Problem Set 3 Solutions

1) Use induction on $n$. For $k = 0$, the empty set is independent by definition. For $k > 0$, we can assume $S \setminus \{ v_1, \ldots, v_{k-1} \}$ independent by induction. Since $v_k \notin \text{Span}(S)$ (and, in particular, $v_k \notin S$), it follows that $S \setminus \{ v_1, \ldots, v_{k-1} \} \cup \{ v_k \}$ is independent, by Theorem 1.7 in your book.

2) a) By problem 1, we can choose an independent sequence $(v_1, \ldots, v_k)$ by choosing each vector in succession, subject to the condition $v_j \notin \text{Span}(v_1, \ldots, v_{j-1})$ for each $j$ (when $j = 1$, this means $v_1 \neq 0$, since $\text{Span}(\emptyset) = \{0\}$). Then each $S = (v_1, \ldots, v_{j-1})$ is independent, hence $\text{dim}(\text{Span}(v_1, \ldots, v_{j-1})) = j - 1$. By PS 2, Problem 5, it follows that $\text{dim}(\text{Span}(v_1, \ldots, v_{j-1})) = 2^{j-1}$, and $|S| = 2^k$, so there are $2^k - 2^{j-1}$ choices for $v_j$. Hence

$$Q(n,k) = (2^k - 1)(2^{k-2})(2^{k-4}) \cdots (2^{k-2^{k-1}})$$

b) Part (a) implies that the number of sequences $(v_1, \ldots, v_k)$ that are bases of any given $k$-dimensional subspace $W \subseteq V(\mathbb{F}_2)^n$ is $Q(n,k)$. Every independent sequence $(v_1, \ldots, v_k)$ is a basis of one such $W$, namely $W = \text{Span}(v_1, \ldots, v_k)$, and since $Q(n,k)$ of these are bases for each $W$, the number of $k$-dimensional subspaces is $Q(n,k)/Q(n,k)$.

c) $(2^{10} - 1)(2^{10} - 2)(2^{10} - 2^2)(2^{10} - 2^3)/(2^{5} - 1)(2^{5} - 2)(2^{5} - 2^2)(2^{5} - 2^3)(2^{5} - 2^4)$

$= 109,221,651$

3) a) Lagrange interpolation implies that there exists $f(x) \in P_{n-1}(\mathbb{F})$ such that $(f(c_1), \ldots, f(c_n))$ is any specified vector in $\mathbb{F}^n$.

Since $P_m(\mathbb{F}) \cong P_{m-1}(\mathbb{F})$ for $m > n-1$, this shows that $E$ is onto.

b) Since $E : P_n(\mathbb{F}) \to \mathbb{F}^n$ is onto, $\text{dim}(P_n(\mathbb{F})) = n+1$, and $\text{dim}(\mathbb{F}^n) = n$, Theorem 2.3 in the book gives nullity$(E) = 1$. Now $E(f) = 0$ means every $c_i$ is a root of $f$, so the space of such polynomials has dimension 1. The polynomial $g(x) = (x-c_1)(x-c_2) \cdots (x-c_n)$ belongs to $\ker(E)$ and is not the 0 polynomial, so $g(x)$ spans $\ker(E)$, i.e., every polynomial $g(x) \in \ker(E)$ is a scalar multiple of $f$. 

(4) Following the hint, define \( T: P_{n-d}(F) \to P_n(F) \) by 
\[ T(p(x)) = p(w)f(x). \]
Since \( \deg(p(x)) = d \), multiplying any \( f(x) \in P_{n-d}(F) \)
by \( p(x) \) gives \( p(x)f(x) \in P_n(F) \), so the definition makes sense.

\( T \) is linear because 
\[ p(x)(af(x) + bg(x)) = ap(x)f(x) + b p(x)g(x). \]
Finally, if \( f(x) \in P(F) \) has degree \( m > n - d \) then \( p(x)f(x) \)
has degree \( m + d > n \), so \( p(x)f(x) \notin P_n(F) \). By definition
any \( g(x) \in P_n(F) \) divisible by \( p(x) \) is \( g(x) = p(x)h(x) \) for some
\( h(x) \), and the preceding sentence shows that \( f(x) \notin P_{n-d}(F) \).

Hence \( W = R(T) \), which proves (a) by Thm. 2.1 in the book.

For (b), since \( p(x) \neq 0 \) (as its degree is \( \geq 1 \)), we have
\( f(x) \neq 0 \Rightarrow p(x)f(x) \neq 0 \), or equivalently \( p(x)f(x) = 0 \Rightarrow f(x) = 0 \).
In other words, \( N(T) = \{0\} \). Then Thm 2.3 gives
\[ N(T) + R(T) = \dim(P_{n-d}(F)) = n - d + 1, \]
so \( \dim(W) = n - d + 1 \).

(5) a) Since \( \bar{w} + \bar{z} = \bar{w} + \bar{z} \), we get
\[ T((\bar{w}_1, \ldots, \bar{w}_n) + (\bar{z}_1, \ldots, \bar{z}_n)) = (\bar{w}_1, \ldots, \bar{w}_n) + (\bar{z}_1, \ldots, \bar{z}_n) = T(\bar{w}) + T(\bar{z}) \]
i.e. \( T \) is additive.

b) If \( a \in R \) is real, then \( \bar{a} \bar{z} = \bar{a} \bar{z} = a \bar{z} \), so
\[ T(a(\bar{z}_1, \ldots, \bar{z}_n)) = a(\bar{z}_1, \ldots, \bar{z}_n) = a T(\bar{z}_1, \ldots, \bar{z}_n). \]
Combined with a), this means \( T \) is a linear transformation
of vector spaces over \( \mathbb{R} \).

(6) a) Since the rows of \( A + B \) are the sums of a row of \( A \) and corresponding
row of \( B \), and since the rows of \( CA \) are \( c \times n \) times the rows of \( A \),
it's clear that \( S(A + B) = S(A) + S(B) \) and \( S(cA) = c S(A) \).

b) (Assuming \( n > 0 \)) we can get any vector in \( F^n \) as the sum
of rows of a matrix with first row \( \bar{v} \) and all other rows \( \bar{0} \). So
\( S \) is onto, \( R(S) = F^n \).

c) By definition, \( N(S) \) is the set \( N \) in part (d).

d) Since \( N = N(S) \), \( N \) is a subspace by Thm. 2.1, and Thm 2.3 gives
\[ \dim(N) + \dim(F^n) = \dim(F^m), \text{ hence } \dim(N) = (m-1)n. \]
a) By Thm 2.3,
\[ \dim R(T) = \dim(v) - \dim(N(T)) \leq \dim(v) \]
\[ \leq \dim(w) \quad \text{by assumption} \]
\[ < \dim(w) \quad \text{since } \dim(N(T)) \leq 0 \]
Hence \( R(T) \neq W \), i.e. \( T \) is not onto.

b) Similarly,
\[ \dim N(T) = \dim(v) - \dim(R(T)) \]
\[ \geq \dim(v) - \dim(w) \quad \text{since } R(T) \subseteq W, \]
\[ \geq \dim(v) - \dim(w) \quad \text{so } \dim(R(T)) \leq \dim w \]
\[ > 0 \quad \text{by assumption} \]
Hence \( N(T) \neq 0 \), so \( T \) is not 1-to-1 by Thm 2.4

(or, more directly, because \( T^{-1}(\{0\}) = N(T) \) by definition and \( N(T) \neq 0 \) implies \( N(T) \) has more than one element.)