Math 110—Linear Algebra
Fall 2009, Haiman
Problem Set 3

Due Monday, Sept. 21 at the beginning of lecture.

1. Show that if vectors $v_1, \ldots, v_k$ in a vector space V have the properties that $v_1 \neq 0$, and each v_i is not in the span of the preceding ones, then the vectors are linearly independent. Conversely, show that if $v_1, \ldots, v_k$ is an ordered list of linearly independent vectors, then it has the above properties.

2. (a) Find a formula for the number $Q(n,k)$ of (ordered) sequences $(v_1, v_2, \ldots, v_k)$ of linearly independent vectors in V, where V is a vector space of dimension n over $\mathbb{F}_2$, and $k \leq n$. [Hint: Use the previous problem and Problem Set 2, Problem 5.]

(b) Prove that the number of k-dimensional subspaces of $(\mathbb{F}_2)^n$ is given by $Q(n,k)/Q(k,k)$, for $k \leq n$.

(c) Calculate the number of 5-dimensional subspaces of $(\mathbb{F}_2)^{10}$.

3. Let $c_1, \ldots, c_n$ be distinct elements of a field $\mathbb{F}$. Define the function $E : P_m(\mathbb{F}) \to \mathbb{F}^n$ by $E(f(x)) = (f(c_1), \ldots, f(c_n))$.

(a) Use Lagrange interpolation to prove that E is onto if $m \geq n-1$.

(b) Find the nullity of E if $m = n$. Deduce that if $f(x)$ is a polynomial of degree at most n such that every c_i is a root of f, then f must be a scalar multiple of $(x-c_1)(x-c_2)\cdots(x-c_n)$.

4. Let $p(x) \in P(\mathbb{F})$ be a polynomial of degree d (exactly). Given $n \geq d$, let W be the set of polynomials in $P_n(\mathbb{F})$ which are divisible by $p(x)$.

(a) Prove that W is a subspace of $P_n(\mathbb{F})$.

(b) Find $\dim(W)$.

[Hint for both parts: show that W is equal to the range of a linear transformation $T\colon P_{n-d}(\mathbb{F}) \to P_n(\mathbb{F})$ given by $T(f(x)) = p(x)f(x)$.]

5. (a) Show that the function $T\colon \mathbb{C}^n \to \mathbb{C}^n$ given by $T((z_1, \ldots, z_n)) = (\overline{z_1}, \ldots, \overline{z_n})$ is additive (satisfies the first property in the definition of *linear transformation*) but not linear. Here $\overline{z}$ denotes the complex conjugate $a - bi$ of $z = a + bi$.

(b) Show that if we regard $\mathbb{C}^n$ as a vector space over $\mathbb{R}$ instead of $\mathbb{C}$, then T is linear.

6. Let $S\colon M_{m\times n}(\mathbb{F}) \to \mathbb{F}^n$ be the function that sends a matrix A to the sum of its rows. Assume m and n are non-zero.

(a) Prove that S is a linear transformation.

(b) Find the range of S.

(c) Find the nullspace S.

(d) Let N be the set of matrices $A \in M_{m\times n}(\mathbb{F})$ such that every column of A sums to zero. Use the preceding parts of this problem to prove that N is a subspace of $M_{m\times n}(\mathbb{F})$, and find its dimension.

7. Section 2.1 Exercise 17.