A REMARKABLE q,t-CATALAN SEQUENCE AND q-LAGRANGE INVERSION

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ABSTRACT. We introduce a rational function $C_n(q,t)$ and conjecture that it always evaluates to a polynomial in q,t with non-negative integer coefficients summing to the familiar Catalan number $\frac{1}{n+1}\binom{2n}{n}$. We give supporting evidence by computing the specializations $D_n(q) = C_n(q,1/q)\, q^{\binom{n}{2}}$ and $C_n(q) = C_n(q,1) = C_n(1,q)$. We show that, in fact, $D_n(q)$ q-counts Dyck words by the major index and $C_n(q)$ q-counts Dyck paths by area. We also show that $C_n(q,t)$ is the coefficient of the elementary symmetric function e_n in a symmetric polynomial $DH_n(x;q,t)$ which is the conjectured Frobenius characteristic of the module of diagonal harmonic polynomials. On the validity of certain conjectures this yields that $C_n(q,t)$ is the Hilbert series of the diagonal harmonic alternants. It develops that the specialization $DH_n(x;q,1)$ yields a novel and combinatorial way of expressing the solution of the q-Lagrange inversion problem studied by Andrews [2], Garsia [5] and Gessel [11]. Our proofs involve manipulations with the Macdonald basis $\{P_\mu(x;q,t)\}_\mu$ which are best dealt with in Λ-ring notation. In particular we derive here the Λ-ring version of several symmetric function identities.

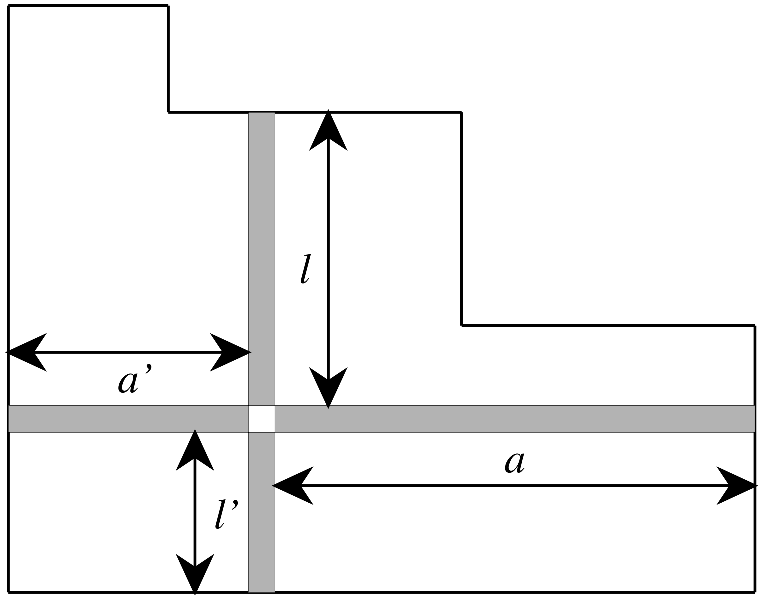
Introduction

Our q,t-Catalan sequence is defined by setting

$$C_n(q,t) = \sum_{\mu \vdash n} \frac{t^{2\Sigma l} q^{2\Sigma a}\ (1-t)(1-q)\ \prod^{0,0}\ (1-q^{a'}t^{l'})\ \sum\ q^{a'}t^{l'}}{\prod (q^a - t^{l+1})(t^l - q^{a+1})} \qquad I.1$$

where the sum is over all partitions of n. All products and sums in the μ^{th} summand are over the cells of μ and the parameters l, l', a, a' denote the *leg, coleg, arm* and *coarm* of a given cell. That is, for a given cell s, when μ is depicted by the French convention (see the figure below) l, l', a, a' give the number of cells that are strictly *north, south, east* and *west* of s in μ. The symbol $\prod^{0,0}$ is to express that this product omits the corner cell with $l' = a' = 0$.

[1]Work carried out under NSF grant support.



On the surface, I.1 appears to define only a rational function of q and t. Nevertheless, computer data and representation theoretical considerations lead us to conjecture that $C_n(q,t)$ evaluates for all n to a polynomial with non-negative integer coefficients and total degree $\binom{n}{2}$. In fact, a perusal of the tables in the appendix quickly reveals that the coefficients of the resulting polynomial always add up to the familiar Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{I.2}$$

To support our conjecture we shall show here that the specializations

$$D_n(q) = C_n(q, 1/q)\, q^{\binom{n}{2}} \quad \text{and} \quad C_n(q) = C_n(q,1) = C_n(1,q) \tag{I.3}$$

are themselves familiar q-analogues of the Catalan number. More precisely we show that

$$D_n(q) = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q \tag{I.4}$$

and that $C_n(q)$ satisfies the recurrence

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q)\, C_{n-k}(q) \quad (\text{with } C_0 = 1) \tag{I.5}$$

The proof of these identities is obtained by computations involving bases of symmetric polynomials, which are best expressed in Λ-ring notation, a device which we pause to explain very briefly, referring the reader to [3] or [8] for a fuller account. We represent an *alphabet* of variables by a formal sum, usually denoted with a capital letter, as for instance

$$X = x_1 + x_2 + \cdots + x_n.$$

More generally, alphabets are allowed contain monomials as well as variables, so that multiplication of alphabets makes sense. Now, given a symmetric function f, the *plethystic substitution* of X into f, denoted $f[X]$, is merely

$$f[X] = f(x_1, x_2, \ldots, x_n).$$

One extends this to alphabets involving negative letters by writing f (uniquely) as a polynomial in the power sums p_k and setting

$$p_k[X - Y] = p_k[X] - p_k[Y].$$

This is well-defined and also yields

$$p_k[XY] = p_k[X]p_k[Y].$$

More generally, there is no difficulty in extending the notion to infinite alphabets, so that we may consider such expressions as

$$f[\tfrac{X}{1-q}] = f[X + qX + q^2X + \cdots];$$

note that the replacement $f[X] \mapsto f[\frac{X}{1-q}]$ is inverse to $f[X] \mapsto f[X(1-q)]$, as we would expect.

It develops that the proof of I.5 is intimately related to the q-Lagrange inversion problem studied by Andrews [2], Garsia [5] and Gessel [11]. To see how this comes about we need to review some material concerning the Macdonald bases $\{P_\mu(x;q,t)\}_\mu$, $\{Q_\mu(x;q,t)\}_\mu$ defined in [19]. Recall that $\{P_\mu(x;q,t)\}_\mu$ is the unique family of polynomials which is triangularly related (in dominance order) to the Schur function basis and satisfies a *Cauchy* identity which, in Λ-ring notation, can be written in the form

$$h_n[XY\tfrac{1-t}{1-q}] = \sum_{\mu \vdash n} Q_\mu(x;q,t)\, P_\mu(y;q,t), \qquad I.6$$

with

$$Q_\mu(x;q,t) = \frac{\prod(1-q^a t^{l+1})}{\prod(1-q^{a+1}t^l)} P_\mu(x;q,t), \qquad I.7$$

where both products run over all the cells of μ and the parameters l, a are respectively the leg and the arm of the cell as defined above. Macdonald also sets

$$J_\mu(x;q,t) = \prod(1-q^a t^{l+1})\; P_\mu(x;q,t) = \prod(1-q^{a+1}t^l)\; Q_\mu(x;q,t), \qquad I.8$$

and conjectures that $J_\mu(x;q,t)$ has an expansion of the form

$$J_\mu(x;q,t) = \sum_{\lambda \vdash n} S_\lambda[X(1-t)]\; K_{\lambda\mu}(q,t) \qquad I.9$$

where $S_\lambda(x)$ is the customary Schur function, and $K_{\lambda\mu}(q,t)$ is a polynomial with non-negative integer coefficients. Here we shall have to deal with the two bases

$$H_\mu(x;q,t) = J_\mu[\tfrac{X}{1-t};q,t] = \sum_{\lambda \vdash n} S_\lambda(x)\; K_{\lambda\mu}(q,t) \qquad I.10$$

and

$$\tilde{H}_\mu(x;q,t) = H_\mu(x;q,1/t)\, t^{n(\mu)} = \sum_{\lambda \vdash n} S_\lambda(x)\, \tilde{K}_{\lambda\mu}(q,t) \qquad I.11$$

where as in [19] we set

$$n(\mu) = \sum l = \sum l' = \sum (i-1)\mu_i = \sum \binom{\mu_i'}{2}. \qquad I.12$$

In fact, to simplify our notation let us also set

$$\begin{array}{rl} \sum q^{a'} t^{l'} & = B_\mu(q,t) \\ \prod^{0,0} (1 - q^{a'} t^{l'}) & = \Pi_\mu(q,t) \\ \prod (1 - q^a t^{l+1}) & = h_\mu(q,t) \\ \prod (1 - t^l q^{a+1}) & = h'_\mu(q,t) \\ \prod (q^a - t^{l+1}) & = \tilde{h}_\mu(q,t) \\ \prod (t^l - q^{a+1}) & = \tilde{h}'_\mu(q,t). \end{array} \qquad I.13$$

This given we may simply write

$$C_n(q,t) = \sum_{\mu \vdash n} \frac{t^{2n(\mu)} q^{2n(\mu')}\, (1-t)(1-q)\, \Pi_\mu(q,t)\, B_\mu(q,t)}{\tilde{h}_\mu(q,t)\, \tilde{h}'_\mu(q,t)}. \qquad I.14$$

Now it develops that this expression is none other than the coefficient of the familiar elementary symmetric function $e_n(x)$ in the Schur function expansion of the symmetric polynomial

$$DH_n(x;q,t) = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu(x;q,t)\, t^{n(\mu)} q^{n(\mu')}\, (1-t)(1-q)\, \Pi_\mu(q,t)\, B_\mu(q,t)}{\tilde{h}_\mu(q,t)\, \tilde{h}'_\mu(q,t)}. \qquad I.15$$

In fact, it can be shown (see Theorem 2.2 below) that

$$\tilde{H}_\mu(x;q,t)\,|_{S_{1^n}} = \tilde{K}_{1^n,\mu}(q,t) = t^{n(\mu)} q^{n(\mu')}, \qquad I.16$$

where the vertical bar denotes extraction of a coefficient—in this case the coefficient of S_{1^n}—in the Schur function expansion of the symmetric function $\tilde{H}_\mu(x;q,t)$. Equation I.16 yields

$$C_n(q,t) = DH_n(x;q,t)|_{S_{1^n}}. \qquad I.17$$

Given this expression for $C_n(q,t)$ in terms of $DH_n(x;q,t)$, we shall show that equations I.4 and I.5 follow from the more general specializations

$$DH_n(x;q,1/q)\, q^{\binom{n}{2}} = \frac{1}{[n+1]_q}\, e_n[X \tfrac{1-q^{n+1}}{1-q}] \qquad I.18$$

and

$$DH_n(x;q,1) = \sum_{\mu \vdash n} \left(\prod_i q^{\binom{\mu_i}{2}} h_{\mu_i}[\tfrac{X}{1-q}] \right) f_\mu[1-q], \qquad I.19$$

where f_μ denotes the so-called *forgotten* basis element indexed by μ.

The connection with q-Lagrange inversion derives from the fact that if for convenience we set

$$k_n(x;q) = k_n(q) = DH_n(x;q,1) \qquad I.20$$

then the formal series

$$f(z) = z \sum_{n\geq 0} k_n(q)\, q^n z^n \qquad I.21$$

is the q-Lagrange inverse of the series

$$F(z) = \sum_{n\geq 1} F_n\, z^n = \frac{z}{E(z)} \quad \text{where} \quad E(z) = \sum_{n\geq 0} e_n(x)\, z^n. \qquad I.22$$

More precisely we show that

$$\sum_{n\geq 1} F_n\, f(z)f(zq)\cdots f(zq^{n-1}) = z. \qquad I.23$$

The contents of this paper are divided into four sections. In the first we review the q-Lagrange inversion results of [5] and recast them in Λ-ring notation. As a byproduct we also obtain a novel and combinatorial way of expressing the solution of the Andrews-Garsia-Gessel q-Lagrange inversion problem. In the second section we prove the identities I.18 and I.19 and derive from them I.4 and I.5. In the third section we briefly review the representation theoretical background that underlies all our computations. We refer the reader to [7],[8], and [19] for further information covering this material. The main object of this section is to show that all the conjectures given in [12] concerning the module of diagonal harmonic polynomials can be replaced by the single statement that $DH_n(x;q,t)$ is the bivariate Frobenius characteristic of this module. Recent results [13] by the second author have also brought forward a 2-parameter family $C_n^{(m)}(q,t)$ of rational expressions which reduce to $C_n(q,t)$ for $m=1$. Computer data leads to the conjecture that $C_n^{(m)}(q,t)$ is (for $m,n\geq 1$) a polynomial with non-negative coefficients adding up to $\binom{mn+n}{n}/(1+nm)$. In the fourth section we present some evidence supporting this conjecture by extending some of the methods and results of Sections 1 and 2. We conclude by presenting some further open problems and conjectures which arise in the study of this new family.

Acknowledgement

We are indebted here to A. Lascoux who more than decade ago showed the first author how to recast the solution of the classical Lagrange inversion problem as a symmetric function identity.

1. q-Lagrange inversion in Λ-ring notation

The general q-Lagrange inversion problem we are concerned with may be stated as follows. We are given a formal power series

$$F(z) = \sum_{k\geq 1} F_k\, z^k \quad (F_1 = 1) \qquad 1.1$$

and we are seeking a formal power series

$$f(z) = \sum_{k\geq 1} f_k\, z^k \quad (f_1 = 1) \tag{1.2}$$

which satisfies the equation

$$\sum_{k\geq 1} F_k\, f(z)f(zq)\cdots f(zq^{k-1}) = z. \tag{1.3}$$

Note that equating the coefficients of z^n we obtain the sequence of identities

$$f_n = -\sum_{k=2}^{n} F_k\, f(z)f(zq)\cdots f(zq^{k-1})\,|_{z^n} \quad (n \geq 2) \tag{1.4}$$

which recursively determine all the coefficients of $f(z)$. Thus the solution of 1.3 exists and is unique. Our task is to show that a very useful form of the solution may be obtained by rewriting one of the basic identities given in [5] in Λ-ring notation. To this end we need to recall some of the contents of [5].

We begin with the following fundamental fact:

Proposition 1.1

For any two sequences $\{\theta_n\}_n$ and $\{\phi_n\}_n$ we have

$$\sum_{n\geq 0} \theta_n\, f(z)f(zq)\cdots f(zq^{n-1}) = \sum_{n\geq 0} \phi_n\, z^n \tag{1.5}$$

if and only if

$$\sum_{n\geq 0} \theta_n\, z^n = \sum_{n\geq 0} \phi_n\, F(z)F(z/q)\cdots F(z/q^{n-1}). \tag{1.6}$$

See [5], Theorem 1.1 for the proof of this result.

In what follows we work with the collection $\mathcal{FP}(q)$ of all formal power series

$$\theta(z) = \sum_{n\geq 0} \theta_n(q)\, z^n \tag{1.7}$$

whose coefficients $\theta_n(q)$ are rational functions of q. This collection is closed under all standard operations on formal power series including taking logarithms, exponentials, functional composition and q-Lagrange inversion. For instance, given the equation

$$\sum_{n\geq 0} \theta_n\, z^n = \exp \sum_{k\geq 1} \frac{p_k}{k}\, z^k, \quad (\,\theta_0 = 1) \tag{1.8}$$

one can compute the p_k 's from the θ_n 's or vice versa by means of the Newton formulas

$$p_k(\theta) = det \begin{pmatrix} \theta_1 & 1 & 0 & \cdots & \cdots & 0 \\ 2\theta_2 & \theta_1 & 1 & \cdots & \cdots & 0 \\ 3\theta_3 & \theta_2 & \theta_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \cdots & \cdots & \cdots & \cdots & \theta_1 & 1 \\ k\theta_k & \theta_{k-1} & \cdots & \cdots & \theta_2 & \theta_1 \end{pmatrix} \tag{1.9}$$

and

$$\theta_k(p) = det \begin{pmatrix} p_1 & -1 & 0 & \cdots & \cdots & 0 \\ p_2 & p_1 & -2 & \cdots & \cdots & 0 \\ p_3 & p_2 & p_1 & -3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \cdots & \cdots & \cdots & \cdots & p_1 & -k+1 \\ p_k & p_{k-1} & \cdots & \cdots & p_2 & p_1 \end{pmatrix} \qquad 1.10$$

which are easily derived by differentiating 1.8, equating coefficients of equal powers of z, and applying Cramer's rule.

In particular, the *starring operations* of [5]

$$\begin{aligned} \theta(z) &\rightarrow \theta^*(z) = \exp \textstyle\sum_{k\geq 1} \frac{p_k(\theta)}{k} \frac{z^k}{1-q^k}, \\ \theta(z) &\rightarrow {}^*\theta(z) = \exp \textstyle\sum_{k\geq 1} \frac{p_k(\theta)}{k} \frac{z^k}{1-(1/q)^k} \end{aligned} \qquad 1.11$$

send an element $\theta(z) \in \mathcal{FP}(q)$ with $\theta_0 = 1$ into another such element. It is easy to verify from 1.11 that we have (when $\theta_0 = 1$)

$$\theta(z)\theta(zq)\cdots\theta(zq^{n-1}) = \frac{\theta^*(z)}{\theta^*(zq^n)} \quad \text{and} \quad \theta(z)\theta(z/q)\cdots\theta(z/q^{n-1}) = \frac{{}^*\theta(z)}{{}^*\theta(z/q^n)}. \qquad 1.12$$

Two other basic operators on $\mathcal{FP}(q)$ introduced in [5] are *roofing* and *unroofing*, respectively defined by setting

$$^\wedge\theta(z) = \sum_{n\geq 0} \theta_n \, q^{-\binom{n}{2}} \, z^n \quad \text{and} \quad {}^\vee\theta(z) = \sum_{n\geq 0} \theta_n \, q^{\binom{n}{2}} \, z^n. \qquad 1.13$$

Un-roofing was introduced in [5] to *untangle* q-products of the form

$$A \otimes_q B\,(z) = \sum_{h\geq 0} A_h \, z^h \, B(z/q^h) \qquad 1.14$$

More precisely, we have

$$^\vee(\,A \otimes_q B)\,(z) = (^\vee A)(z)\,(^\vee B)(z), \qquad 1.15$$

as can be easily verified by equating coefficients of z^n and using the simple identity

$$\binom{h+k}{2} = \binom{h}{2} + \binom{k}{2} + h\,k.$$

It is standard practice in the classical Lagrange inversion theory to write the given formal series $F(z)$ in the form

$$F(z) = \frac{z}{E(z)} = \frac{z}{\sum_{n\geq 0} E_n \, z^n} \quad (E_0 = 1). \qquad 1.16$$

This given, one of the expressions for the solution of 1.3 given in [5] may be written in the form

$$f(z) = z \, \frac{^{\vee *}E(zq)}{^{\vee *}E(z)}, \qquad 1.17$$

a result we will shortly re-derive in proving Theorem 1.1 below.

The main difficulty we encounter in applying formula 1.17 consists in the problem of inverting the formal series ${}^{\vee *}E(z)$. It develops that this inversion can be easily carried out by recasting 1.17 as a symmetric function identity. To see how this comes about we need to make a slight change of notation. Because of the algebraic independence of the elementary symmetric functions $e_n(x)$, there is no loss in assuming that the coefficients E_n can be written in the form

$$E_n = e_n(x_1, x_2, x_3, \ldots) = e_n(x),$$

for a suitable infinite alphabet $x = \{x_1, x_2, x_3, \ldots\}$. This point of view enables us to derive from 1.17 a remarkable expression for the coefficients of the solution of 1.3. To state the result in a form convenient for our applications we rewrite our unknown series in the form

$$f(z) = z\ K(zq) = z \sum_{n \geq 0} k_n(q)\ z^n\ q^n. \qquad 1.18$$

This simply amounts to setting

$$f_n = k_{n-1}(q) q^{n-1}. \qquad 1.19$$

This given we can show that 1.17 is equivalent to the following sequence of identities:

Theorem 1.1

For $n = 1, 2, \ldots$ we have

$$k_n(q) = \sum_{\mu \vdash n} \left(\prod_i q^{\binom{\mu_i}{2}}\ h_{\mu_i}[\tfrac{X}{1-q}] \right)\ f_\mu[1-q], \qquad 1.20$$

where f_μ is the 'forgotten' basis element indexed by μ.

Proof

Proposition 1.1 gives that 1.3 is equivalent to

$$\sum_{k \geq 1} f_k\ F(z) F(z/q) \cdots F(z/q^{k-1}) = z. \qquad 1.21$$

Rewriting $F(z)$ according to 1.16, this becomes

$$\sum_{k \geq 1} f_k\ \frac{z^k q^{-\binom{k}{2}}}{E(z) E(z/q) \cdots E(z/q^{k-1})} = z.$$

Multiplying both sides by ${}^*E(z)$ and using the second equation in 1.12 with θ replaced by E gives

$$\sum_{k \geq 1} f_k\ z^k q^{-\binom{k}{2}} {}^*E(z/q^k) = z\ {}^*E(z).$$

By 1.15, unroofing both sides untangles this q-product into the equation

$$f(z)\ {}^{\vee *}E(z) = {}^{\vee}\left(z\ {}^*E(z)\right) = z\ {}^{\vee *}E(zq).$$

This establishes 1.17. Substituting the expression in 1.18 for $f(z)$ and replacing z by z/q yields the identity

$$K(z) = \frac{{}^{\vee *}E(z)}{{}^{\vee *}E(z/q)}. \tag{1.22}$$

To invert the series ${}^{\vee *}E(z/q)$ we start by writing ${}^{*}E(z/q)$ in exponential form and use Λ-ring notation. This yields the following sequence of equalities:

$$\begin{aligned} {}^{*}E(z/q) &= \exp\ \textstyle\sum_{k\geq 1}\ \frac{p_k(x)}{k}\frac{(-1)^{k-1}(z/q)^k}{1-(1/q)^k} \\ &= \exp\ \textstyle\sum_{k\geq 1}\ \frac{p_k(x)}{k}\frac{(-1)^k z^k}{1-q^k} \\ &= \exp\ \textstyle\sum_{k\geq 1}\ \frac{p_k[\frac{X}{1-q}]}{k}(-z)^k \\ &= \textstyle\sum_{m\geq 0}\ h_m[\frac{X}{1-q}]\ (-z)^m \end{aligned}$$

where in the present context $p_k(x)$ and $h_m(x)$ simply denote the *power sum* and *complete homogeneous* symmetric functions in the alphabet $x = \{x_1, x_2, \ldots\}$. This yields us the expression

$$ {}^{\vee *}E(z/q) = \sum_{m\geq 0} h_m[\tfrac{X}{1-q}]\ q^{\binom{m}{2}}(-z)^m. \tag{1.23}$$

The same reasoning that allowed us to write 1.16 yields that we shall have

$$q^{\binom{m}{2}}\ h_m[\tfrac{X}{1-q}] = h_m[A] \tag{1.24}$$

for some suitable infinite alphabet A. Substituting in 1.23 we get

$$\begin{aligned} {}^{\vee *}E(z/q) &= \textstyle\sum_{m\geq 0}\ h_m[A]\ (-z)^m \\ &= \exp\left(\textstyle\sum_{k\geq 1}\ \frac{p_k[A]}{k}\ (-z)^k\right), \end{aligned} \tag{1.25}$$

where again, $p_k[A]$ denotes the power sum in the alphabet A. In this form ${}^{\vee *}E(z/q)$ is easily inverted. Namely, we have

$$\frac{1}{{}^{\vee *}E(z/q)} = \exp\ \left(-\textstyle\sum_{k\geq 1}\ \frac{p_k[A]}{k}\ (-z)^k\right).$$

Multiplying this by 1.25 with z replaced by zq and using 1.22 gives

$$\begin{aligned} K(z) &= \exp\ \left(\textstyle\sum_{k\geq 1}\ \frac{p_k[A]}{k}(-1)^k\ (q^k-1)\ z^k\right) \\ &= \exp\ \left(\textstyle\sum_{k\geq 1}\ \frac{p_k[A]}{k}(-1)^{k-1}\ (1-q^k)\ z^k\right) \\ &= \exp\ \left(\textstyle\sum_{k\geq 1}\ \frac{p_k[A(1-q)]}{k}(-1)^{k-1}\ z^k\right) \\ &= \textstyle\sum_{m\geq 0}\ e_m[A(1-q)]\ z^m \end{aligned}$$

In particular, equating coefficients of z^n we derive that

$$k_n(q) = e_n[A(1-q)]. \tag{1.26}$$

On the other hand, a simple argument based on the fact that the elementary and forgotten bases are dual with respect to the Hall inner product (see [20]) yields us the identity

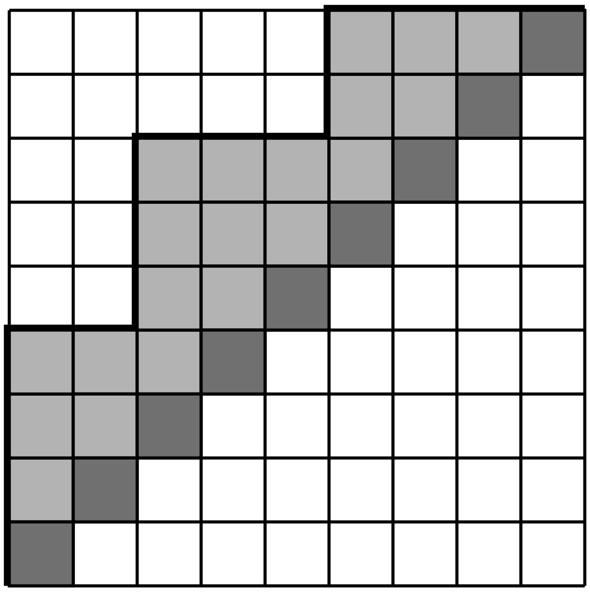
$$e_n[A(1-q)] = \sum_{\mu \vdash n} h_\mu[A]\, f_\mu[1-q]. \qquad 1.27$$

and 1.20 follows then immediately from the definition 1.24 of the alphabet A.

Remark 1.1

We should point out that 1.24 and our referring to A as an *alphabet* is only a device to guide us into the proper use of matrices relating the various bases of symmetric functions. For instance, if $\{c_n\}_n$ is any sequence whatsoever, writing it in the form $c_n = h_n[A]$ allows us to denote by $e_n[A]$ the sequence $\{d_n\}_n$ which is related to c_n in the same manner the sequence $\{e_n(x)\}_n$ of elementary symmetric functions is related to the sequence $\{h_n(x)\}_n$ of homogeneous symmetric functions. In particular, since in any alphabet x with more than 2 letters $e_2(x) = h_1^2(x) - h_2(x)$, then $e_2[A]$ is only a convenient way to refer to the polynomial $d_2 = c_1^2 - c_2$.

Our next task is to show that the coefficient $k_n(q)$ in the solution 1.20 of our q-Lagrange inversion problem has also a remarkable combinatorial interpretation. To do this we need some notation. We recall that the points of the x,y-plane with integral coordinates are called *lattice points* and the squares with lattice vertices are usually referred to as *lattice squares*. The lattice squares with unit side will be referred to here simply as *cells*. The cells with vertices (i,i), $(i+1,i+1)$ will be called *diagonal cells*. The lattice square with vertices $(0,0)$, (n,n) will be denoted by $SQ[n]$. The collection of all diagonal cells in $SQ[n]$ will be referred to as the *diagonal* of $SQ[n]$. This given, we let $\mathcal{D}_n$ denote the collection of all lattice paths from $(0,0)$ to (n,n) which proceed by NORTH and EAST steps and constantly remain weakly above the diagonal of $SQ[n]$. Paths in $\mathcal{D}_n$ are referred to as *Dyck* paths; it is well known that $\mathcal{D}_n$ has cardinality equal to the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Given $D \in \mathcal{D}_n$ we let $a(D)$ denote the number of cells below D and strictly above the diagonal. We may interpret $a(D)$ as the *area* between D and the diagonal of $SQ[n]$. The figure below shows an element of $\mathcal{D}_9$, the diagonal of $SQ[9]$ (dark shading) and the area between (light shading).



To each path $D \in \mathcal{D}_n$ we associate a vector $I(D) = (i_1 \leq i_2 \leq \cdots \leq i_n)$, where i_k is the x-coordinate of the unique NORTH edge $(i_k, k-1) \to (i_k, k)$ in the path. It is easy to see that a vector $I = (i_1 \leq i_2 \leq \cdots \leq i_n)$ comes from a path of $\mathcal{D}_n$ if and only if its coordinates satisfy the conditions

$$i_k \leq k-1 \qquad (k = 1, 2, \ldots, n-1).$$

For instance in the example above

$$I(D) = (0, 0, 0, 0, 2, 2, 2, 5, 5).$$

It is also easy to see that the quantity

$$i_1 + i_2 + \cdots + i_n$$

gives the number of cells of $SQ[n]$ that are weakly above D. This immediately gives the identity

$$a(D) = \binom{n}{2} - i_1 - i_2 - \cdots - i_n. \qquad 1.28$$

We can also represent the vector $I(D)$ in the form $0^{\alpha_0} 1^{\alpha_1} \cdots (n-1)^{\alpha_{n-1}}$ where $\alpha_i = \alpha_i(D)$ gives the number of NORTH steps of D along the line $x = i$. In our example this would be $\alpha_0 = 4$, $\alpha_2 = 3$, $\alpha_5 = 2$, all other $\alpha_i = 0$.

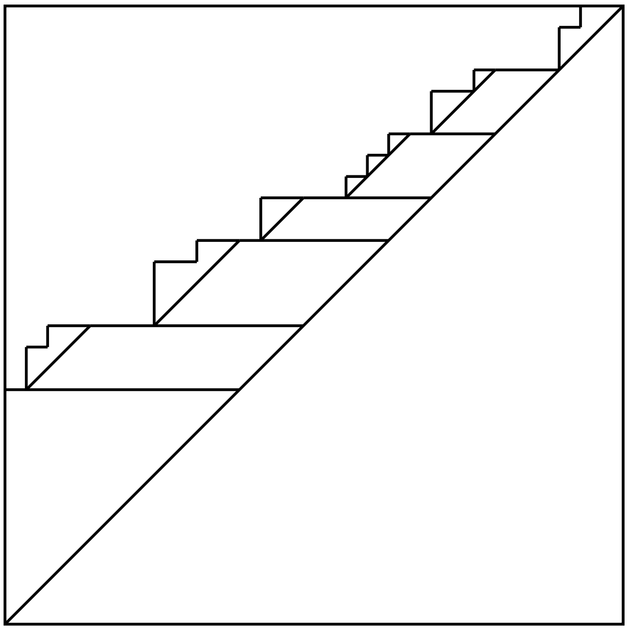
It develops that the coefficient $k_n(q)$ can be obtained by an appropriate q-counting of the elements of $\mathcal{D}_n$. This connection is a simple consequence of a factorization of the paths of $\mathcal{D}_n$. More precisely, given $D \in \mathcal{D}_n$ and given that $\alpha_0(D) = k$ we can break D up (see figure below) into a vertical step of length k followed by a sequence of paths

$$D_{m_1}, D_{m_2}, \ldots, D_{m_k}$$

each preceded by an EAST step, with $D_{m_i} \in \mathcal{D}_{m_i}$ and

$$m_1 + m_2 + \cdots + m_k = n - k \qquad (m_i \geq 0) \tag{1.29}$$

Of course, if one of the summands here is 0 the corresponding path must be assumed to be consisting of a single lattice point and all we see is the EAST step that precedes it. Moreover, we require that the path D_{m_i} has its first and last vertices on the line $x = i + y - k$. A look at the figure below should give a clear idea on how this factorization should be carried out.



Symbolically, we can represent this factorization by writing

$$D = V_k + \sum_{i=1}^{k} (E_i + D_{m_i}) \tag{1.30}$$

where V_k denotes the initial vertical portion of D consisting of the first k NORTH steps, and the symbol E_i represents the EAST step that precedes D_{m_i}. Note that the number of cells weakly east of D_{m_i} and strictly west of the diagonal of $SQ[n]$ is given by $a(D_{m_i}) + (k-i)m_i$, thus the factorization in 1.30 yields us the identity

$$a(D) = \binom{k}{2} + \sum_{i=1}^{k} \left(\, a(D_{m_i}) + (k-i)m_i \, \right), \tag{1.31}$$

where the $\binom{k}{2}$ is contributed by the cells weakly east of V_k. This given, we can state our combinatorial interpretation of $k_n(q)$ in the following form.

Theorem 1.2

$$k_n(q) = \sum_{D \in \mathcal{D}_n} q^{a(D)} \prod_{i=0}^{n-1} e_{\alpha_i(D)}(x) \tag{1.32}$$

Proof

Denote for a moment the right-hand side of 1.32 by $\phi_n(q)$. The factorization 1.30 and the identity 1.31 imply that we must have

$$\phi_n(q) = \sum_{k=1}^{n} q^{\binom{k}{2}}\ e_k(x) \sum_{m_1+\cdots+m_k=n-k} \prod_{i=1}^{k} \sum_{D_{m_i}\in\mathcal{D}_{m_i}} q^{a(D_{m_i})+(k-i)m_i} \prod_{j=0}^{m_i-1} e_{\alpha_j(D_{m_i})}(x)$$

From this we immediately derive that $\phi_n(q)$ satisfies the recursion

$$\phi_n(q) = \sum_{k=1}^{n} q^{\binom{k}{2}}\ e_k(x) \sum_{m_1+\cdots+m_k=n-k} \prod_{i=1}^{k} q^{(k-i)m_i}\ \phi_{m_i}(q), \qquad 1.33$$

where we must adopt the convention that $\phi_0(q) = 1$. This given, to complete our argument we need only show that $k_n(q)$ itself satisfies the same recursion. Our point of departure is equation 1.21, which after multiplying both sides by $E(z)$ and using 1.16 becomes

$$\sum_{n\geq 1} f_n\ z\ F(z/q)\cdots F(z/q^{n-1}) = z\ E(z).$$

Canceling the common factor z and making the replacement $z \to zq$ we obtain

$$\sum_{n\geq 0} f_{n+1}\ F(z)\cdots F(z/q^{n-1}) = E(zq).$$

We can use Proposition 1.1 once more and get

$$\sum_{n\geq 0} f_{n+1}\ z^n = \sum_{n\geq 0} e_n(x)\ q^n\ f(z)f(zq)\cdots f(zq^{n-1}). \qquad 1.34$$

But now 1.18 and 1.19 allow us to rewrite 1.34 in the form

$$\sum_{n\geq 0} k_n(q)\ q^n\ z^n = \sum_{n\geq 0} e_n(x)\ q^n\ q^{\binom{n}{2}}\ z^n\ K(zq)K(zq^2)\cdots K(zq^n).$$

Making the replacement $z \to z/q$ we finally obtain

$$\sum_{n\geq 0} k_n(q)\ z^n = \sum_{n\geq 0} e_n(x)\ q^{\binom{n}{2}}\ z^n\ K(z)K(zq)\cdots K(zq^{n-1}).$$

Equating coefficients of z^n yields the recursion

$$k_n(q) = \sum_{k=1}^{n} e_k(x)\ q^{\binom{k}{2}} \sum_{m_1+\cdots+m_k=n-k} \prod_{i=1}^{k} q^{(i-1)m_i} k_{m_i}(q)$$

which is plainly equivalent to the one in 1.33. This completes our proof.

2. Specializations and further identities

To proceed with our developments we need to review some further material from the theory of Macdonald polynomials. We recall that in [19] Macdonald introduces an operator on the space Λ_n of symmetric polynomials in the variables $\{x_1, x_2, \ldots, x_n\}$ by setting for $P \in \Lambda_n$

$$\delta_1 \, P(x) = \sum_{i=1}^{n} \, A_i(x) \, T_q^{(i)} \, P(x) \tag{2.1}$$

with

$$A_i(x) = \prod_{j \neq i} \, \frac{tx_i - x_j}{x_i - x_j} \tag{2.2}$$

and

$$T_q^{(i)} \, P(x) = P(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n). \tag{2.3}$$

It is shown there that the eigenvalues of δ_1 are given by the sums

$$\gamma_\mu = \sum_{i=1} \, t^{n-i} \, q^{\mu_i} \qquad (\mu \vdash n). \tag{2.4}$$

We can see, that as long as q and t are generic, the γ_μ *'s* are all distinct. This allows Macdonald to construct $P_\mu(x; q, t)$ as the unique polynomial satisfying the conditions

$$\begin{array}{ll} a) & \delta_1 P_\mu(x; q, t) = \gamma_\mu P_\mu(x; q, t) \\ b) & \quad P_\mu(x; q, t) \mid_{S_\mu(x)} = 1 \end{array} \tag{2.5}$$

The polynomials $\tilde{H}_\mu(x; q, t)$ which enter in our formula I.15 have a similar characterization. It will be convenient here to express this characterization in Λ-ring notation. A first step in this direction is provided by the following basic identity.

Theorem 2.1

For any symmetric polynomial $P(x)$ we have

$$\delta_1 \, P(x) = \frac{1}{1-t} P(x) + \frac{t^n}{t-1} \, P[X + \tfrac{q-1}{tz}] \Omega[z(t-1)X] \mid_{z^0} \tag{2.6}$$

where

$$\Omega[z(t-1)X] = \prod_i \, \frac{1 - zx_i}{1 - ztx_i}. \tag{2.7}$$

Proof

Before we proceed with our argument we must recall that all Λ-ring identities may be deduced from the single basic principle that the operation $p_k[\cdot]$ is a ring homomorphism:

$$p_k[X + Y] = p_k[X] + p_k[Y] \quad \text{and} \quad p_k[XY] = p_k[X] p_k[Y].$$

Thus if we define

$$\Omega(x) = \prod_i \frac{1}{1 - x_i} = \exp \left(\sum_{k \geq 1} \frac{p_k(x)}{k} \right) \tag{2.8}$$

then 2.7 becomes a simple consequence of the identity

$$p_k[z(t-1)X] = z^k\ (t^k-1)\sum_i\ x_i^k. \tag{2.9}$$

This given, to show 2.6 it is best to verify it on the power symmetric function basis. Note that since we may write for any integer $m \geq 1$

$$T_q^{(i)} p_m(x) = p_m[X+(q-1)x_i] = p_m[X] + p_m[q-1]x_i^m,$$

we see that for any partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 1)$ we have (from 2.1)

$$\begin{aligned} \delta_1\ p_\mu(x) &= \textstyle\sum_{i=1}^n\ A_i(x)\ \prod_{r=1}^k\ (\ p_{\mu_r}[X] + p_{\mu_r}[q-1]\ x_i^{\mu_r}\ ) \\ &= \textstyle\sum_{S\subseteq[1,k]}\ \prod_{r\notin S} p_{\mu_r}[X] \prod_{r\in S} p_{\mu_r}[q-1] \sum_{i=1}^n\ A_i(x)\ x_i^{\sum_{r\in S}\mu_r}. \end{aligned} \tag{2.10}$$

Note next that the kernel in 2.7 has the z-partial fraction expansion

$$\Omega[z(t-1)X] = \frac{1}{t^n} + \frac{t-1}{t^n}\sum_{i=1}^n \frac{A_i(x)}{1-tzx_i}.$$

which gives that for an integer m we have

$$\sum_{i=1}^n\ A_i(x)\ x_i^m = \begin{cases} \frac{t^n-1}{t-1} & \text{if } m=0 \\ \frac{t^n}{t-1}\ \frac{1}{(tz)^m}\ \Omega[zX(t-1)]\ |_{z^0} & \text{if } m\geq 1. \end{cases}$$

Using this identity for $m = \sum_{r\in S}\mu_r$ we can rewrite 2.10 in the form

$$\begin{aligned} \delta_1\ p_\mu(x) &= \tfrac{1}{1-t}\ p_\mu(x) + \tfrac{t^n}{t-1}\ \textstyle\sum_{S\subseteq[1,k]}\ \prod_{r\notin S} p_{\mu_r}[X] \prod_{r\in S} \frac{p_{\mu_r}[q-1]}{(tz)^{\mu_r}}\Omega[zX(t-1)]\ |_{z^0} \\ &= \tfrac{1}{1-t}\ p_\mu(x) + \tfrac{t^n}{t-1}\ \textstyle\prod_{r=1}^n\ \Big(\ p_{\mu_r}[X] + p_{\mu_r}[\frac{q-1}{tz}]\ \Big)\ \Omega[zX(t-1)]\ |_{z^0} \end{aligned}$$

which reduces to 2.6 by the additivity of the power symmetric function.

Macdonald in [19] derives a number of specializations for his polynomial $P_\mu(x;q,t)$. One of them plays an important role here. In Λ-ring notation it can be stated as follows.

Proposition 2.1

$$P_\mu[\tfrac{1-u}{1-t};q,t] = \frac{\prod(t^{l'}-q^{a'}\ u)}{\prod(1-q^a t^{l+1})} \tag{2.11}$$

This yields the following specialization for our polynomial $\tilde{H}_\mu(x;q,t)$.

Corollary 2.1

$$\tilde{H}_\mu[1-u;q,t] = \prod(1-q^{a'}t^{l'}\ u) \tag{2.12}$$

In particular we must have

$$\sum_{r=0}^{n-1}\ u^r\ \tilde{K}_{(1^r,n-r),\mu}(q,t) = \prod{}^{0,0}\ (1+q^{a'}t^{l'}\ u) \tag{2.13}$$

Proof

Multiplying both sides of 2.11 by $h_\mu(q,t)$ and using I.10 and I.8 gives

$$H_\mu[1-u;q,t]=J_\mu[\tfrac{1-u}{1-t};q,t]=\prod(t^{l'}-q^{a'}\ u).$$

Replacing t by $1/t$ and multiplying by $t^{n(\mu)}$ we get from I.11

$$\tilde{H}_\mu[1-u;q,t]=t^{n(\mu)}\ \prod(t^{-l'}-q^{a'}\ u)=\prod(1-t^{l'}q^{a'}\ u),$$

as desired. Now, using I.11 again, this may be rewritten as

$$\sum_\lambda\ S_\lambda[1-u]\ \tilde{K}_{\lambda\mu}(q,t)=\prod(1-t^{l'}q^{a'}\ u). \tag{2.14}$$

However, it is well known and easy to show that

$$S_\lambda[1-u]=\begin{cases}(-u)^r\ (1-u) & \text{if } \lambda=(1^r,n-r)\\ 0 & \text{otherwise.}\end{cases} \tag{2.15}$$

Thus 2.14 reduces to

$$\sum_{r=0}^{n-1}(-u)^r(1-u)\ K_{(1^r,n-r),\mu}(q,t)=\prod\ (1-t^{l'}q^{a'}\ u)$$

and 2.13 follows upon division by $1-u$ and changing the sign of u.

We are now in a position to derive our Λ-ring characterization of the polynomial $\tilde{H}_\mu(x;q,t)$. To this end let us set for any symmetric polynomial $P(x)$

$$\Delta_1\ P(x)=P[X]-P[X+\tfrac{(1-q)(1-t)}{z}]\ \Omega[-zX]\ |_{z^0}. \tag{2.16}$$

This given we have

Theorem 2.2

$$\begin{array}{ll} a) & \Delta_1\ \tilde{H}_\mu(x;q,t)=(1-t)(1-q)B_\mu(q,t)\ \tilde{H}_\mu(x;q,t)\\ b) & \qquad\qquad \tilde{H}_\mu(x;q,t)\ |_{e_n(x)}=q^{n(\mu')}t^{n(\mu)}.\end{array} \tag{2.17}$$

Proof

Using Theorem 2.1 we we can rewrite a) of 2.5 in the form

$$\frac{1}{1-t}\ P_\mu(x;q,t)+\frac{t^n}{t-1}\ P_\mu[X+\tfrac{q-1}{tz};q,t]\ \Omega[zX(t-1)]\ |_{z^0}=\gamma_\mu\ P_\mu(x;q,t).$$

Making the replacement $X\ \rightarrow\ \frac{X}{1-t}$, multiplying both sides by $h_\mu(q,t)$ and using I.8 we get

$$\frac{1}{1-t}\ J_\mu[\tfrac{X}{1-t};q,t]+\frac{t^n}{t-1}\ J_\mu[\tfrac{X-(1-q)(1-t)/tz}{1-t};q,t]\ \Omega[-zX]\ |_{z^0}=\gamma_\mu\ J_\mu[\tfrac{X}{1-t};q,t],$$

which is converted by I.10 and 2.4 into

$$\frac{1}{1-t}\ H_\mu[X;q,t] + \frac{t^n}{t-1}\ H_\mu[X - \tfrac{(1-q)(1-t)}{tz};q,t]\ \Omega[-zX]\ |_{z^0} = \left(\sum_{i=1}^{n}\ t^{n-i}\ q^{\mu_i}\right)\ H_\mu[X;q,t].$$

Making the replacement $t\ \rightarrow\ 1/t$, multiplying by $t^{n(\mu)+n-1}$ and using I.11 gives

$$\tfrac{t^n}{t-1}\ \tilde{H}_\mu(x;q,t) + \tfrac{1}{1-t}\ \tilde{H}_\mu[X + \tfrac{(1-q)(1-t)}{z};q,t]\ \Omega[-zX]\ |_{z^0} = \left(\sum_{i=1}^{n}\ t^{i-1}\ q^{\mu_i}\right)\ \tilde{H}_\mu[X;q,t]. \qquad 2.18$$

Now note that for a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0)$ we have from I.13

$$(1-q)\ B_\mu(q,t) = \sum_{i=1}^{n} t^{i-1}(1-q^{\mu_i}) = \tfrac{t^n-1}{t-1} - \sum_{i=1}^{n} t^{i-1}q^{\mu_i}$$

or better

$$\sum_{i=1}^{n} t^{i-1}q^{\mu_i} = \tfrac{t^n-1}{t-1} - (1-q)B_\mu(q,t).$$

Substituting this in 2.18 we finally obtain

$$\tfrac{t^n}{t-1}\ \tilde{H}_\mu[X;q,t] + \tfrac{1}{1-t}\ \tilde{H}_\mu[X + \tfrac{(1-q)(1-t)}{z};q,t]\ \Omega[-zX]\ |_{z^0} = \left(\tfrac{t^n-1}{t-1} - (1-q)B_\mu(q,t)\right)\ \tilde{H}_\mu[X;q,t]$$

which is easily reduced to 2.17 a). Finally, equating coefficients of u^{n-1} in both sides of 2.13 we get that the coefficient of the Schur function $S_{1^n}(x) = e_n(x)$ is precisely as asserted in 2.17 b). This completes our proof.

Note that since the polynomials $B_\mu(q,t)$ are all distinct, 2.17 a) fixes $\tilde{H}_\mu(x;q,t)$ up to a multiplicative constant. Clearly, this freedom is removed by the knowledge of any one of the coefficients in the Schur function expansion I.11. Thus we see that 2.17 a) together with 2.17 b) uniquely determine $\tilde{H}_\mu(x;q,t)$. In particular we must have

Corollary 2.2

$$\tilde{H}_\mu(x;t,q) = \tilde{H}_{\mu'}(x;q,t) \qquad 2.19$$

Proof

The definition 2.16 makes the operator Δ_1 symmetric in t and q. Thus from 2.16 a) we get

$$\Delta_1\ \tilde{H}_\mu(x;t,q) = (1-t)(1-q)B_\mu(t,q)\tilde{H}_\mu(x;t,q).$$

On the other hand 2.17 b) becomes

$$\tilde{H}_\mu(x;t,q)\ |_{e_n(x)} = t^{n(\mu')}q^{n(\mu)}.$$

However, since $B_\mu(t,q) = B_{\mu'}(q,t)$ we see that the polynomial $\tilde{H}_\mu(x;t,q)$ satisfies precisely the conditions which characterize $\tilde{H}_{\mu'}(x;q,t)$. Thus 2.19 must hold true as asserted.

We are now in a position to establish our specializations I.18 and I.19. Both of them have straighforward but tedious verifications. However, we will follow here a more indirect path to them since in the process we will collect some rather remarkable identities. We start by noting that our polynomials $\{\tilde{H}_\mu(x;q,t)\}_\mu$ satisfy the following *Cauchy* formula :

Theorem 2.3

$$e_n[\tfrac{X\,Y}{(1-t)(1-q)}] = \sum_\mu \frac{\tilde{H}_\mu(x;q,t)\ \tilde{H}_\mu(y;q,t)}{\tilde{h}_\mu(q,t)\ \tilde{h}'_\mu(q,t)} \tag{2.20}$$

Proof

Making the replacements $X \to \frac{X}{1-t}$ and $Y \to \frac{Y}{(1-t)}$ in I.6 and using I.8 and I.10 we get

$$h_n[\tfrac{X\,Y}{(1-t)(1-q)}] = \sum_\mu \frac{H_\mu(x;q,t)\ H_\mu(y;q,t)}{h_\mu(q,t)\ h'_\mu(q,t)} \tag{2.21}$$

Making the replacement $t \to 1/t$ and using I.11 gives

$$h_n[\tfrac{-tX\,Y}{(1-t)(1-q)}] = \sum_\mu \frac{\tilde{H}_\mu(x;q,t)\ \tilde{H}_\mu(y;q,t)}{h_\mu(q,1/t)\ h'_\mu(q,1/t)} t^{-n(\mu)}\ t^{-n(\mu)}. \tag{2.22}$$

Now note that

$$\begin{aligned} h_\mu(q,1/t) &= \prod(1-q^a t^{-l-1}) = (-1)^n t^{-n(\mu)-n}\tilde{h}_\mu(q,t), \\ h'_\mu(q,1/t) &= \prod(1-q^{a+1}t^{-l}) = t^{-n(\mu)}\tilde{h}'_\mu(q,t). \end{aligned}$$

Thus, since for any alphabet A we have $h_n[-tA] = (-t)^n e_n[A]$, we easily see that 2.20 follows from 2.22 upon cancelling the factor $(-t)^n$ from both sides.

Formula 2.20 has the following immediate specialization:

Corollary 2.3

$$e_n[\tfrac{X(1-u)}{(1-t)(1-q)}] = \sum_\mu \frac{\tilde{H}_\mu(x;q,t)\ \prod(1-q^{a'}t^{l'}u)}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}. \tag{2.23}$$

Proof

Just make the replacement $Y \to (1-u)$ and use 2.12.

A further specialization is obtained by eliminating the presence of u in 2.23. This gives

Corollary 2.4

$$(-1)^{n-1}\ p_n[\tfrac{X}{(1-t)(1-q)}] = \sum_\mu \frac{\tilde{H}_\mu(x;q,t)\Pi_\mu(q,t)}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)} \tag{2.24}$$

Proof

Note that each summand of 2.23 has the factor $(1-u)$ in the numerator and reduces to the corresponding summand in 2.24 (see I.13) if we divide this factor out and set $u=1$. On the other hand, we have

$$e_n[\tfrac{X(1-u)}{(1-t)(1-q)}] = (-1)^{n-1}\ p_n[\tfrac{X}{(1-t)(1-q)}]\ \tfrac{1-u^n}{n} + O[(1-u)^2] \tag{2.25}$$

where the symbol $O[(1-u)^2]$ is to indicate that the remainder is a sum of terms all divisible by $(1-u)^2$ at the very least. This is simply due to the fact that the expansion of e_n in terms of power sums can be written in the form

$$e_n(x) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} p_i(x)^{m_i(\sigma)}$$

where $m_i(\sigma)$ is the number of cycles of length i in σ. Thus the first term in the right hand side of 2.25 represents the contribution of the full cycles of S_n and the rest comes from the remaining permutations. This given, we see that we can divide both sides of 2.23 by $(1-u)$, set $u = 1$, and obtain 2.24 as desired.

We can easily see from 2.17 that the Δ_1 image of 2.24 produces a right hand side that is remarkably close to the right hand side of defining formula I.15. It turns out that this calculation yields the following beautiful identity:

Theorem 2.4

$$e_n(x) = \sum_{\mu} \frac{\tilde{H}_\mu(x;q,t)\,(1-t)(1-q)\Pi_\mu(q,t)B_\mu(q,t)}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}. \qquad 2.26$$

Proof

As we already observed, in view of 2.17, the right hand side of 2.26 is the result of applying Δ_1 to the right hand side of 2.24. Thus we only have to compute the image of the left hand side. However, due to the linearity of the power symmetric function under plethysm we immediately get from 2.16:

$$\begin{aligned}\Delta_1 p_n[\tfrac{X}{(1-t)(1-q)}] &= p_n[\tfrac{X}{(1-t)(1-q)}] - \Big(\, p_n[\tfrac{X}{(1-t)(1-q)}] + p_n[\tfrac{(1-t)(1-q)/z}{(1-t)(1-q)}]\,\Big)\,\Omega[-zX]\,\big|_{z^0} \\ &= -\; p_n[\tfrac{1}{z}]\;\Omega[-zX]\,\big|_{z^0} = -\tfrac{1}{z^n}\Omega[-zX]\,\big|_{z^0} = (-1)^{n-1}e_n(x)\end{aligned}$$

which gives 2.26.

It develops that both I.18 and I.19 can now be established by placing information extracted from 2.26 back into I.15.

Theorem 2.5

$$DH_n(x;q,1/q)\; q^{\binom{n}{2}} = \frac{1}{[n+1]_q}\; e_n[X\tfrac{1-q^{n+1}}{1-q}] \qquad 2.27$$

Proof

From I.8 and I.10 we get that

$$\tilde{H}_\mu(x;q,1/q) = H_\mu(x;q,q)q^{-n(\mu)} = h_\mu(q,q)\; q^{-n(\mu)}\; P_\mu[\tfrac{X}{1-q};q,q]$$

Now, Macdonald showed [19] that the polynomial $P_\mu(x;q,t)$ reduces to the ordinary Schur function when $t = q$. This gives

$$\tilde{H}_\mu(x;q,1/q) = h_\mu(q,q)\; q^{-n(\mu)}\; S_\mu[\tfrac{X}{1-q}].$$

Thus the specialization $t = 1/q$ reduces 2.26 and I.15 to the form

$$\begin{aligned} e_n(x) &= \textstyle\sum_\mu \ S_\mu[\frac{X}{1-q}] \ A_\mu(q), \\ DH_n(x;q,1/q) &= \textstyle\sum_\mu \ S_\mu[\frac{X}{1-q}] \ q^{-n(\mu)+n(\mu')} \ A_\mu(q). \end{aligned} \qquad 2.28$$

Here the rational function $A_\mu(q)$ can be identified without computation from the *dual* Cauchy formula

$$e_n(x) = e_n[\tfrac{X}{1-q}(1-q)] = \sum_\mu \ S_\mu[\tfrac{X}{1-q}] \ S_{\mu'}[1-q].$$

This gives that

$$DH_n(x;q,1/q) \ q^{\binom{n}{2}} = \sum_\mu \ S_\mu[\tfrac{X}{1-q}] \ q^{\binom{n}{2}-n(\mu)+n(\mu')} \ S_{\mu'}[1-q].$$

In particular, from 2.15 we derive that this sum need only be carried out over hook shapes. A simple calculation gives that when $\mu' = (1^r, n-r)$,

$$\binom{n}{2} - n(\mu) + n(\mu') = n \ r$$

and using 2.15 with $u = q$ we get

$$DH_n(x;q,1/q) \ q^{\binom{n}{2}} = \sum_{\mu'=(1^r,n-r)} \ S_\mu[\tfrac{X}{1-q}] \ (-1)^r q^{nr+r}(1-q) \ . \qquad 2.29$$

On the other hand, 2.15 again with $u = q^{n+1}$ and $\mu' = (1^r, n-r)$ gives

$$S_{\mu'}[1-q^{n+1}] = (-1)^r q^{nr+r}(1-q^{n+1}).$$

This permits us to write 2.29 in the form

$$DH_n(x;q,1/q) \ q^{\binom{n}{2}} = \sum_\mu \ S_\mu[\tfrac{X}{1-q}] \ S_{\mu'}[1-q^{n+1}] \ \tfrac{1-q}{1-q^{n+1}} \ ,$$

and 2.27 is now a consequence of the dual Cauchy formula

$$e_n[X\tfrac{1-q^{n+1}}{1-q}] = \sum_\mu \ S_\mu[\tfrac{X}{1-q}] \ S_{\mu'}[1-q^{n+1}]. \qquad 2.30$$

This result yields more than I.4. More precisely we have:

Corollary 2.5

For any partition λ

$$DH_n(x;q,1/q) \ q^{\binom{n}{2}} \ |_{S_\lambda(x)} = \frac{1}{[n+1]_q} \ S_{\lambda'}[\tfrac{1-q^{n+1}}{1-q}]. \qquad 2.31$$

In particular

$$C_n(q,1/q) \ q^{\binom{n}{2}} = \frac{1}{[n+1]_q} \ S_n[\tfrac{1-q^{n+1}}{1-q}] = \frac{1}{[n+1]_q} \ \begin{bmatrix} 2n \\ n \end{bmatrix}_q . \qquad 2.32$$

Proof

By splitting X from the rest of the argument of e_n in 2.30 we obtain yet another dual Cauchy formula. Namely

$$e_n[X\tfrac{1-q^{n+1}}{1-q}] = \sum_\mu S_\mu[X]\ S_{\mu'}[\tfrac{1-q^{n+1}}{1-q}],$$

and 2.31 is obtained by substituting this in 2.27 and equating coefficients of $S_\lambda(x)$. To prove 2.32 we note that the first equality follows from 2.31 because, as we have already observed, 2.17 b) gives $C_n(q,t) = DH_n(x;q,t)\ |_{e_n(x)}$. But then the second equality is a simple consequence of the q-binomial expansion

$$\Omega[x\tfrac{1-q^{n+1}}{1-q}] = \sum_m \begin{bmatrix} m+n \\ n \end{bmatrix}_q x^m$$

and the corresponding Cauchy identity.

Theorem 2.6

$$DH_n(x;q,1) = \sum_{\mu\vdash n} \left(\prod_i q^{\binom{\mu_i}{2}} h_{\mu_i}[\tfrac{X}{1-q}] \right) f_\mu[1-q], \qquad 2.33$$

Proof

In [19] Macdonald gives the specialization

$$P_\mu(x;1,t) = \prod_{i=1}^h e_{\mu_i'}(x)$$

where $\mu' = (\mu_1', \mu_2', \ldots, \mu_h')$ denotes the partition conjugate to μ. From this, using I.8 and I.10, we derive

$$H_\mu(x;1,t) = \prod_{i=1}^h (t)_{\mu_i'}\ e_{\mu_i'}[\tfrac{X}{1-t}], \qquad 2.34$$

where as is customary, for any integer m and any parameter t we set

$$(t)_m = (1-t)(1-t^2)\cdots(1-t^m).$$

Making the replacement $t \to 1/t$ in 2.34, and using I.11 gives

$$\tilde{H}_\mu(x;1,t) = t^{n(\mu')} \prod_{i=1}^n (1/t)_{\mu_i'}\ e_{\mu_i'}[\tfrac{-t\,X}{1-t}] = (-1)^n t^{-n} \prod_{i=1}^n (t)_{\mu_i'}\ e_{\mu_i'}[\tfrac{-t\,X}{1-t}].$$

Since as we have observed, for any alphabet A we have $e_n[-tA] = (-t)^n\ h_n[A]$, we are led to the specialization

$$\tilde{H}_\mu(x;1,t) = \prod_{i=1}^n (t)_{\mu_i'}\ h_{\mu_i'}[\tfrac{X}{1-t}].$$

But now the symmetry expressed by Corollary 2.2 yields us also the other specialization

$$\tilde{H}_\mu(x;q,1) = \prod_{i=1}^n (q)_{\mu_i}\ h_{\mu_i}[\tfrac{X}{1-q}]. \qquad 2.35$$

This given, we are in a position to let $t \to 1$ in 2.26 and I.15. To this end we note that the coefficient of $\tilde{H}_\mu(x;q,t)$ in 2.26 may be rewritten in the form

$$A_\mu(q,t) = \frac{(1-t)(t)_{k-1}\ \prod^{a'\neq 0}(1-t^{l'}q^{a'})\ \sum_{i=1}^k\ t^{i-1}(1-q^{\mu_i})}{\prod_{i=1}^s (t)_{\alpha_i} \prod^{a\neq 0}(q^a - t^{l+1})\ \prod (t^l - q^{a+1})}$$

where k denotes the number of parts of μ and $\alpha_1, \alpha_2, \ldots, \alpha_s$ denote the lengths of the successive vertical segments of the east boundary of the diagram of μ. Since both numerator and denominator of $A_\mu(q,t)$ have the factor $(1-t)^k$ as a common divisor, we can cancel it out and set $t=1$ to get

$$A_\mu(q,1) = \frac{(k-1)!\ \prod_{i=1}^k (q)_{\mu_i - 1}\ \sum_{i=1}^k (1-q^{\mu_i})}{\prod_{i=1}^s\ \alpha_i!\ (-1)^{n-k} \prod_{i=1}^k (q)_{\mu_i-1} \prod (q)_{\mu_i}}.$$

Substituting 2.35 and this into 2.26 and making the appropriate cancellations we are finally led to the expansion

$$e_n(x) = \sum_\mu \prod_{i=1}^n\ h_{\mu_i}[\tfrac{X}{1-q}](-1)^{n-k}\ \frac{(k-1)!\ \sum_{i=1}^k (1-q^{\mu_i})}{\prod_{i=1}^s\ \alpha_i!}. \tag{2.36}$$

On the other hand since for any two alphabets A and B we have the dual Cauchy formula

$$e_n[A\ B] = \sum_\mu\ h_\mu[A]\ f_\mu[B]$$

with $f_\mu[B]$ denoting the *forgotten* basis element, we can use it with $A = \frac{X}{1-q}$ and $B = 1-q$ and get that

$$e_n(x) = \sum_\mu \prod_{i=1}^k\ h_{\mu_i}[\tfrac{X}{1-q}] f_\mu[1-q].$$

Comparing with 2.36 we are led to the conclusion that

$$f_\mu[1-q] = (-1)^{n-k}\ \frac{(k-1)!\ \sum_{i=1}^k (1-q^{\mu_i})}{\prod_{i=1}^s\ \alpha_i!}.$$

Since the coefficients of $\tilde{H}_\mu(x;q,t)$ in 2.26 and I.15 only differ by the factor $t^{n(\mu)}q^{n(\mu')}$ we can pass to the limit as $t \to 1$ in I.15 and derive the specialization

$$DH_n(x;q,1) = \sum_\mu\ q^{n(\mu')}\ \prod_{i=1}^k\ h_{\mu_i}[\tfrac{X}{1-q}] f_\mu[1-q]$$

which is just another way of writing 2.33. This completes our proof.

Remark 2.1

This theorem establishes the claim we made in the introduction that the formal series $f(z)$ defined by I.20 and I.21 is indeed the solution of the q-Lagrange equation in I.23. In particular, equating coefficients of $e_n(x)$ in 1.32, we deduce that the specialization $C_n(q) = C_n(q,1)$ may be given the combinatorial interpretation

$$C_n(q) = \sum_{D\in\mathcal{D}_n} q^{a(D)},$$

from which the recursion in I.5 can be easily derived.

We close this section by establishing a number of identities closely related to our q,t-Catalan which are also of independent interest. To do this we need to extract some further information from the theory of Macdonald polynomials.

Theorem 2.7

$$\tilde{H}_\mu(x;q,t) = \omega \tilde{H}_\mu(x;1/q,1/t)\; q^{n(\mu')}t^{n(\mu)} \qquad 2.37$$

where ω is the involution that interchanges the elementary and the homogeneous symmetric function bases. In particular, we have

$$\tilde{K}_{\lambda,\mu}(q,t) = \tilde{K}_{\lambda',\mu}(1/q,1/t)\; q^{n(\mu')}t^{n(\mu)} \qquad 2.38$$

Proof

We recall that in [19] Macdonald proved that

$$\omega\; P_\mu(x;q,t) = Q_{\mu'}(x;t,q).$$

Making the substitution $x \rightarrow \frac{X}{1-t}$, multiplying both sides by $h_\mu(q,t)$ and noting that $h_\mu(q,t) = h_{\mu'}(t,q)$, formulas I.8 and I.10 give

$$\omega\; H_\mu(x;q,t) = H_{\mu'}(x;t,q).$$

This is converted by I.11 into

$$\omega\; \tilde{H}_\mu(x;q,1/t)\; t^{n(\mu)} = \tilde{H}_{\mu'}(x;t,1/q)\; q^{n(\mu')}.$$

Using 2.19 we get

$$\omega\; \tilde{H}_\mu(x;q,1/t)\; t^{n(\mu)} = \tilde{H}_\mu(x;1/q,t)\; q^{n(\mu')},$$

and 2.37 follows upon replacing q by $1/q$. The identity in 2.38 is then obtained by equating the coefficients of $S_\lambda(x)$.

Theorem 2.8

$$\begin{aligned} a)\quad & e_n[\tfrac{X}{(1-t)(1-q)}] = \textstyle\sum_\mu \frac{\tilde{H}_\mu(x;q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \\ b)\quad & h_n[\tfrac{X}{(1-t)(1-q)}] = \textstyle\sum_\mu \frac{q^{n(\mu')}t^{n(\mu)}\tilde{H}_\mu(x;q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \end{aligned} \qquad 2.39$$

Proof

Formula a) is an immediate consequence of 2.20. In fact, when the alphabet Y reduces to a single letter z, the left hand side of 2.20 becomes

$$z^n\; e_n[\tfrac{X}{(1-t)(1-q)}].$$

To evaluate the right hand side we note that, when $Y = \{z\}$, I.11 reduces to

$$\tilde{H}_\mu(z;q,t) = z^n\ \tilde{K}_{(n),\mu}(q,t).$$

On the other hand since 2.17 b) essentially states that

$$\tilde{K}_{1^n,\mu}(q,t) = q^{n(\mu')}\ t^{n(\mu)}$$

then 2.38 for $\lambda = (n)$ gives

$$\tilde{K}_{(n),\mu}(q,t) = q^{-n(\mu')}t^{-n(\mu)}q^{n(\mu')}\ t^{n(\mu)} = 1 \qquad 2.40$$

Thus

$$\tilde{H}_\mu(z;q,t) = z^n$$

and we see that in this case 2.20 is none other than 2.39 a) multiplied by z^n.

It develops that after making the substitutions $t \to 1/t$, $q \to 1/q$ formula 2.39 a) becomes 2.39 b). Indeed, if we do this we get

$$t^n\ q^n\ e_n[\tfrac{X}{(1-t)(1-q)}] = \sum_\mu \frac{\tilde{H}_\mu(x;1/q,1/t)}{\tilde{h}_\mu(q,t)\ t^{-n(\mu)-n}q^{-n(\mu')}\tilde{h}'_\mu(q,t)\ t^{-n(\mu)}q^{-n(\mu')-n}}.$$

Cancelling the common factor $t^n\ q^n$ and using 2.37 we get

$$e_n[\tfrac{X}{(1-t)(1-q)}] = \sum_\mu \frac{\omega\tilde{H}_\mu(x;q,t)\ q^{n(\mu')}\ t^{n(\mu)}}{\tilde{h}_\mu(q,t)\ \tilde{h}'_\mu(q,t)}.$$

and 2.39 b) follows by applying the involution ω to both sides of this relation.

Theorem 2.9

$$\begin{array}{ll} a) & \Delta_1\ e_n[\tfrac{X}{(1-t)(1-q)}] = e_1[X]\ e_{n-1}[\tfrac{X}{(1-t)(1-q)}] \\ b) & \Delta_1\ h_n[\tfrac{X}{(1-t)(1-q)}] = \sum_{k=0}^{n-1}\ (-1)^{n-1-k}\ h_k[\tfrac{X}{(1-t)(1-q)}]\ e_{n-k}[X] \end{array} \qquad 2.41$$

Proof

Note that for any two alphabets A, B we have the addition formulas

$$\begin{array}{ll} a) & e_n[A+B] = \sum_{k=0}^n\ e_k[A]\ e_{n-k}[B] \\ b) & h_n[A+B] = \sum_{k=0}^n\ h_k[A]\ h_{n-k}[B] \end{array} \qquad 2.42$$

Using 2.42 a) with $A = \frac{X}{(1-t)(1-q)}$ and $B = 1/z$, from the definition 2.16 we immediately obtain

$$\Delta_1\ e_n[\tfrac{X}{(1-t)(1-q)}] = e_n[\tfrac{X}{(1-t)(1-q)}] - \sum_{k=0}^{n} e_k[\tfrac{X}{(1-t)(1-q)}]\ e_{n-k}[1/z]\ \Omega[-\tfrac{X}{z}]\ |_{z^0}. \qquad 2.43$$

However, since

$$e_{n-k}[1/z] = \begin{cases} 1 & \text{for } n-k=0, \\ 1/z & \text{for } n-k=1, \\ 0 & \text{for } n-k \geq 2, \end{cases}$$

2.43 reduces to

$$\begin{aligned}\Delta_1\, e_n[\tfrac{X}{(1-t)(1-q)}] &= -\, e_{n-1}[\tfrac{X}{(1-t)(1-q)}]\ \Omega[-\tfrac{X}{z}]\ \vert_z \\ &= e_{n-1}[\tfrac{X}{(1-t)(1-q)}]\ e_1[X]\end{aligned}$$

as desired. Using 2.42 b) in an analogous manner we get

$$\Delta_1\, h_n[\tfrac{X}{(1-t)(1-q)}] = h_n[\tfrac{X}{(1-t)(1-q)}] - \sum_{k=0}^{n} h_k[\tfrac{X}{(1-t)(1-q)}]\ h_{n-k}[1/z]\ \Omega[-\tfrac{X}{z}]\ \vert_{z^0}$$

which reduces to 2.41 b) since here $h_{n-k}[1/z] = (1/z)^{n-k}$ and

$$\Omega[-\tfrac{X}{z}]\ \vert_{z^{n-k}} = (-1)^{n-k}\ e_{n-k}[X].$$

The machinery we have put together allows us to derive an interesting collection of identities. Remarkably, it develops that many of the sums that can be obtained by deleting some of the factors in the q,t-Catalan summand evaluate to familiar expressions. We give below a representative sample.

Theorem 2.10

$$\begin{array}{llcl}
a) & \displaystyle\sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu')}\ (1-t)(1-q)B_\mu(q,t)\ \Pi_\mu(q,t)}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} & = & 1 \\
b) & \displaystyle\sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu')}\ \Pi_\mu(q,t)}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} & = & \displaystyle\frac{1}{(1-t^n)(1-q^n)} \\
c) & \displaystyle\sum_{\mu\vdash n} \frac{(1-t)(1-q)B_\mu(q,t)}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} & = & \displaystyle\sum_{\nu\vdash n-1} \frac{t^{n(\nu)}\ q^{n(\nu')}}{h_\nu(t)h_\nu(q)} \\
d) & \displaystyle\sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu')}\ (1-t)(1-q)B_\mu(q,t)}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} & = & \displaystyle\sum_{\nu\vdash n-1} \frac{t^{n(\nu)}\ q^{n(\nu)}}{h_\nu(t)h_\nu(q)} \\
e) & \displaystyle\sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu')}}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} & = & \displaystyle\sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu)}}{h_\mu(t)h_\mu(q)} \\
f) & \displaystyle\sum_{\mu\vdash n} \frac{1}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} & = & \displaystyle\sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu')}}{h_\mu(t)h_\mu(q)} \\
g) & \displaystyle\sum_{\mu\vdash n} \frac{t^{2n(\mu)}\ q^{2n(\mu')}}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} & = & \displaystyle\sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu')}}{h_\mu(t)h_\mu(q)} \\
h) & \displaystyle\sum_{\mu\vdash n} \frac{t^{2n(\mu)}\ q^{2n(\mu')}\ (1-t)(1-q)B_\mu(q,t)}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} & = & \displaystyle\sum_{k=0}^{n-1} (-1)^{n-1-k}\ e_k[\tfrac{1}{(1-t)(1-q)}]
\end{array} \qquad 2.44$$

where $h_\mu(t)$ and $h_\mu(q)$ denote the standard hook products, for instance

$$h_\mu(t) = h_\mu(t,t) = \Pi(1-t^{a+l+1}).$$

Proof

a) Just equate the coefficients of $e_n(x)$ in both sides of 2.26 and use 2.17 b).

b) The Schur function expansion of the power symmetric function $p_n(x)$ may be written in the form

$$p_n(X) = \sum_{\lambda \vdash n} \chi^{\lambda}_{(n)} \; S_{\lambda}(x)$$

where $\chi^{\lambda}_{(n)}$ denotes the value of irreducible character χ^{λ} at the n-cycles. In particular this gives that

$$p_n[\tfrac{X}{(1-t)(1-q)}]|_{e_n(x)} = \frac{1}{(1-t^n)(1-q^n)} \; p_n(x) \;|_{e_n(x)} = \frac{(-1)^{n-1}}{(1-t^n)(1-q^n)}.$$

Thus 2.44 b) follows by equating the coefficients of $e_n(x)$ in 2.24 and using 2.17 b).

c) Applying Δ_1 to 2.39 a) and using 2.17 a) we get

$$\Delta_1 \; e_n[\tfrac{X}{(1-t)(1-q)}] = \sum_{\mu \vdash n} \frac{\tilde{H}_{\mu}(x;q,t)\;(1-t)(1-q)B_{\mu}(q,t)}{\tilde{h}_{\mu}(q,t)\tilde{h}'_{\mu}(q,t)} \tag{2.45}$$

which gives (using 2.40)

$$\Delta_1 \; e_n[\tfrac{X}{(1-t)(1-q)}] \;|_{S_{(n)}(x)} = \sum_{\mu \vdash n} \frac{(1-t)(1-q)B_{\mu}(q,t)}{\tilde{h}_{\mu}(q,t)\tilde{h}'_{\mu}(q,t)}. \tag{2.46}$$

which is the left hand side of c). On the other hand we have

$$e_1[X] \; e_{n-1}[\tfrac{X}{(1-t)(1-q)}] = \sum_{\nu \vdash n-1} e_1[X] S_{\nu}[X] \; S_{\nu'}[\tfrac{1}{(1-t)(1-q)}] \tag{2.47}$$

and since

$$e_1(x) \; S_{\nu}(x) \;|_{S_{(n)}(x)} = \begin{cases} 1 & \text{if } \nu = (n-1) \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

we derive that

$$e_1[X] \; e_{n-1}[\tfrac{X}{(1-t)(1-q)}] \;|_{S_{(n)}(x)} = e_{n-1}[\tfrac{1}{(1-t)(1-q)}], \tag{2.48}$$

and 2.44 c) follows by combining 2.41 a) with 2.46, 2.48 and the dual Cauchy identity

$$e_{n-1}[\tfrac{1}{(1-t)(1-q)}] = \sum_{\nu \vdash n-1} \frac{t^{n(\nu)} \; q^{n(\nu')}}{\tilde{h}_{\nu}(q,t)\tilde{h}'_{\nu}(q,t)}. \tag{2.49}$$

d) Equating coefficients of $e_n(x)$ on both sides of 2.45 gives

$$\Delta_1 \; e_n[\tfrac{X}{(1-t)(1-q)}] \;|_{e_n(x)} = \sum_{\mu \vdash n} \frac{t^{n(\mu)} \; q^{n(\mu')} \;(1-t)(1-q)B_{\mu}(q,t)}{\tilde{h}_{\mu}(q,t)\tilde{h}'_{\mu}(q,t)}, \tag{2.50}$$

which is the left hand side of d). On the other hand since

$$e_1(x) \; S_{\nu}(x) \;|_{e_n(x)} = \begin{cases} 1 & \text{if } \nu = 1^{n-1} \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

2.47 now gives

$$e_1[X]\ e_{n-1}[\tfrac{X}{(1-t)(1-q)}]\ |_{e_n(x)} = S_{(n-1)}[\tfrac{1}{(1-t)(1-q)}], \qquad 2.51$$

and 2.44 d) follows by combining 2.41 a) with 2.50, 2.51 and the Cauchy identity

$$S_{(n-1)}[\tfrac{1}{(1-t)(1-q)}] = \sum_{\nu\vdash n-1} \frac{t^{n(\nu)}\ q^{n(\nu)}}{\tilde{h}_\nu(q,t)\tilde{h}'_\nu(q,t)}. \qquad 2.52$$

e) Equating coefficients of $e_n(x)$ in 2.39 a) and using 2.17 b) gives

$$e_n[\tfrac{X}{(1-t)(1-q)}]|_{e_n(x)} = \sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu')}}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}.$$

On the other hand, from the dual Cauchy formula

$$e_n[\tfrac{X}{(1-t)(1-q)}] = \sum_{\mu\vdash n} S_\mu[X]\ S_{\mu'}[\tfrac{1}{(1-t)(1-q)}] \qquad 2.53$$

we get

$$e_n[\tfrac{X}{(1-t)(1-q)}]|_{e_n(x)} = S_{(n)}[\tfrac{1}{(1-t)(1-q)}],$$

and 2.44 e) follows from the classical identity

$$S_{(n)}[\tfrac{1}{(1-t)(1-q)}] = \sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu)}}{h_\mu(t)\ h_\mu(q)}.$$

Note that the same identity could be obtained by equating coefficients of $S_{(n)}$ $(= h_n)$ in 2.39 b).

f) Equating coefficients of $S_{(n)}$ in 2.39 a) and using 2.40 gives

$$e_n[\tfrac{X}{(1-t)(1-q)}]|_{e_n(x)} = \sum_{\mu\vdash n} \frac{1}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}.$$

On the other hand, from 2.53 we get

$$e_n[\tfrac{X}{(1-t)(1-q)}]\ |_{S_{(n)}} = e_n[\tfrac{1}{(1-t)(1-q)}] = \sum_{\mu\vdash n} \frac{t^{n(\mu)}\ q^{n(\mu')}}{h_\mu(t)\ h_\mu(q)}. \qquad 2.54$$

This establishes 2.44 f).

g) Equating coefficients of e_n in 2.39 b) and using 2.17 b) gives

$$h_n[\tfrac{X}{(1-t)(1-q)}]|_{e_n(x)} = \sum_{\mu\vdash n} \frac{t^{2n(\mu)}\ q^{2n(\mu')}}{\tilde{h}_\mu(q,t)\tilde{h}'_\mu(q,t)}.$$

But then from the standard Cauchy formula

$$h_n[\tfrac{X}{(1-t)(1-q)}] = \sum_{\mu\vdash n} S_\mu[X]\ S_\mu[\tfrac{1}{(1-t)(1-q)}] \tag{2.55}$$

we derive that

$$h_n[\tfrac{X}{(1-t)(1-q)}]|_{e_n(x)} = e_n[\tfrac{1}{(1-t)(1-q)}]$$

and 2.44 g) follows from the dual Cauchy formula in 2.54.

h) Finally, equating coefficients of e_n in 2.41 b) we get

$$\begin{aligned}\Delta_1\ h_n[\tfrac{X}{(1-t)(1-q)}]|_{e_n(x)} &= \textstyle\sum_{\mu\vdash n} \frac{t^{2n(\mu)}\ q^{2n(\mu')}\ (1-t)(1-q)B_\mu(q,t)}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} \\ &= \textstyle\sum_{k=0}^{n-1} (-1)^{n-1-k}\ h_k[\frac{X}{(1-t)(1-q)}]\ e_{n-k}[X]|_{e_n(x)}.\end{aligned} \tag{2.56}$$

But 2.55 rewritten for $n=k$ gives

$$h_k[\tfrac{X}{(1-t)(1-q)}]\ e_{n-k}[X]|_{e_n(x)} = \sum_{\nu\vdash k} S_\nu[X]\ e_{n-k}[X]|_{e_n(x)}\ S_\nu[\tfrac{1}{(1-t)(1-q)}],$$

and 2.44 h) immediately follows since for $\nu\vdash k$

$$S_\nu[X]\ e_{n-k}[X]|_{e_n(x)} = \begin{cases}1 & \text{if } \nu=1^k \text{ and}\\ 0 & \text{otherwise}\end{cases}$$

This completes our proof.

3. Diagonal Harmonics

We shall deal here with certain subspaces of the ring $\mathbf{Q}[X,Y]$ of polynomials with rational coefficients in the two sets of variables $X=\{x_1,x_2,\ldots,x_n\}$ and $Y=\{y_1,y_2,\ldots,y_n\}$. For a given exponent vector $p=(p_1,p_2,\cdots,p_n)$ we set

$$x^p = x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n} \quad\text{and}\quad |p| = p_1+p_2+\cdots+p_n. \tag{3.1}$$

If

$$P(x,y) = \sum_{p,q} c_{p,q}\ x^p\ y^q \tag{3.2}$$

then we let

$$\pi_{h,k}\ P = \sum_{|p|=h,|q|=k} c_{p,q}\ x^p\ y^q \tag{3.3}$$

and refer to it as the *bihomogeneous component of P of bidegree (h,k).* A subspace $\mathbf{M}\subseteq \mathbf{Q}[X,Y]$ which contains all the bihomogeneous components of each of its elements is said to be *bigraded.* If $\mathbf{M}$ is bigraded then it has the direct sum decomposition

$$\mathbf{M} = \bigoplus_{h\geq 0}\bigoplus_{k\geq 0}\ \mathcal{H}_{h,k}(\mathbf{M}), \tag{3.4}$$

where $\mathcal{H}_{h,k}(\mathbf{M})$ consists of the bihomogeneous elements of $\mathbf{M}$ of bidegree (h,k) or, equivalently the space

$$\mathcal{H}_{h,k}(\mathbf{M}) = \{\pi_{h,k} P : \ P \in \mathbf{M}\}. \tag{3.5}$$

We refer to it as the *bihomogeneous component of* $\mathbf{M}$ *of bidegree* (h,k).

Recall that the *diagonal action* of the symmetric group S_n on $\mathbf{Q}[X,Y]$ is defined by setting for $\sigma \in S_n$ and $P \in \mathbf{Q}[X,Y]$

$$\sigma\ P = P(x_{\sigma_1}, x_{\sigma_2} \ldots, x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2} \ldots, y_{\sigma_n}). \tag{3.6}$$

Note that if a bigraded subspace $\mathbf{M} \subseteq \mathbf{Q}[X,Y]$ is invariant under this action, then all its bihomogeneous components are also invariant. In this case we have an associated *bigraded character* $\Pi_{\mathbf{M}}(q,t)$ which is defined as the bivariate generating function of the characters of the components $\mathcal{H}_{h,k}(\mathbf{M})$. In symbols

$$\Pi_{\mathbf{M}}(q,t) = \sum_{h\geq 0} \sum_{k \geq 0} t^h\ q^k \ \mathrm{char}\ \mathcal{H}_{h,k}(\mathbf{M}). \tag{3.7}$$

We also have an associated *bigraded Frobenius characteristic* $C_{\mathbf{M}}(x;q,t)$ which is simply the image of $\Pi_{\mathbf{M}}(q,t)$ under the Frobenius map. In symbols

$$C_{\mathbf{M}}(x;q,t) = F\ \Pi_{\mathbf{M}}(q,t) = \frac{1}{n!} \sum_{\sigma \in S_n} \Pi_{\mathbf{M}}(\sigma;q,t)\ p_{\lambda(\sigma)}(x), \tag{3.8}$$

where $\Pi_{\mathbf{M}}(\sigma;q,t)$ denotes the value of this character at σ and $p_{\lambda(\sigma)}(x)$ is the power basis element indexed by the partition $\lambda(\sigma)$ which gives the cycle structure of σ. Since the Schur function $S_\lambda(x)$ is the Frobenius image of the irreducible S_n-character χ^λ we then have the two parallel expansions

$$\begin{array}{lrl} a) & \Pi_{\mathbf{M}}(q,t) & = \sum_{\lambda \vdash n}\ \chi^\lambda\ C_{\lambda,\mathbf{M}}(q,t) \\ b) & C_{\mathbf{M}}(x;q,t) & = \sum_{\lambda \vdash n}\ S_\lambda(x)\ C_{\lambda,\mathbf{M}}(q,t) \end{array} \tag{3.9}$$

where the $C_{\lambda,\mathbf{M}}(q,t)$ is the bivariate generating function of the multiplicity of χ^λ in the various bihomogeneous components of $\mathbf{M}$. In particular, when $\mathbf{M}$ is finite dimensional, $C_{\lambda,\mathbf{M}}(q,t)$ will necessarily be a polynomial with non-negative integer coefficients.

This circumstance yields a representation theoretical approach to the Macdonald conjecture concerning the coefficients $K_{\lambda\mu}(q,t)$. Note that I.11 and 2.38 yield

$$K_{\lambda\mu}(q,t) = K_{\lambda'\mu}(1/q,1/t)\ q^{n(\mu')}t^{n(\mu)}. \tag{3.10}$$

Thus if $K_{\lambda\mu}(q,t)$ is a polynomial it must necessarily be of degree $n(\mu')$ in q and degree $n(\mu)$ in t. This shows that the Macdonald conjecture is equivalent to the statement that the $\tilde{K}_{\lambda\mu}(q,t)$ themselves are polynomials with positive integer coefficients. In particular we may prove the Macdonald conjecture by constructing (for each μ) a bigraded module whose Frobenius characteristic is given by the polynomial $\tilde{H}_\mu(x;q,t)$. This observation has been the point of departure in a continuing investigation that has brought forward a number of problems that are of interest in their own right. In particular, it ultimately brought us to a path which led to the q,t-Catalan and the calculations of the present paper. To see how all this comes about we need some further ingredients.

Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$ be a partition of n and let

$$(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)$$

be the pairs (l', a') of the various cells of the diagram of μ arranged in lexicographic order. Set

$$\Delta_\mu(x,y) = det \ \| x_i^{p_j} y_i^{q_j} \|. \qquad 3.11$$

This given, we let $\mathbf{M}_\mu[X,Y]$ be the space spanned by all the partial derivatives of $\Delta_\mu(x,y)$. In symbols

$$\mathbf{M}_\mu[X,Y] = \mathcal{L}[\partial_x^p \, \partial_y^q \ \Delta_\mu(x,y) \,]. \qquad 3.12$$

Since the polynomial $\Delta_\mu(x,y)$ is bihomogeneous of bidegree $(n(\mu), n(\mu'))$ and alternates in sign under the diagonal action, we can easily deduce from 3.12 that $\mathbf{M}_\mu[X,Y]$ is, under this action, a bigraded S_n-module. We may then write its Frobenius characteristic in the form

$$C_{\mathbf{M}_\mu}(x; q, t) = \sum_\lambda \ S_\lambda(x) \ C_{\lambda,\mu}(q,t). \qquad 3.13$$

Supported by extensive computer explorations and strong theoretical evidence, in [7] we conjectured that indeed we have

$$C_{\mathbf{M}_\mu}(x; q, t) = \tilde{H}_\mu(x; q, t). \qquad 3.14$$

This equality, which we refer to as the $C = \tilde{H}$ conjecture, is one of several that can be found in [7], where we presented (for each μ) a number of different constructions all of which, upon the validity of the $C = \tilde{H}$ conjecture, should lead to the same bigraded submodule of $\mathbf{Q}[X,Y]$.

Clearly, 3.14 is equivalent to the identities

$$C_{\lambda,\mathbf{M}_\mu}(q,t) = \tilde{K}_{\lambda\mu}(q,t).$$

Now it is shown in [9] and [8] that this does hold true for all μ when λ is a hook and for all λ when μ is a hook, a 2-row or a 2-column partition. Moreover, it has recently also been verified (by different independent approaches in [1] and [21]) for all partitions of the form $\mu = (1^k, 2, n-k-2)$.

It is a classical construction of the invariant theorists that with every finite matrix group action on the polynomial ring there is an associated space of harmonic polynomials which are solutions of all corresponding non-trivial homogeneous invariant polynomial differential operators. In the case of the diagonal action, the corresponding space of harmonic polynomials, which we shall refer to as *Diagonal Harmonics* may be simply defined as the solution space

$$\mathbf{DH}_n[X,Y] = \{ \ P(x,y) : \sum_{i=1}^n \ \partial_{x_i}^h \partial_{y_i}^k \ P = 0 \quad \forall \quad h + k \geq 1 \}. \qquad 3.15$$

It is easy to see that we have here yet another bigraded subspace of $\mathbf{Q}[X,Y]$ which is also invariant under the diagonal action.

Its discovery prompted the second author to carry out an extensive computer exploration of the space $\mathbf{DH}_n[X,Y]$. The resulting data suggested a number of surprising conjectures concerning various specializations of the bigraded character of $\mathbf{DH}_n[X,Y]$. These conjectures are described in full detail in [12].

Now very recently, in an algebraic geometrical setting suggested by C. Procesi, the second author, assuming the $C=\tilde{H}$ conjecture and a number of other desirable algebraic geometrical facts, was led to conjecture that the Frobenius characteristic of $\mathbf{DH}_n[X,Y]$ is none other than the symmetric polynomial defined by I.15. That is, using the present notation, we should have

$$C_{\mathbf{DH}_n}(x;q,t)=DH_n(x;q,t). \tag{3.16}$$

Note that in particular, this implies that our q,t-Catalan must necessarily be the Hilbert series of the alternating part of $\mathbf{DH}_n[X,Y]$.

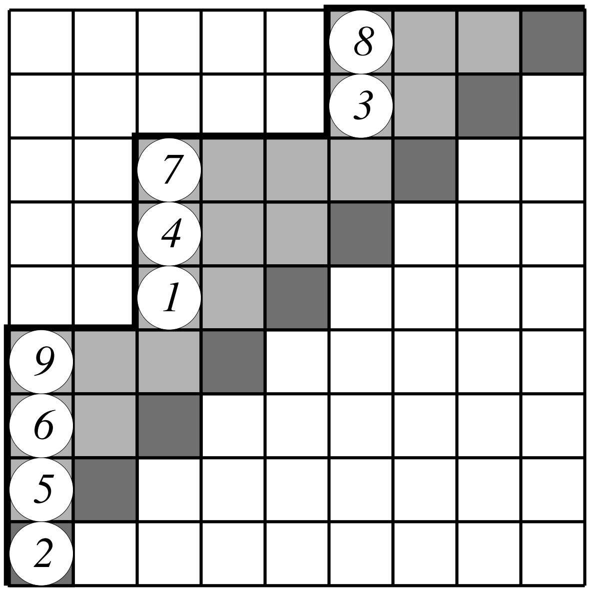
It is precisely this development that led us to the calculations we have presented here. Indeed, we were thus forced to investigate to what extent the implications of this identity could be conciliated with the various conjectures ventured in [12]. We shall presently see that all of the conjectures in [12] are, in fact, only specializations of 3.16 and they may be replaced by the single identity in 3.16. Since this identity was derived through algebraic geometrical considerations which are entirely independent of the calculations of the present paper and the computer explorations that led to the conjectures in [12], this complete agreement may be viewed as the most remarkable evidence in support of the $C=\tilde{H}$ conjecture. To fully appreciate what we are asserting here we need to briefly review some of the contents of [12].

The most fascinating discovery advanced in [12] is that the diagonal action of S_n on $\mathbf{DH}_n[X,Y]$ appears to be equivalent to a sign-twisted version of the the action of S_n on the so called *Parking Functions* of Konheim and Weiss [15]. In fact, there is even a graded refinement of this, but to state it we need to know some properties of Parking Functions.

This concept, which also arises in computer science in the theory of *hashing* [14], can be defined picturesquely as follows. On a *one way* street there are n parking spaces, labeled $1,2,\ldots,n$ in succession. There are n drivers who plan to park on this street. Each of the drivers has a preferred parking space in mind. Say the i^{th} driver wishes to park in parking space f_i. We call the map $i\to f_i$ a *Preference Function.* The cars arrive one at a time. The i^{th} car proceeds to parking space f_i and, if it is free, the driver parks there. However, this place may already be occupied. If that happens, the driver will proceed (in the legal direction) to the first available parking space and park there. A *Parking Function* is simply a Preference Function under which all cars will be able to park. It is easy to see that not all Preference Functions are Parking Functions. For instance if less than 4 drivers wish to park in the first 4 parking spaces then more than $n-4$ prefer to park in the last $n-4$ parking spaces and they cannot all park. However, this type of occurrence is the only thing that can go wrong. More precisely, it can be shown that a Preference Function is a Parking Function if and only if for all k there are at least k drivers who prefer one of the first k places. In symbols

$$\#\{i \,:\, f_i \leq k\} \geq k \quad \forall \; k = 1, 2, \ldots, n-1 \tag{3.17}$$

There is a convenient way to depict a Parking Function, which reveals many of its properties.



In the $n \times n$ lattice square $SQ[n]$ of Section 1 we represent the drivers that prefer the k^{th} place by labeled circles, stacked on the k^{th} column starting at the lattice square at height equal to one plus the number of drivers who prefer one of the first $k-1$ parking places. The figure above illustrates the Parking Function

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 6 & 3 & 1 & 1 & 3 & 6 & 1 \end{pmatrix}$$

Driver i is represented by a circle labelled i, and we agree to place the labels in increasing order along each column. This manner of representing a Parking Function brings into evidence that there is a path $D \in \mathcal{D}_n$ associated to each Parking Function. This is simply the graph of the function $k \;\to\; \#\{i \;:\; f_i \leq k\}$. The condition in 3.17 assures that this path remains weakly above the diagonal. We can reverse the process and construct all Parking Functions in the following manner. We first choose a path $D \in \mathcal{D}_n$. Next we draw circles to the right of the vertical steps in the path. Finally we choose the labels that are to fall in each column and place them there in increasing order. Note that if D has s vertical segments of lengths $d_1, d_2, \ldots, d_s$ then there are exactly

$$\binom{n}{d_1\ d_2\ \cdots\ d_s} \tag{3.18}$$

ways of placing the labels in the circles. It will be convenient to denote by $\mathcal{P}_n$ the collection of all parking functions on the one way street with n parking spaces, and by $\mathcal{P}_n(D)$ the Parking

Functions constructed in this manner from the path D. Now there is a natural way to let S_n act on $\mathcal{P}_n$. For a given $\sigma \in S_n$ we simply replace the label i by σ_i and rearrange the circles so that the labels again increase along the columns. In other words we assign the driver of car σ_i what used to be the preference of the driver of car i. It is clear from our construction that under this action $\mathcal{P}_n$ breaks up into $\frac{1}{n+1}\binom{2n}{n}$ orbits, given by the subcollections $\mathcal{P}_n(D)$ as D varies in $\mathcal{D}_n$. This implies that the corresponding S_n-representation contains precisely $\frac{1}{n+1}\binom{2n}{n}$ occurrences of the trivial representation.

The area $a(D)$ under a Dyck path D corresponding to parking function f will be briefly referred to as the *weight* of f. Note that since the corresponding Dyck path does not change under the S_n action, the weight itself will also remain invariant. This given, the Parking Function conjecture made in [12] can be expressed as follows

Conjecture

The action of S_n on the diagonal harmonics of $\mathbf{DH}_n[X,Y]$ which are homogeneous of degree k in the Y variables is a sign-twisted version of the action on the collection of parking functions of weight k.

Now it is not difficult to show (see [12]) that this conjecture is equivalent to the symmetric function identity

$$C_{\mathbf{DH}_n}(x;q,1) = \sum_{D\in\mathcal{D}_n} q^{a(D)} \prod_{i=1}^{n} e_{\alpha_i(D)}(x). \tag{3.19}$$

Putting this together with 1.32, 1.20 and 2.33 we see that the Parking Function conjecture (Conjecture 2.6.4 of [12]) is thus simply the special case $t = 1$ of 3.16. Note further that since the coefficient of e_n in 3.19 is

$$C_n(q,1) = \sum_{D\in\mathcal{D}_n} q^{a(D)}, \tag{3.20}$$

we can deduce that if 3.16 holds then the alternants of $\mathbf{DH}_n$ have a Hilbert series which specializes for $t = 1$ to the q-Catalan number $C_n(q)$ defined by I.5. The conjecture that the dimension of the alternating part of $\mathbf{DH}_n$ is given by the Catalan number was also noted in [12]. Of course, it is also a direct consequence of the ungraded Parking Function conjecture since if we sign-twist a representation with $\frac{1}{n+1}\binom{2n}{n}$ occurrences of the trivial we will get a representation with the $\frac{1}{n+1}\binom{2n}{n}$ occurrences of the sign representation.

Conjecture 2.5.1 of [12] states that the coefficient of the irreducible character χ^λ in $\Pi_n(q,1/q)\, q^{\binom{n}{2}}$ is given by the Schur function specialization

$$\frac{S_{\lambda'}(1,q,\ldots,q^n)}{1+q+\cdots+q^n}. \tag{3.21}$$

Note that it is not even obvious that this rational function simplifies to a polynomial for general λ, let alone to a polynomial with positive integer coefficients. However, Proposition 2.5.2 of [12] assures us that this is indeed the case. In Λ-ring notation this conjecture translates into the symmetric function identity

$$C_{\mathbf{DH}_n}(x;q,1/q)\, q^{\binom{n}{2}} = DH_n(x;q,1/q)\, q^{\binom{n}{2}} = \frac{1}{[n+1]_q}\, e_n[X\tfrac{1-q^{n+1}}{1-q}]$$

which, as we have seen, is the content of Corollary 2.5. Thus we see again that this second conjecture from [12] is another specialization of 3.16.

Using conjecture 3.16 we get complete information about each of the invariant subspaces $\mathcal{H}_{h,k}(\mathbf{DH}_n)$. In particular, we can easily calculate for large values of n the bivariate Hilbert series of $\mathbf{DH}_n$. In fact, it is not difficult to see that this Hilbert series is given by the formula

$$F_{\mathbf{DH}_n}(q,t) = \partial_{p_1}^n\, C_{\mathbf{DH}_n}(x;q,t), \tag{3.22}$$

where again p_1 denotes the first power symmetric function. So assuming 3.16 we obtain that

$$F_{\mathbf{DH}_n}(q,t) = \sum_{\mu\vdash n} \frac{F_\mu(q,t)\, t^{n(\mu)}q^{n(\mu')}\,(1-t)(1-q)\,\Pi_\mu(q,t)\,B_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}, \tag{3.24}$$

where

$$F_\mu(q,t) = \partial_{p_1}^n\, \tilde{H}_{p_1}(x;q,t). \tag{3.25}$$

Now in [9] (see also [8]) we established a recursion from which $F_\mu(q,t)$ is easily computed. Our computations, carried out for $n \le 16$, exhibit $F_\mu(q,t)$ as a beautiful polynomial with integer coefficients giving further support to the Macdonald conjecture as well as our $C = \tilde{H}$ conjecture, the latter implying that $F_\mu(q,t)$ should give the Hilbert series of our modules $\mathbf{M}_\mu[X,Y]$. In this manner we can also easily evaluate 3.24, the result of course agreeing perfectly with the tables in [12] of the actual values of $F_{\mathbf{DH}_n}(q,t)$ for $n \le 7$.

The present work has a closely related extension which leads to further combinatorial constructs and conjectures. It develops that for each integer $m \ge 0$ the two sequences

$$C_n^{(m)}(q,t) = \sum_{\mu\vdash n} \frac{t^{(m+1)n(\mu)}q^{(m+1)n(\mu')}\,(1-t)(1-q)\,\Pi_\mu(q,t)\,B_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \tag{3.26}$$

and

$$DH_n^{(m)}(x;q,t) = \sum_{\mu\vdash n} \frac{\tilde{H}_\mu(x;q,t)\, t^{mn(\mu)}q^{mn(\mu')}\,(1-t)(1-q)\,\Pi_\mu(q,t)\,B_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \tag{3.27}$$

admit a treatment that follows closely our treatment of $C_n(q,t)$ and $DH_n(x;q,t)$. To see how this comes about we need some definitions. Let $\mathcal{J} = \mathcal{J}_n$ denote the ideal generated by the *polarized* power sums

$$\sum_{i=1}^n x_i^h y_i^k \quad (h+k \ge 1)$$

and let $\mathcal{A} = \mathcal{A}_n$ denote the ideal generated by the polynomials $P(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_n)$ which alternate in sign under the diagonal action. For convenience let us set

$$\mathbf{R}^{(m)}[X;Y] = \mathcal{A}^{m-1}/\mathcal{A}^{m-1}\mathcal{J}.$$

and twist its natural S_n-action by the $(m-1)^{st}$ power of the alternating representation, so that the generators of this module, which are the minimal generators of $\mathcal{A}^{m-1}$, become S_n-invariant. It is not hard to show that the S_n-alternating part of $\mathbf{R}^{(m)}[X;Y]$ is naturally isomorphic (except for a sign twist) to the space spanned by any minimal set of generators for $\mathcal{A}^m$. Let us call this space $\mathbf{\Gamma}^{(m)}$. In symbols

$$\mathbf{\Gamma}^{(m)}[X,Y] \cong \mathcal{A}^m/\mathcal{M}\mathcal{A}^m \quad \text{where} \quad \mathcal{M}=(x_1,y_1,\ldots,x_n,y_n).$$

Clearly $\mathbf{R}^{(m)}[X;Y]$ is a bigraded S_n-module under the action described and $\mathbf{\Gamma}^{(m)}[X;Y]$ carries only the sign representation. This given, we let $C_{\mathbf{R}^{(m)}}(x;q,t)$ denote the Frobenius characteristic of $\mathbf{R}^{(m)}[X;Y]$ and let $F_{\mathbf{\Gamma}^{(m)}}(q,t)$ be the Hilbert series of $\mathbf{\Gamma}^{(m)}[X;Y]$. From the preceding remarks, we have

$$F_{\mathbf{\Gamma}^{(m)}}(q,t) = C_{\mathbf{R}^{(m)}}(x;q,t)\,|_{e_n(x)}. \tag{3.28}$$

Now the same algebraic geometrical considerations which produced 3.16 have led the second author to conjecture that

$$C_{\mathbf{R}^{(m)}}(x;q,t) = DH_n^{(m)}(x;q,t). \tag{3.29}$$

Combining this with 3.28 and I.16 we deduce the further conjecture that

$$C_n^{(m)}(q,t) = F_{\mathbf{\Gamma}^{(m)}}(q,t), \tag{3.30}$$

which in particular implies that also $C_n^{(m)}(q,t)$ must be a polynomial with non-negative integer coefficients. It can be shown (see [12]) that $\mathbf{R}^{(m)}[X;Y]$ for $m=1$ reduces to a bigraded S_n-module $\mathbf{Q}[X,Y]/\mathcal{J}$ equivalent to $\mathbf{DH}_n[X,Y]$. Moreover, it is a special case of the relationship between $\mathbf{R}^{(m)}$ and $\mathbf{\Gamma}^{(m)}$ that any basis of diagonal harmonic alternants minimally generates $\mathcal{A}$. This given, we see that 3.29 and 3.30 are natural extensions of our conjectures concerning $\mathbf{DH}_n(x;q,t)$ and $C_n(q,t)$. We should thus suspect that some of our manipulations in Sections 1 and 2 can be carried out also for an arbitrary $m\geq 1$. We should also expect some interesting combinatorial descriptions for the specializations at $t=1/q$ and $t=1$ for both $DH_n^{(m)}(x;q,t)$ and $C_n^{(m)}(q,t)$. It develops that this is the case up to a point. We shall see that an exploration of these two constructs leads to some very interesting questions.

4. The extended family $C_n^{(m)}(q,t)$: Results and problems.

In this section we give a brief overview of what can be proved concerning $DH^{(m)}(x;q,t)$ and $C_n^{(m)}(q,t)$. For brevity and to avoid repetitions we shall omit some of the details here, especially when they can be easily filled in by imitating our previous arguments.

To begin with we have an analogue of Theorem 2.5. Namely

Theorem 4.1

$$DH_n^{(m)}(x;q,1/q)\; q^{m\binom{n}{2}} = \frac{1}{[mn+1]_q} e_n[X\tfrac{1-q^{mn+1}}{1-q}] \tag{4.1}$$

Proof

The same calculations that led us to 2.29 now yield

$$DH_n^{(m)}(x;q,1/q)\, q^{m\binom{n}{2}} = \sum_{\mu'=(1^r,n-r)} S_\mu[\tfrac{X}{1-q}]\,(-1)^r\, q^{mnr+r}\,(1-q).$$

Using 2.15 with $u = q^{mn+1}$ this can be rewritten in the form

$$DH_n^{(m)}(x;q,1/q)\, q^{m\binom{n}{2}} = \sum_{\mu'=(1^r,n-r)} S_\mu[\tfrac{X}{1-q}]S_{\mu'}[1-q^{mn+1}]\,\frac{1-q}{1-q^{mn+1}},$$

which easily yields 4.1 by an application of the corresponding dual Cauchy identity.

We thus obtain a complete analogue of Corollary 2.5:

Corollary 4.1

For any partition λ

$$DH_n^{(m)}(x;q,1/q)\, q^{m\binom{n}{2}}|_{S_\lambda(x)} = \frac{1}{[nm+1]_q}\, S_{\lambda'}[\tfrac{1-q^{mn+1}}{1-q}]. \qquad 4.2$$

In particular

$$C_n^{(m)}(q,1/q)\, q^{m\binom{n}{2}} = \frac{1}{[mn+1]_q}\begin{bmatrix} mn+n \\ n\end{bmatrix}_q. \qquad 4.3$$

In the case $m=1$, as we have already mentioned, the right hand side of 4.3 q-counts Dyck words by the major index statistic. However, we don't know what the extension of this property might be for arbitrary m. There is at least one clue. The lattice path setting of Section 1 has a natural extension here. Denote by $RE^{(m)}[n]$ the lattice rectangle with vertices $(0,0)$, (mn,n). Proceeding in an analogous manner we shall let $\mathcal{D}_n^{(m)}$ denote the collection of lattice paths that consist of NORTH and EAST steps and constantly remain above the diagonal joining $(0,0)$ to (mn,n). Now it easy to show that, for $q=1$, the right hand side of 4.3 gives the cardinality of $\mathcal{D}_n^{(m)}$. Given a path $\pi\in\mathcal{D}_n^{(m)}$ let us label each EAST step by an a and each NORTH step by a b and let $w(\pi)$ denote the word obtained by reading these letters out of π from left to right. We might suspect that in general the right hand side of 4.3 q-counts these words by the major index statistic as it does for $m=1$. This is not so even for $m=2$. We must then leave it as an open problem to find the statistic on words that works in the general case.

Our next task is to derive a combinatorial interpretation for the specializations at $t=1$ of $DH^{(m)}(x;q,t)$ and $C_n^{(m)}(q,t)$. We start with the following identity which can be established by only minor changes in the proof of Theorem 2.6

Theorem 4.2

$$DH_n^{(m)}(x;q,1) = \sum_{\mu\vdash n}\left(\prod_i q^{m\binom{\mu_i}{2}} h_{\mu_i}[\tfrac{X}{1-q}]\right) f_\mu[1-q], \qquad 4.4$$

To conform as closely as possible to the notation of Sections 1 and 2, let us denote the symmetric function in 4.4 by $k_n^{(m)}(q)$. For a given $\pi\in\mathcal{D}_n^{(m)}$ let $a(\pi)$ denote the number of lattice squares below π that are above the diagonal of $RE^{(m)}(n)$ and let $\alpha_i(\pi)$ denote, as before, the number of NORTH steps of π on the line $x = i$. This given, we have the following extension of the identity in 1.32.

Theorem 4.3

$$k_n^{(m)}(q) = \sum_{\pi \in \mathcal{D}_n^{(m)}} q^{a(\pi)} \prod_{i=0}^{mn-1} e_{\alpha_i(\pi)}(x) \tag{4.5}$$

Obvious as this may be to conjecture, given 1.32, its proof turns out to be not entirely routine. In fact, it will involve some rather surprising uses of our Λ-ring and Lagrange inversion machinery. For the moment we shall let $\phi_n^{(m)}(q)$ denote the right hand side of 4.5 and shall obtain the equality $k_n^{(m)}(q) = \phi_n^{(m)}(q)$ as the ultimate consequence of a number of auxiliary propositions which will progressively change both of them into a common final expression.

Our basic ingredient here is the formal series

$$F(z) = \sum_{k \geq 1} F_k \, z^k = \frac{z}{E(z)E(z/q)\cdots E(z/q^{m-1})}. \tag{4.6}$$

where, as in Section 1,

$$E(z) = \sum_{n \geq 0} e_n(x) \, z^n.$$

Let us also define $f(z) = \sum_{k \geq 1} f_k \, z^k$ as the solution of the Lagrange inversion problem

$$\sum_{k \geq 1} F_k \, f(z)f(zq^m)\cdots f(zq^{(k-1)m}) = z. \tag{4.7}$$

Note that by Proposition 1.1 (with q replaced by q^m) this is equivalent to setting

$$\sum_{k \geq 1} f_k \, F(z)F(z/q^m)\cdots F(z/q^{(k-1)m}) = z. \tag{4.8}$$

Finally, following 1.19 let us also set

$$g_n = k_{n-1}^{(m)}(q) \, q^{n-1}. \tag{4.9}$$

This amounts to setting as in 1.18

$$g(z) = \sum_{n \geq 1} g_n \, z^n = z \, K^{(m)}(zq) = z \sum_{n \geq 0} k_n^{(m)}(q) \, q^n \, z^n. \tag{4.10}$$

Now it turns out that $f(z)$ and $g(z)$ are not equal, as we might have been inclined to believe. Rather we have

Proposition 4.1

The formal power series $g(z)$ is the solution of the inversion problem

$$\sum_{k \geq 1} g_k \, F(z)F(z/q^m)\cdots F(z/q^{(k-1)m}) = \frac{z}{E(z)E(z/q)\cdots E(z/q^{m-2})}. \tag{4.11}$$

Proof

Using the notation of Section 1 we may write $F(z)$ in the form

$$F(z) = z\, \frac{{}^*E(z/q^m)}{{}^*E(z)}. \qquad 4.12$$

Substituting this into 4.11 and making the necessary cancelations yields

$$\sum_{k\geq 1} g_k\, z^k\, q^{-m\binom{k}{2}}\, \frac{{}^*E(z/q^{km})}{{}^*E(z)} = \frac{z}{E(z)E(z/q)\cdots E(z/q^{m-2})},$$

and a multiplication by ${}^*E(z)$ gives

$$\sum_{k\geq 1} g_k\, z^k\, q^{-m\binom{k}{2}}\, {}^*E(z/q^{km}) = z\, {}^*E(z/q^{m-1}). \qquad 4.13$$

Since now q^m plays the role of q, the natural unroofing operation we must work with here should be

$$^{\vee}A(z) = \sum_{n\geq 0} A_n\, q^{m\binom{n}{2}}\, z^n.$$

In fact, this untagles q^m-products of the form

$$A\ \otimes_{q^m} B\ (z) = \sum_{k\geq 0} A_k\, z^k\, B(z/q^{km}).$$

More precisely, we have

$$^{\vee}(\ A\ \otimes_{q^m} B)(z) = (^{\vee}A)(z)\ (^{\vee}B)(z).$$

This given, unroofing both sides of 4.13 we get

$$g(z)\ ^{\vee *}E(z) = z\ ^{\vee *}\check{E}(zq).$$

Using 4.10, cancelling z and making the substitution $z \to z/q$ we get

$$K^{(m)}(z) = \frac{^{\vee *}E(z)}{^{\vee *}E(z/q)}.$$

We can now follow almost verbatim the steps in the proof of 1.20. The only change is that now the alphabet A must be chosen so that

$$q^{m\binom{n}{2}}\ h_n[\tfrac{X}{1-q}] = h_n[A] \quad (\forall n \geq 0) \qquad 4.14$$

We are thus led again to the equation

$$k_n^{(m)}(q) = e_n[A(1-q)].$$

The dual Cauchy formula 1.27 then gives

$$k_n^{(m)}(q) = \sum_{\mu\vdash n} h_\mu[A]\, f_\mu[1-q]$$

which by 4.14 is seen to be equivalent to our original definition 4.4. This completes our proof.

We shall next examine $f(z)$ and its relation to $g(z)$. To this end it is more convenient to set

$$f(z) = z\, H(zq) = z \sum_{k\geq 0} H_k(q)\, q^k\, z^k. \qquad 4.15$$

This given, we obtain the following recursion for the coefficients of $H(z)$.

Proposition 4.2

$$H_n(q) = \sum_{k=1}^{n} q^{m\binom{k}{2}}\, e_k[X\tfrac{1-q^m}{1-q}] \sum_{n_1+\cdots+n_k=n-k} \prod_{i=1}^{k} q^{m(k-i)n_i}\, H_{n_i}(q), \qquad 4.16$$

Proof

Multiplying 4.8 by the denominator of $F(z)$ in 4.6, making the replacement $z \to zq^m$ and cancelling out z gives

$$\sum_{k\geq 0} f_{k+1}\, F(z)F(z/q^m)\cdots F(z/q^{(k-1)m}) = E(zq)E(zq^2)\cdots E(zq^m) = \sum_{k\geq 0} e_k[qX\tfrac{1-q^m}{1-q}]\, z^k.$$

Using again Proposition 1.1 we deduce (by 4.15) that this identity is equivalent to

$$\sum_{k\geq 0} q^k\, H_k(q)\, z^k = \sum_{k\geq 0} e_k[qX\tfrac{1-q^m}{1-q}]\, z^k\, q^{m\binom{k}{2}}\, H(zq)H(zq^{1+m})\cdots H(zq^{1+(k-1)m}).$$

The substitution $z \to z/q$ simplifies this to

$$\sum_{k\geq 0} H_k(q)\, z^k = \sum_{k\geq 0} e_k[X\tfrac{1-q^m}{1-q}]\, z^k\, q^{m\binom{k}{2}}\, H(z)H(zq^m)\cdots H(zq^{(k-1)m})$$

and our desired identity 4.16 immediately follows by equating coefficients of z^n.

Formula 4.16 reveals that $H_n(q)$ may be also be obtained by appropriately q-counting ordinary Dyck paths. In fact, note that 4.16 is essentially 1.33 with q^m replacing q and $e_k[X\frac{1-q^m}{1-q}]$ replacing $e_k(x)$. This observation immediately yields the following corollary of Proposition 4.2.

Proposition 4.3

$$H_n(q) = \sum_{D\in\mathcal{D}_n} q^{m\, a(D)} \prod_{i=0}^{n-1} e_{\alpha_i(D)}[X\tfrac{1-q^m}{1-q}] \qquad 4.17$$

The Λ-ring expression $e_k[X\frac{1-q^m}{1-q}]$ may also be given a lattice path interpretation. To see this, let $\Gamma(m,k)$ denote the collection of lattice paths (which proceed by EAST and NORTH steps) that start at $(0,0)$ with an EAST step and end at (m,k). For a path $\gamma \in \Gamma(m,k)$ we let $a(\gamma)$ denote the number of lattice squares below γ and let $\alpha_i(\gamma)$ denote as before the number of NORTH steps of γ on the line $x=i$. This given, we have

Proposition 4.4

$$e_k[X\tfrac{1-q^m}{1-q}] = \sum_{\gamma\in\Gamma(m,k)} q^{a(\gamma)}\; e_{\alpha_1(\gamma)}(x)e_{\alpha_2(\gamma)}(x)\cdots e_{\alpha_m(\gamma)}(x). \qquad 4.18$$

Proof

The addition formula for the elementary symmetric function e_k gives

$$e_k[X\tfrac{1-q^m}{1-q}] = \sum_{k_0+k_1+\cdots+k_{m-1}=k} e_{k_0}(x)\; q^{k_1}e_{k_1}(x)\; q^{2k_2}e_{k_2}(x)\cdots q^{(m-1)k_{m-1}}e_{k_{m-1}}(x). \qquad 4.19$$

Given a choice of $k_0,k_1,\ldots,k_{m-1}$ with $k_0+k_1+\cdots+k_{m-1}=k$ let $\gamma(k_0,k_1,\ldots,k_{m-1})$ denote the path of $\Gamma(m,k)$ which has k_i NORTH steps on the line $x=m-i$. This establishes a bijection between $\Gamma(m,k)$ and the summands in 4.19. Now it is easy to see that the number of lattice squares under $\gamma(k_0,k_1,\ldots,k_{m-1})$ is given precisely by the sum

$$k_1+2k_2+\cdots+(m-1)k_{m-1}.$$

Since by construction $\alpha_i(\gamma(k_0,k_1,\ldots,k_{m-1}))=k_{m-i}$ we see that 4.18 is just another way of writing 4.19.

This result allows us to rewrite the right-hand side of 4.5 as a sum over $\mathcal{D}_n$. Namely we have

Proposition 4.5

$$\begin{aligned}\phi_n^{(m)}(q) &= \textstyle\sum_{\pi\in\mathcal{D}_n^{(m)}} q^{a(\pi)} \prod_{i=0}^{mn-1} e_{\alpha_i(\pi)}(x)\\ &= \textstyle\sum_{D\in\mathcal{D}_n} q^{m\,a(D)}\; e_{\alpha_0(D)}(x)\; e_{\alpha_1(D)}[X\tfrac{1-q^m}{1-q}]\cdots e_{\alpha_{n-1}(D)}[X\tfrac{1-q^m}{1-q}]\end{aligned} \qquad 4.20$$

Proof

Given a path $\pi\in\mathcal{D}_n^{(m)}$ and an integer $i\in[0,n]$ let (mi,y_i) be the point of π with x-coordinate mi and highest y-coordinate. It is easy to see that there is a unique path in $\mathcal{D}_n$ which passes through the points

$$(0,0),(0,y_0),(1,y_0),(1,y_1),\ldots,(i,y_{i-1}),(i,y_i),\ldots,(n,y_{n-1}),(n,y_n).$$

Let us denote this path by $D(\pi)$.

For a given $D\in\mathcal{D}_n$ we can construct all the paths $\pi\in\mathcal{D}_n^{(m)}$ for which $D(\pi)=D$ by the following procedure. We first take each EAST step of D and replace it by m successive EAST steps to obtain a path $\pi(D)\in\mathcal{D}_n^{(m)}$ which passes through the points

$$(0,0),(0,y_0),(m,y_0),(m,y_1),\ldots,(i\,m,y_{i-1}),(i\,m,y_i),\ldots,(n\,m,y_{n-1}),(n\,m,y_n).$$

Then between the vertices $((i-1)\,m,y_{i-1})$ and $(i\,m,y_i)$ of $\pi(D)$ we insert a subpath that proceeds by EAST and NORTH steps and *starts with an* EAST *step*. Clearly, the area under the resulting element $\pi \in \mathcal{D}_n^{(m)}$ decomposes into the area under the path $\pi(D)$, which is equal to $m\, a(D)$, plus the area between the chosen subpath and $\pi(D)$ itself. Now it is not difficult to see that Proposition 4.4 implies that summing over all possible choices of subpaths produces the identity

$$\sum_{\substack{\pi\in\mathcal{D}_n^{(m)} \\ D(\pi)=D}} q^{a(\pi)} \prod_{i=0}^{nm-1} e_{\alpha_i(\pi)}(x) = q^{m\, a(D)}\; e_{\alpha_0(D)}(x)\; e_{\alpha_1(D)}[X\tfrac{1-q^m}{1-q}] \cdots e_{\alpha_{n-1}(D)}[X\tfrac{1-q^m}{1-q}]. \qquad 4.21$$

But then summing over all $D \in \mathcal{D}_n$ yields the equality in 4.20 as desired.

An immediate corollary of this result is a recursion expressing $\phi_n^{(m)}(q)$ in terms of the coefficients of $f(z)$. This may be stated as follows.

Proposition 4.6

$$\begin{aligned} \phi_n^{(m)}(q) &= \textstyle\sum_{D\in\mathcal{D}_n} q^{m\, a(D)}\; e_{\alpha_0(D)}(x)\; e_{\alpha_1(D)}[X\frac{1-q^m}{1-q}] \cdots e_{\alpha_{n-1}(D)}[X\frac{1-q^m}{1-q}] \\ &= \textstyle\sum_{k=1}^{n} q^{m\binom{k}{2}}\; e_k(x)\; \sum_{n_1+\cdots+n_k=n-k} \prod_{i=1}^{k} q^{m(k-i)n_i}\; H_{n_i}(q), \end{aligned} \qquad 4.22$$

Proof

This is yet another consequence of the path factorization. To avoid conflict of symbols, we shall replace m_i by n_i in both 1.30 and 1.31, and otherwise use the same conventions we made in Section 1. Thus we write

$$D = V_k + \sum_{i=1}^{k} (E_i + D_{n_i}) \quad (D_{n_i} \in \mathcal{D}_{n_i}), \qquad 4.23$$

and

$$a(D) = \binom{k}{2} + \sum_{i=1}^{k} (\; a(D_{n_i}) + (k-i)n_i\;) \qquad 4.24$$

The idea is to collect together and sum all terms corresponding to paths D which factorize as in 4.23 for a fixed choice of k and $n_1, n_2, \ldots, n_k$. We see that in each of these terms the first vertical segment of a path $D \in \mathcal{D}_n$ (that is V_k in 4.23) contributes the factor $e_k(x)$, and each of the other vertical segments of D contributes a factor of the form $e_m[X\frac{1-q^m}{1-q}]$. Now we easily see from 4.17 that the sum over $D_{n_i} \in \mathcal{D}_{n_i}$ of $q^{m\, a(D_{n_i})}$ times all the factors contributed by the vertical segments of D_{n_i} must necessarily condense into the coefficient $H_{n_i}(q)$. Taking into account that 4.24 gives

$$q^{m\, a(D)} = q^{m\binom{k}{2}} \prod_{i=1}^{k} q^{m(k-i)n_i + m\, a(D_{n_i})}$$

we see that the sum of all the D-terms which correspond to a fixed choice of k and $n_1, n_2, \ldots, n_k$ will produce the monomial

$$e_k(x)\; q^{m\binom{k}{2}} \prod_{i=1}^{k} q^{m(k-i)n_i} H_{n_1}(q) H_{n_2}(q) \cdots H_{n_k}(q),$$

and 4.22 necessarily follows by summing over all possible choices of k and $n_1, n_2, \ldots, n_k$.

This completes our collection of auxiliary identities and we can proceed to the

Proof of Theorem 4.3

We are left to show that also $k_n^{(m)}(q)$ is equal to the righthand side of 4.22. Our starting point is the inversion problem in 4.11. Multiplying both sides of 4.11 by the denominator of $F(z)$ in 4.6 and cancelling out z we get

$$\sum_{k\geq 1} g_k\, F(z/q^m)\cdots F(z/q^{(k-1)m}) = E(z/q^{m-1}).$$

Making the replacement $z \to zq^m$ and changing the summation variable, we may rewrite this as

$$\sum_{k\geq 0} g_{k+1}\, F(z)F(z/q^m)\cdots F(z/q^{(k-1)m}) = E(zq). \qquad 4.25$$

Using the basic inversion result of Proposition 1.1 (with q^m replacing q) and the definition 4.7 of $f(z)$ we can convert 4.25 into the identity

$$\sum_{k\geq 0} g_{k+1}\, z^k = \sum_{k\geq 0} e_k(x)\, q^k\, f(z)f(zq^m)\cdots f(zq^{(k-1)m}).$$

Rewriting $g(z)$ and $f(z)$ by means of 4.10 and 4.15 we are led to

$$K^{(m)}(zq) = \sum_{k\geq 0} e_k(x)\, q^k\, z^k\, q^{m\binom{k}{2}}\, H(zq)H(zq^{1+m})\cdots H(zq^{1+(k-1)m}).$$

The substitution $z \to z/q$ simplifies this to

$$K^{(m)}(z) = \sum_{k\geq 0} e_k(x)\, z^k\, q^{m\binom{k}{2}}\, H(z)H(zq^m)\cdots H(zq^{(k-1)m}),$$

from which we immediately derive (equating coefficients of z^n) that $k_n^{(m)}(q)$ is indeed equal to the right hand side of 4.22 as desired.

As a corollary of Theorem 4.3 we obtain the following combinatorial interpretation for the specialization of $C_n^{(m)}(q,t)$ at $t = 1$.

Theorem 4.4

$$C_n^{(m)}(q,1) = \sum_{\pi\in\mathcal{D}_n^{(m)}} q^{a(\pi)}. \qquad 4.26$$

Proof

Using 2.17 we see from the definitions 3.26 and 3.27 that $C_n^{(m)}(q,t)$ is equal to the coefficient of $e_n(x)$ in the Schur function expansion of $DH_n^{(m)}(x;q,t)$. Thus, 4.26 is obtained by equating coefficients of $e_n(x)$ in both sides of 4.5.

Tables of $C_n^{(m)}(q,1)$ may be easily computed through a recurrence which extends I.5. This recurrence is best expressed in terms of the generating function

$$\Phi^{(m)}(z) = 1 + \sum_{n\geq 1} z^n\, C_n^{(m)}(q,1), \qquad 4.27$$

and it may be stated as follows.

Theorem 4.5

$$\Phi^{(m)}(z) = 1 + z\, \Phi^{(m)}(z)\Phi^{(m)}(zq)\cdots \Phi^{(m)}(zq^m). \tag{4.28}$$

Proof

We recall that a *k-ary tree* is a planar tree all of whose nodes have either 0 or exactly k children. We shall denote the collection of k-ary trees with n internal nodes by $\mathcal{T}_n^{(k)}$. We see that $\mathcal{T}_n^{(2)}$ is simply the collection of customary *binary trees*. Thus the cardinality of $\mathcal{T}_n^{(2)}$ is given by the Catalan number. More generally we have

$$\mid \mathcal{T}_n^{(m+1)} \mid = \frac{1}{1+mn}\binom{mn+n}{n} \tag{4.29}$$

It develops that 4.26 is a q-analogue of 4.29. In fact, the summation in 4.26 can also be interpreted as a q-counting of $\mathcal{T}_n^{(m+1)}$ according to a suitable *area* statistic. This can be seen as follows. Given a tree $T \in \mathcal{T}_n^{(k)}$, let us label each of its leaves by a and all the other nodes by b. Reading these labels in the depth-first traversal of T yields a word in the alphabet $\{a, b\}$ which we shall refer to as *the word of* T and denote by $w(T)$. There is a standard way to visualize words constructed in this manner from a planar tree. We simply associate to T a lattice path $\pi(T)$ whose steps are governed by the letters of $w(T)$. The idea is to replace each b by a raising step given by the vector $(1, k-1)$ and each a by a down step given by $(1, -1)$. Then as we read one by one (from left to right) the letters of $w(T)$ we progressively construct all the edges of $\pi(T)$. It is well known [18] and easy to show that the resulting lattice path will remain above the x-axis, for all but the last step (which necessarily corresponds to the last leaf of T in the depth-first order). This condition is necessary and sufficient for a lattice path π with steps given by $(1, k-1)$ and $(1, -1)$ to be the path of a tree $T \in \mathcal{T}_n^{(k)}$. The case of interest here is when $k = m + 1$. In fact, from the remarks above we can see that there is a natural bijection between $\mathcal{T}_n^{(m+1)}$ and $\mathcal{D}_n^{(m)}$. Given a tree $T \in \mathcal{T}_n^{(m+1)}$ we simply replace each $(1, m)$-step of $\pi(T)$ by a NORTH step and each $(1, -1)$ step (except the last) by an EAST step. Clearly, this results in a path $\pi \in \mathcal{D}_n^{(m)}$. Moreover, it is not difficult to show that the area $a(\pi)$, as defined to give 4.26, counts also the number of lattice points (i, j) $(j \geq 0)$ strictly below those vertices of $\pi(T)$ that are starting points of $(1, m)$-steps. Denoting the latter number by $\mathcal{A}(T)$ we can thus rewrite 4.26 in the form

$$C_n^{(m)}(q, 1) = \sum_{T \in \mathcal{T}_n^{(m+1)}} q^{\mathcal{A}(T)}. \tag{4.30}$$

In particular, if we let $\mathcal{T}^{(m+1)}$ denote the collection of all $(m+1)$-ary trees, we must also have

$$\Phi^{(m)}(z) = \sum_{T \in \mathcal{T}^{(m+1)}} z^{n(T)}\, q^{\mathcal{A}(T)}, \tag{4.31}$$

where $n(T)$ denotes the number of internal nodes of T. The latter of course is also equal to the number of $(1, m)$-steps in $\pi(T)$.

The functional equation in 4.28 follows from a factorization of the paths $\pi(T)$ which is best explained in terms of the corresponding factorization of the words $w(T)$. Note that unless a $T \in \mathcal{T}^{(m+1)}$ consists of a single leaf, its root will have $m+1$ children and will necessarily be labelled by a b. Since the letters of T are derived from a depth-first order reading, we have, by definition,

$$w(T) = b\, w(T_1)w(T_2)\cdots w(T_{m+1}). \tag{4.32}$$

Here $w(T_i)$ denotes the word of the i^{th} subtree of T, the latter being the tree rooted at the i^{th} child of the root of T. Now when we convert $w(T)$ into $\pi(T)$ the path $\pi(T_1)$ will be attached at the tip of the vector $(1,m)$. As a result the contribution to $\mathcal{A}(T)$ coming from the letters of $w(T_1)$ is $\mathcal{A}(T_1)+m\; n(T_1)$. Similarly, it is not difficult to see that the letters of $w(T_i)$ will contribute $\mathcal{A}(T_i)+(m+1-i)n(T_i)$ to $\mathcal{A}(T)$. To summarize, 4.32 yields that

$$\mathcal{A}(T) = \sum_{i=1}^{m+1} \left(\mathcal{A}(T_i)+(m+1-i)n(T_i)\right). \tag{4.33}$$

Since

$$n(T) = 1+\sum_{i=1}^{m+1} n(T_i),$$

we deduce from 4.32 and 4.33 that

$$z^{n(T)}\, q^{\mathcal{A}(T)} = z \prod_{i=1}^{m+1} q^{\mathcal{A}(T_i)} \left(z\, q^{m+1-i}\right)^{n(T_i)},$$

and the functional equation in 4.28 is thus obtained by summing over all possible choices of $T_1,\, T_2, \ldots,\, T_{m+1}$.

This combinatorial fact brings us to the central problem concerning $C_n^{(m)}(q,t)$ We see that upon the validity of the conjecture 3.30, there must be for each n and m two statistics, $\alpha(T)$, $\beta(T)$ on trees in $\mathcal{T}_n^{(m+1)}$, both having the same distribution as $\mathcal{A}(T)$, and such that

$$C_n^{(m)}(q,t) = \sum_{T\in\mathcal{T}_n^{(m+1)}} q^{\alpha(T)}\, t^{\beta(T)}. \tag{4.34}$$

Moreover we see from 4.3 that these statistics must also yield the identity

$$\sum_{T\in\mathcal{T}_n^{(m+1)}} q^{\alpha(T)\,+\,m\binom{n}{2}\,-\,\beta(T)} = \frac{1}{[mn+1]_q} \begin{bmatrix} mn+n \\ n \end{bmatrix}_q .$$

However, we suspect that the problem of constructing these two statistics might be of an order of difficulty comparable to the construction of the *charge* statistic in the work [17] of Lascoux and Schützenberger. Thus the solution of this problem might require deeper combinatorial methods than are being used in most of the bijective combinatorics of present day literature. In particular we view our q,t-Catalan as a considerably more complex construct than any of the multivariate Catalan polynomials that have been studied so far (see, *e.g.*, [4]).

As a final remark we should point out that the specialization of $DH_n(x;q,t)$ given in 2.27 may be written in the rather suggestive form

$$DH_n(x;q,1/q)\ q^{\binom{n}{2}} = \frac{1}{[n+1]_q} E(z)E(zq)\cdots E(zq^n)|_{z^n} \tag{4.35}$$

where, as in 1.16, we set

$$E(z) = \sum_{m\geq 0} e_n(x)\ z^n.$$

This makes $DH_n(x;q,1/q)\ q^{\binom{n}{2}}$ appear as yet another q-analogue of the coefficient of z^{n+1} in the solution $f(z) = \sum_{n\geq 1} f_n\ z^n$ of the Lagrange inversion problem

$$\frac{f(z)}{E(f(z))} = z.$$

Indeed, one form of the classical Lagrange formula would give

$$f_{n+1} = \frac{1}{n+1}\ E(z)^{n+1}\ |_{z^n}.$$

Now, 1.20 and 2.33 gave us that the specialization $DH_n(x;q,1) = k_n(q)$ is none other than $1/q^n$ times the coefficient of z^{n+1} in the solution $f(z)$ of the q-Lagrange inversion problem 1.3. This suggests that the symmetric function $DH_n(x;q,t)$ may also appear as a coefficient in the solution of some q,t-analogue of Lagrange inversion. Given the bivariate symmetry exhibited by $DH_n(x;q,t)$ this q,t-analogue should turn out to be quite remarkable. We should point out that that none of the q-analogues that have been given in the literature so far (see the references in [24]) lead to a formula such as in 4.35. Thus the construction of an inversion problem yielding $DH_n(x;q,t)$ should lead to new avenues in Lagrange inversion.

Appendix: Tables of $C_n(q,t)$.

Displayed below are the polynomials $C_n(q,t)$ for $n = 2$ through $n = 6$. For convenience, we have arranged the coefficients of each polynomial into an array: the coefficient of $q^h t^k$ appears in in position (h,k), indexed from $(0,0)$ at the lower left. For example,

1			
	1		
	1	1	
			1

represents the polynomial $q^3 + q^2 t + qt + t^2 + t^3$.

$C_2(q,t)$

1	
	1

$C_3(q,t)$

1			
	1		
	1	1	
			1

$C_4(q,t)$

1						
	1					
	1	1				
	1	1	1			
		1	1	1		
			1	1	1	
						1

$C_5(q,t)$

1										
	1									
	1	1								
	1	1	1							
	1	2	1	1						
		1	2	1	1					
		1	2	2	1	1				
			1	2	2	1	1			
				1	1	2	1	1		
						1	1	1	1	
										1

$$
C_6(q,t) \qquad
\begin{array}{cccccccccccccccc}
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
 & 1 & 1 & & & & & & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & & & & & & \\
 & 1 & 2 & 1 & 1 & & & & & & & & & & & \\
 & 1 & 2 & 2 & 1 & 1 & & & & & & & & & & \\
 & & 2 & 3 & 2 & 1 & 1 & & & & & & & & & \\
 & & 1 & 3 & 3 & 2 & 1 & 1 & & & & & & & & \\
 & & 1 & 2 & 4 & 3 & 2 & 1 & 1 & & & & & & & \\
 & & & 2 & 3 & 4 & 3 & 2 & 1 & 1 & & & & & & \\
 & & & & 2 & 3 & 4 & 3 & 2 & 1 & 1 & & & & & \\
 & & & & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 1 & & & & \\
 & & & & & & 2 & 2 & 3 & 3 & 2 & 1 & 1 & & & \\
 & & & & & & & 1 & 1 & 2 & 2 & 2 & 1 & 1 & & \\
 & & & & & & & & & & 1 & 1 & 1 & 1 & 1 & \\
 & & & & & & & & & & & & & & & 1
\end{array}
$$

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