Math 110 HW2 Solutions
Fall 2008 - Prof. Haiman

1. Suppose \( \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0. \)

The LHS is \( \begin{pmatrix} a+d & a+e \\ b+d & b+e \end{pmatrix} \), so

\[
\begin{align*}
a+d &= b+d = c+d = a+e = b+e = c+e = 0.
\end{align*}
\]

This implies \( a = b = c, d = e, \) and \( d = -a. \) A nontrivial solution is \( a = b = c = 1, d = e = -1, \) and you can check directly that this gives 0 as a linear combination of the given matrices.

2. The identity \( \sin^2(x) = \frac{1}{2} \left( 1 - \cos(2x) \right) \) shows that

\( \{ \sin^2(x), \cos(2x), 1 \} \) is dependent, and hence so is the whole set \( S. \) None of the functions in \( S \) is \( 0, \) and none is a scalar multiple of another, so \( \emptyset \) and all 1- and 2-element subsets of \( S \) are independent.

It remains to show that each of the sets

\[
\{ \sin^2(x), \sin(2x), 1 \} \\
\{ \sin(2x), \cos(2x), 1 \} \\
\{ \sin^2(x), \sin(2x), \cos(2x) \}
\]

is independent. Let's do the last one. Suppose \( a, b, c \) are constants s.t.

\( a \sin^2(x) + b \sin(2x) + c \cos(2x) = 0. \)

We want to prove \( a = b = c = 0. \) Evaluating at \( x = 0, \frac{\pi}{4}, \frac{\pi}{2} \) we get equations

\[
\begin{align*}
c &= 0 \\
\frac{1}{2}a + b &= 0 \\
a - c &= 0
\end{align*}
\]

which easily imply \( a = b = c = 0. \)
The other two subsets can be handled similarly, or by a more clever trick: suppose
\[ a \sin^2(x) + b \sin(2x) + c = 0. \]

Differentiate to get
\[ 2a \sin(x) \cos(x) + 2b \cos(2x) = 0. \]
Since \( 2\sin(x) \cos(x) = \sin(2x) \) and \( \cos(2x) \) aren't proportional, they are independent. Therefore \( a = b = 0 \) and the original equation implies \( c = 0 \).
The set \( \{ \sin(2x), \cos(2x), 1 \} \) can be proved independent using the same trick.

(3) By definition, every \( \text{Span}(S) \) has some expression as \( a_1 v_1 + \ldots + a_n v_n \). We are to prove that if two such expressions are equal, they have the same coefficients. So suppose \( a_1 v_1 + \ldots + a_n v_n = b_1 v_1 + \ldots + b_n v_n \).

Subtracting, we get
\[ (a_1 - b_1) v_1 + \ldots + (a_n - b_n) v_n = 0. \]
Since the set \( S \) is independent this implies every \( a_i - b_i = 0 \), i.e. \( a_i = b_i \) for all \( i \).

(4) The matrices
\[ \{ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, A_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \} \]
form a basis, since any symmetric matrix
\[ M = \begin{bmatrix} a & b & c \\ d & e & f \\ c & f & g \end{bmatrix} \]
has the unique expression \( M = a A_1 + b A_2 + c A_3 + d A_4 + e A_5 + f A_6 \).
(It's clear that \( M \) is given by this expression, and it's unique because the entries on and above the diagonal force all the coefficients.)
So the dimension is 6.
Since \( \dim(V) = n \), \( V \) has a basis of \( n \) vectors \( \{v_1, \ldots, v_n\} \). Every vector \( \vec{v} \in V \) has a unique expression \( \vec{v} = a_1v_1 + \cdots + a_nv_n \) with \( a_i \in \mathbb{F}_2 \). There are two choices for each \( a_i \), hence \( 2^n \) choices for the expression for \( \vec{v} \). By uniqueness, these \( 2^n \) expressions give \( 2^n \) different vectors, i.e., \( V \) has \( 2^n \) elements.