Math 110 HW2 Solutions
Fall 2009 — Prof. Haiman

(1) Suppose $a\begin{pmatrix}1&1\\0&0\\0&0\end{pmatrix}+b\begin{pmatrix}0&0\\1&1\\0&0\end{pmatrix}+c\begin{pmatrix}0&0\\0&0\\1&1\end{pmatrix}+d\begin{pmatrix}1&0\\1&0\\1&0\end{pmatrix}+e\begin{pmatrix}0&1\\0&1\\0&1\end{pmatrix}=0.$

The L.H.S. is $\begin{pmatrix}a+d&a+e\\b+d&b+e\\c+d&c+e\end{pmatrix}$, so

$a+d=b+d=c+d=a+e=b+e=c+e=0.$

This implies $a=b=c$, $d=e$, and $d=-a$. A nontrivial solution is $a=b=c=1$, $d=e=-1$, and you can check directly that this gives 0 as a non-trivial linear combination of the given matrices.

(2) The identity $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$ shows that $\{\sin^2(x),\cos(2x),1\}$ is dependent, and hence so is the whole set S. None of the functions in S is zero, and none is a scalar multiple of another, so $\emptyset$ and all 1- and 2-element subsets of S are independent.

It remains to show that each of the sets

$$\{\sin^2(x),\sin(2x),1\}$$

$$\{\sin(2x),\cos(2x),1\}$$

$$\{\sin^2(x),\sin(2x),\cos(2x)\}$$

is independent. Let's do the last one. Suppose a,b,c are constants s.t.

$$a\sin^2(x)+b\sin(2x)+c\cos(2x)=0.$$

We want to prove $a=b=c=0$. Evaluating at $x=0,\ \pi/4,\ \pi/2$ we get equations

$$c=0$$

$$\tfrac{1}{2}a+b=0$$

$$a-c=0$$

which easily imply $a=b=c=0$.

The other two subsets can be handled similarly, or by a more clever trick: Suppose

$$a\sin^2(x) + b\sin(2x) + c = 0.$$

Differentiate to get

$$2a\sin(x)\cos(x) + 2b\cos(2x) = 0.$$

Since $2\sin(x)\cos(x) = \sin(2x)$ and $\cos(2x)$ aren't proportional, they are independent. Therefore $a=b=0$ and the original equation implies $c=0$.

The set $\{\sin(2x), \cos(2x), 1\}$ can be proved independent using the same trick.

(3) By definition, every $w \in \mathrm{Span}(S)$ has some expression as $a_1v_1 + \dots + a_nv_n$. We are to prove that if two such expressions are equal, they have the same coefficients.

So suppose $a_1v_1 + \dots + a_nv_n = b_1v_1 + \dots + b_nv_n$.

Subtracting, we get

$$(a_1-b_1)v_1 + \dots + (a_n-b_n)v_n = 0.$$

Since the set S is independent this implies every $a_i - b_i = 0$, i.e. $a_i = b_i$ for all i.

(4) The matrices

$$\left\{ A_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, A_2=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, A_3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, A_4=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, A_5=\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, A_6=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix} \right\}$$

form a basis, since any symmetric matrix

$$M=\begin{bmatrix}a&d&e\\d&b&f\\e&f&c\end{bmatrix}$$

has the unique expression $M = aA_1 + bA_2 + cA_3 + dA_4 + eA_5 + fA_6$.

(It's clear that M is given by this expression, and it's unique because the entries on and above the diagonal force all the coefficients.)

So the dimension is 6.

(5) Since $\dim(V) = n$, V has a basis of n vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. Every vector $\vec{v} \in V$ has a unique expression $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$

with $a_i \in \mathbb{F}_2$. There are two choices for each a_i, hence 2^n choices for the expression for $\vec{v}$. By uniqueness, these 2^n expressions give 2^n different vectors, i.e. V has 2^n elements.