Math 110—Linear Algebra
Fall 2009, Haiman
Problem Set 1

Due Friday, Sept. 4, 1:10 pm in lecture, or before that time at your GSI's office or mailbox. Once homework has been collected at the start of the lecture, solutions will be posted on the web page. No homework will be accepted after solutions are posted.

Because we have no money to hire graders, only some of the problems from each set will be selected for grading (which ones will of course not be announced in advance). Solutions to all the problems will be posted on the web page.

In proofs, indicate clearly the steps in your reasoning. At any step justified by some axiom, definition, lemma or theorem, specify which one(s) you are using. You may use without proof any lemmas or theorems proved in the text or in the lectures.

See the course web page for ground rules about collaborating on homework.

1. Let $\mathbb{Q}[i]$ denote the set of complex numbers $a + bi$ such that $a, b \in \mathbb{Q}$.
(a) Prove that the subset $\mathbb{Q}[i]$ of $\mathbb{C}$ is closed under the arithmetic operations $+$, $-$, $\times$.
(b) Prove that $\mathbb{Q}[i]$ with these operations is a field.

2. Prove that $M_{m\times n}(\mathbb{F})$ is a vector space with the operations specified in Sect. 1.2, Example 2.

3. Let V be a vector space over $\mathbb{R}$. Let $W = V \times V = \{(v_1, v_2) \,|\, v_1, v_2 \in V\}$. Define addition in W coordinate-wise, *i.e.*,

$$(v_1, v_2) + (v_1', v_2') = (v_1 + v_1', v_2 + v_2').$$

Define scalar multiplication on W by a complex number $a + bi \in \mathbb{C}$ by the formula

$$(a + bi)(v_1, v_2) = (av_1 - bv_2, av_2 + bv_1).$$

Prove that W with these operations is a vector space over $\mathbb{C}$.

4. Sect. 1.3, Exercise 23.

5. Recall the definition of sum of subspaces from the previous problem.
(a) If W is a subspace of V, what is $W + W$?
(b) Is it true that every three subspaces W_1, W_2, W_3 of a vector space V satisfy

$$(W_1 + W_2) \cap W_3 = (W_1 \cap W_3) + (W_2 \cap W_3)?$$

Either prove this, or give a counterexample.
(c) Prove that if $W_1 \cap W_2 = 0$, then every element $w \in W_1 + W_2$ has a *unique* expression as $w = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$.

6. Let W be the set of polynomials $f(x) \in P(\mathbb{R})$ which satisfy the identify $f(-x) = f(x)$.
(a) Prove that W is a subspace of $P(\mathbb{R})$.
(b) Prove that W is the linear span of the set of monomials $\{x^n \mid n \text{ even}\}$.

7. In each of the following, determine whether or not the vector $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. If it is, write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
(a) In $\mathbb{R}^4$,

$$\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

(b) In $M_{2\times 2}(\mathbb{R})$,

$$\mathbf{u} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}$$

(c) In $P_3(\mathbb{R})$,

$$\mathbf{u} = 2x^3 - x + 1, \quad \mathbf{v} = x^3 + 3x^2 + 3x + 2, \quad \mathbf{w} = 5x^3 + 3x^2 + x + 4$$