1. (5 points each) Determine whether each of the following assertions is true or false. Give a brief explanation for each answer (full proof is not required).

(a) If a linear transformation \( T: V \rightarrow W \) between finite-dimensional vector spaces is 1-to-1, then \( \dim(V) \leq \dim(W) \).

**True.** Since \( \text{null}(T) = 0 \), \( \dim(V) = \text{rank}(T) \) by the dimension theorem.
And \( \text{rank}(T) \leq \dim(W) \) since \( \text{R}(T) \subseteq W \).

(b) If \( V \) and \( W \) are finite-dimensional vector spaces such that \( \dim(V) \leq \dim(W) \), and \( T: V \rightarrow W \) is a linear transformation, then \( T \) is 1-to-1.

**False.** A counterexample is the zero map \( T: \mathbb{R}^n \rightarrow \mathbb{R}^n \) for any \( n > 0 \).

(c) The set of vectors \( (x_1, x_2, x_3, x_4) \) which satisfy \( x_1 = x_4 \) and \( x_2 = x_3 \) is a subspace of \( \mathbb{R}^4 \).

**True.** The simplest reason is that part (d) is also true.

(d) The set of vectors in part (c) is the nullspace of a linear transformation from \( \mathbb{R}^4 \) to some vector space over \( \mathbb{R} \).

**True.** It's the nullspace of \( T: \mathbb{R}^4 \rightarrow \mathbb{R}^2 \) defined by
\[
T((x_1, x_2, x_3, x_4)) = (x_1 - x_4, x_2 - x_3) .
\]
(It's easy to check that \( T \) is linear, but you need not do so to get full credit on the problem)

(e) The set of vectors in part (c) is the nullspace of a linear transformation from \( \mathbb{R}^4 \) to \( \mathbb{R} \) (in other words, a linear functional).

**False.** Since \( T \) in part (d) is onto, it has \( \text{rank}(T) = 2 \), and therefore its nullspace, which is the set of vectors in (c), has dimension 2. But the nullspace of any linear \( S: \mathbb{R}^4 \rightarrow \mathbb{R} \) has dimension \( \geq 3 \) by dimension theorem.

(f) \( \mathbb{Q}^n \) is a subspace of the vector space \( \mathbb{R}^n \) over \( \mathbb{R} \). (\( \mathbb{Q} \) denotes the field of rational numbers.)

**False.** Not closed under scalar multiplication by irrational scalars

(g) \( \mathbb{Q}^n \) is a subspace of \( \mathbb{R}^n \) considered as a vector space over \( \mathbb{Q} \) (with the usual addition, and multiplication by rational scalars).

**True.** Clearly closed under addition, and closed under scalar multiplication since \( (ax_1, ..., ax_n) \in \mathbb{Q}^n \) if \( a \) and all \( x_i \) are rational.
2. Let $S$ be the following subset of $P(\mathbb{R})$:

$$S = \{ f(x) = x^5 + x^2, \ g(x) = x^5 + 2, \ h(x) = x^3, \ j(x) = x^2 - 2 \}$$

(a) (30 points) Find a subset of $S$ which is a basis of $\text{Span}(S)$ and prove that your answer is correct.

There are three possible correct answers: any subset consisting of
$h(x)$ and two elements from $\{ f(x), g(x), j(x) \}$. I'll prove that
$B = \{ f(x), g(x), h(x) \}$ is a basis. The proof for the other bases
is similar.

First we'll show $\text{Span}(B) = \text{Span}(S)$. Since $B \subseteq S$, $\text{Span}(B) \subseteq \text{Span}(S)$,
and to prove $\text{Span}(S) \subseteq \text{Span}(B)$, since $\text{Span}(B)$ is a subspace, it's
enough to prove $S \subseteq \text{Span}(B)$. Thus we only need to show that $j(x)
is in $\text{Span}(B)$, which is true because

$$j(x) = f(x) - g(x).$$

Now we'll show $B$ is linearly independent. Suppose $a, b, c$ are
scalars such that

$$af(x) + bg(x) + ch(x) = 0$$

(identically as polynomials). The left-hand side is

$$(a+b)x^5 + cx^3 + ax^2 + 2b.$$  

For this to be the 0 polynomial we must have $a = b = c = 0$.

(b) (5 points) Find $\text{dim}(\text{Span}(S))$.

$$\text{dim}(\text{Span}(S)) = |B| = 3.$$
3. (30 points) Let $T: V \to W$ be a linear transformation. Prove that if $T$ is 1-to-1, and $v_1, \ldots, v_k \in V$ are linearly independent, then $T(v_1), \ldots, T(v_k)$ are linearly independent.

Suppose $a_1T(v_1) + a_2T(v_2) + \ldots + a_kT(v_k) = 0$.

Since $T$ is linear, the left-hand side is equal to $T(a_1v_1 + a_2v_2 + \ldots + a_kv_k)$.

Since $T$ is 1-to-1, the fact that this is zero implies $a_1v_1 + a_2v_2 + \ldots + a_kv_k = 0$.

Finally, since $v_1, \ldots, v_k$ are linearly independent, this implies that all coefficients $a_i$ are zero.

Hence $T(v_1), \ldots, T(v_k)$ are linearly independent.