Math 110—Linear Algebra
Fall 2009, Haiman
Problem Set 4

Due Monday, Sept. 28 at the beginning of lecture.

1. If $\mathbb{F}$ is any field, and n is a non-negative integer, we can define a corresponding element of $\mathbb{F}$ to be the sum $1 + 1 + \cdots + 1$ in $\mathbb{F}$ with n terms. (We usually denote this element by n as well, although strictly speaking this is bad notation, since the integer n and the element n of $\mathbb{F}$ are two different things.) For example, in the two-element field $\mathbb{F}_2$, 0 stands for 0, 1 for 1, 2 for $1 + 1 = 0$, 3 for $2 + 1 = 1$, and so on: for n even we will have $n = 0$, and for n odd, $n = 1$.

Now let $D \colon P(\mathbb{F}) \to P(\mathbb{F})$ be the unique linear operator whose values on the basis of monomials are given by $D(1) = 0$ and $D(x^n) = nx^{n-1}$ for $n > 0$.

(a) Show that for $\mathbb{F} = \mathbb{R}$, D is differentiation, that is, $D(f(x)) = f'(x)$ for all $f(x) \in P(\mathbb{R})$.

(b) If $\mathbb{F} = \mathbb{F}_2$ is the two-element field, find a simple formula for the "second derivative" $D(D(f(x)))$.

(c) Letting $\mathbb{F}$ be arbitrary once again, prove that for all $f(x) \in P(\mathbb{F})$, we have $D(f(x)) = g(x, x)$, where $g(x, y) = (f(x) - f(y))/(x - y)$. You may assume without proof the fact that the polynomial $f(x) - f(y)$ in two variables is always divisible by $(x - y)$, so $g(x, y)$ is a polynomial.

(d) Prove that the product rule $D(f(x)g(x)) = f(x)D(g(x)) + D(f(x))g(x)$ holds over any field $\mathbb{F}$.

2. Section 2.2, Exercise 2, parts (e,f,g).

3. Let $T \colon V \to V$ be a linear transformation from a vector space V to itself. A subspace $W \subseteq V$ is called *invariant* for T if $T(W) \subseteq W$. In this case, the restriction of T to the domain W is a linear transformation from W to itself, denoted $T|_W : W \to W$.

Let $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis of V, and let $W = \operatorname{Span}(\{v_1, \ldots, v_k\})$. Prove that W is invariant for T if and only if the matrix $[T]_\beta$ has the block form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A is a $k \times k$ matrix, B is a $k \times (n - k)$ matrix, C is an $(n - k) \times (n - k)$ matrix, and 0 denotes the $(n - k) \times k$ zero matrix.

Also show that $[\,T|_W\,]_{\{v_1,\ldots,v_k\}} = A$.