqrt{2}/2) \), \( S(e_2) = (\sqrt{2}/2 \ 0 \ 0) \), \( S(e_3) = e_3 \), so
\[
[S] = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Multiplying gives
\[
[RS] = [R][S] = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.
\]
b) Now \((\text{RS}|-1) \mathbf{v} = 0\) is a system of 3 linear equations in the coefficients of \(\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\), namely

\[
\begin{align*}
\left(\frac{\sqrt{3}}{2} - 1\right) a + \frac{\sqrt{3}}{2} b &= 0 \\
\frac{1}{2} a - \frac{1}{2} b - \frac{\sqrt{3}}{2} c &= 0 \\
\frac{1}{2} a + \frac{1}{2} b + \left(\frac{\sqrt{3}}{2} - 1\right) c &= 0,
\end{align*}
\]

whose solutions are the scalar multiples of \(\mathbf{v} = \begin{pmatrix} 1 \\ -\sqrt{3} \\ 1 \end{pmatrix}\).

(\text{5})

Observe that \(S\) acts the same way on the ordered basis \(\mathbf{v} = \{e_3, e_1, e_2\}\) as \(R\) does on the standard one \(\mathbf{v} = \{e_1, e_2, e_3\}\), i.e. \(S\mathbf{v} = [R]\mathbf{v}\). Then \(B = [S]\mathbf{v} = [I]_\beta^\mathbf{v} [R]\mathbf{v} = [I]_\beta^\mathbf{v} [I]_\beta^\mathbf{v} = [I]_\beta^\mathbf{v} A [I]_\beta^\mathbf{v}\). This tells us that the required matrix is \(Q = [I]_\beta^\mathbf{v} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\). You can then check by hand that \(A = [R]\) and \(B = [S]\) above satisfy \(B = Q^{-1}AQ\). Note that \(Q^{-1} = [I]_\beta^\mathbf{v} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\).

(\text{6})

a) No. For example if \(\mathbf{v} = \mathbf{R}^n\), \(\alpha = \mathbf{v}\) = standard basis, and \(\varepsilon\neq\delta = \) any other basis, we'll have \([I]_\alpha^\mathbf{v} = I_n = [I]_\delta^\mathbf{v}\), but \(\alpha\neq\varepsilon\) and \(\beta\neq\delta\).

b) Yes. \([I]_\alpha^\mathbf{v}\) and \([I]_\delta^\mathbf{v}\) are invertible, so taking inverses, \([I]_\alpha^\mathbf{v} = [I]_\delta^\mathbf{v}\) \(\Rightarrow [I]_\beta^\mathbf{v} = [I]_\delta^\mathbf{v}\). By definition, the columns of \([I]_\alpha^{\mathbf{v}}\) are the coordinates of the basis vectors \(v_j\) in \(\alpha\) with respect to \(\alpha\), i.e., the \(j\)th column is \([v_j]_\alpha\). Similarly the \(j\)th column of \([I]_\delta^{\mathbf{v}}\) is \([w_j]_\alpha\), where \(\mathbf{v} = \{w_1, \ldots, w_n\}\). So equality of the two matrices implies \([v_j]_\alpha = [w_j]_\alpha\) for all \(j\), hence \(v_j = w_j\), i.e., \(\alpha = \delta\).