Math 110—Linear Algebra
Fall 2009, Haiman
Problem Set 5

Due Monday, Oct. 5 at the beginning of lecture.

1. Given vector spaces V and W over a field $\mathbb{F}$, and subspaces $V' \subseteq V$, $W' \subseteq W$, show that each of the following sets of linear transformations is a subspace of $\mathcal{L}(V, W)$:

(a) $\{T \colon V \to W \text{ such that } V' \subseteq N(T)\}$

(b) $\{T \colon V \to W \text{ such that } R(T) \subseteq W'\}$

2. Let V be a finite-dimensional vector space and $T \colon V \to V$ a linear transformation. Prove that if $T^2 = 0$, then $\dim(R(T)) \leq \dim(V)/2 \leq \dim(N(T))$.

3. Section 2.3, Exercise 9.

4. Recall that the *trace* $\operatorname{tr}(A)$ of a square matrix A is defined to be the sum of the diagonal entries A_{ii}. Show that for all $A \in M_{m\times n}(\mathbb{F})$ and $B \in M_{n\times m}(\mathbb{F})$, we have

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

(Note that AB is $m \times m$ and BA is $n \times n$, so they are square matrices.)

5. Let A be a matrix over $\mathbb{F}$. Prove that $\operatorname{rank}(L_A) = 1$ if and only if there exist a non-zero row vector X and a non-zero column vector Y such that $A = YX$.