# GEOMETRY OF $q$ AND $q, t$-ANALOGS IN COMBINATORIAL ENUMERATION

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## Introduction

The aim of these lectures was to give an overview of some combinatorial, symmetric-function theoretic, and representation-theoretic developments during the last several years in the theory of Hall-Littlewood and Macdonald polynomials. The motivating problem for all these developments was Macdonald's 1988 positivity conjecture [20, 21]. The positivity conjecture asserts that certain polynomials have non-negative integer coefficients, and so it naturally raised the question of how to understand Macdonald polynomials combinatorially. This question remains open, even after the proof of the positivity conjecture in [16], using methods from algebraic geometry. The latest developments, which will be discussed at the end of these notes, for the first time promise progress on the combinatorial side of the problem.

The lectures start with basics and proceed towards a discussion of the most recent combinatorial advances. Along the way, I have taken as my central topic the $q$ and $q, t$-analogs of classical combinatorial themes such as Catalan numbers, enumeration of trees and parking functions, and Lagrange inversion. The surprising connection between these themes and the theory of Macdonald polynomials was one of the most beautiful discoveries to emerge from work on the positivity conjecture. This topic also serves nicely to motivate the combinatorial conjectures discussed in the final lecture.

The subject as a whole has grown far beyond what can be covered in a series of introductory lectures. Omitted entirely are the algebraic geometrical aspects [2, 14, 16, 17]. Also omitted is a treatment of the full list of other quantities, not quite so immediately connected with classical enumeration, which are also expressed by formulas involving Macdonald polynomials, and are known or conjectured to be Schur-positive, for which combinatorial interpretations are still sought [1]. Yet another direction not touched on here is the link with representation theory of Cherednik algebras and their degenerations [4, 5, 10, 11]. A more advanced but less up-to-date survey of some of these topics can be found in [18].

My heartfelt thanks go to Alexander Woo, who conducted discussion and exercise sessions associated with the lectures and did most of the work in preparing these notes. In the process he greatly improved the exposition, worked out missing details, and took pains to

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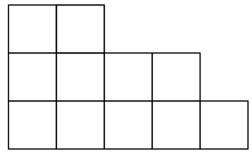

FIGURE 1. Young diagram for $\lambda = (542)$

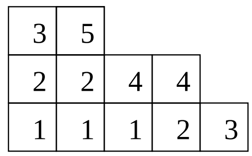


FIGURE 2. SSYT of shape $\lambda = (542)$ and content $\mu = (33221)$

clarify those points which proved most troublesome for students in the discussion sections. Credit for whatever good qualities the following notes may possess is mostly due to him.

–M.H.

## 1. Kostka numbers and $q$-analogs

1.1. **Definition of Kostka numbers.** Let $n$ be a nonnegative integer. A **partition** of $n$ is a non-decreasing sequence of nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. The number $n$ is known as the **size** of $\lambda$ and denoted $|\lambda|$. Assuming we have written $\lambda$ so that $\lambda_l \neq 0$, the number $l$ is the **length** of $\lambda$ and denoted $l(\lambda)$.

We can associate to any partition a pictorial representation called the **Young diagram**, or sometimes the **Ferrers diagram**. It consists of boxes $(i, j)$ in the first quadrant such that $j < \lambda_{i+1}$. For example, the Young diagram for the partition $\lambda = (5, 4, 2)$ is in Figure 1. Note the standard convention in the literature, which we follow, is that boxes are labelled (row, column) as in upside-down matrix coordinates.

To keep notation simple, we will frequently write $\lambda$ to indicate its diagram when there is no possibility of confusion.

A **semistandard Young Tableaux** (abbreviated SSYT) of **shape** $\lambda$ is a filling of the boxes of the diagram of $\lambda$ by positive integers, that is, a function $T : \text{diagram}(\lambda) \to \mathbb{Z}_{>0}$, such that rows are non-decreasing as one moves to the right, and columns are strictly increasing as one moves up. For example, Figure 2 is a SSYT of shape $(5, 4, 2)$.

The **content** $\mu$ of a tableau is the sequence $\mu_1, \ldots, \mu_k$ with $\mu_i = \#(T^{-1}(i))$. It is obviously a composition of $n$ (that is, a sequence $\mu_1, \ldots, \mu_k$ of nonnegative integers such that their sum is $n$). The SSYT in Figure 2 has content $\mu = (3, 3, 2, 2, 1)$.

The **Kostka number** $K_{\lambda\mu}$ is then the number of SSYT of shape $\lambda$ and content $\mu$. Kostka numbers (and, by extension, Young tableaux) have significance in the theory of symmetric functions, and in the representation theory of $S_n$ and of $\mathrm{GL}_n$. We will visit these interpretations of $K_{\lambda\mu}$ in order.

1.2. $K_{\lambda\mu}$ **in symmetric functions.** We have the following lemma whose proof consists of finding a simple bijection and is left as an exercise:

**Lemma 1.** *$K_{\lambda\mu}$ is independent of the order of the parts of $\mu$.*

This states that, for example, $K_{(41),(221)} = K_{(41),(212)} = K_{(41),(122)}$. Therefore, we can, and by convention usually will, consider $K_{\lambda\mu}$ only in the case where $\mu$ is a partition.

To an SSYT $T$, we can associate the monomial $x^T := \prod_{c\in\lambda} x_{T(c)}$ in $\mathbb{Z}[x_1, x_2, \ldots]$ or $\mathbb{C}[x_1, x_2, \ldots]$. In this product we have written $c \in \lambda$ to indicate that $c$ is a cell in the diagram of $\lambda$. Now we can associate to each partition $\lambda$ the **Schur function**

$$s_\lambda = \sum_{\mathrm{SSYT}(\lambda)} x^T.$$

By our definition, this is a "polynomial" in infinitely many variables, and, by Lemma 1, it is symmetric. The Schur functions form a basis for the ring of symmetric functions. Although they may seem unmotivated at first, in light of what follows, the Schur functions should probably be considered the most natural basis for the ring of symmetric functions.

Perhaps the most obvious basis for the ring of symmetric functions consists of the **monomial symmetric functions**, defined by

$$m_\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k} + \text{all symmetric terms}.$$

By our definition of $s_\lambda$, we have

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu,$$

where the sum is taken over all partitions $\mu$, or equivalently all partitions $\mu$ of size $|\lambda|$.

1.3. $S_n$ **representations.** Let $V$ be a finite dimensional $S_n$ representation, that is, a finite dimensional $\mathbb{C}$-vector space with a linear action by $S_n$. For any partition $\mu$ of $n$, there is the **Young subgroup** $S_\mu = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_k} \subseteq S_n$, where the $S_{\mu_1}$ factor permutes the first $\mu_1$ letters, the $S_{\mu_2}$ factor permutes the $\mu_1 + 1$-th through $\mu_1 + \mu_2$-th letters, and so on. Now let $V^{S_\mu}$ denote the subspace of $V$ fixed by every element of $S_\mu$. Then there is a symmetric function associated to $V$, called the **Frobenius characteristic** of $V$, defined by

$$F_V(\mathbf{x}) = \sum_{|\mu|=n} (\dim V^{S_\mu}) m_\mu(\mathbf{x}).$$

(This is not quite the usual definition of $F_V$, as for example in [21, 22], but it can easily be seen to be equivalent to the usual one.)

A representation is said to be **irreducible** if it has no proper sub-representations. By a classical theorem of Maschke, any representation of a finite group (over $\mathbb{C}$) splits as the direct sum of irreducible representations. Therefore, it suffices to study the irreducible representations. For $S_n$, we have the following theorem of Frobenius.

**Theorem 1.** *The irreducible representations $V_\lambda$ of $S_n$ are determined up to isomorphism by their Frobenius characteristics, and $F_{V_\lambda}(\mathbf{x}) = s_\lambda(\mathbf{x})$.*

Note that Frobenius characteristic is additive on direct sums, so this theorem essentially describes the representation theory (over $\mathbb{C}$) of $S_n$ completely.

As examples, we have the two one-dimensional irreducible representations of $S_n$.

**Example 1.** *Let $V = \mathbb{C}$, with $S_n$ acting trivially. (In other words, every element of $S_n$ acts as the identity.) Then $\dim V^{S_\mu} = 1$ for every $S_\mu$, so $F_V(\mathbf{x}) = \sum_{|\mu|=n} m_\mu(\mathbf{x})$. This representation is clearly irreducible, so $F_V(\mathbf{x}) = s_\lambda(\mathbf{x})$ for some partition of $n$. Since the partition $(n)$ has the property that, for any $\mu$ of size $n$, there is exactly one SSYT of shape $(n)$ and content $\mu$, we have $F_V(\mathbf{x}) = s_{(n)}(\mathbf{x})$.*

**Example 2.** *Now let $V = \mathbb{C}$, but with $S_n$ acting by sign. That is, let $w \in S_n$ act by the identity if $w$ is an even permutation but by $-1$ if $w$ is odd. Except for $\mu = (1, 1, \ldots, 1) = (1^n)$, every $S_\mu$ has an odd permutation, so $\dim V^{S_\mu} = 0$ except when $\mu = (1^n)$. Therefore, $F_V(\mathbf{x}) = m_{(1^n)}(\mathbf{x})$. The unique partition $\lambda$ which admits only one symmetry class of SSYT's of the given shape is $\lambda = (1^n)$, so $F_V(\mathbf{x}) = s_{(1^n)}(\mathbf{x})$.*

The symmetric functions associated with these examples have special importance and therefore have their own names. The symmetric function $s_{(n)}$ is known as the **complete homogeneous symmetric function** (of degree $n$) and is denoted $h_n$. The symmetric function $s_{(1^n)}$ is known as the **elementary symmetric function** and is denoted $e_n$.

There is another interpretation of $K_{\lambda\mu}$ in terms of the representation theory of $S_n$, which we will only sketch briefly. Let $W_\mu$ be the set of words of content $\mu$, that is, words with $\mu_1$ 1's, $\mu_2$ 2's, and so on, and let $S_n$ act on these words by permuting the positions of their letters. Extending by linearity gives an $S_n$ representation on $\mathbb{C} \cdot W_\mu$. Then we have that

**Proposition 1.**

$$F_{\mathbb{C}\cdot W_\mu}(\mathbf{x}) = \sum_\lambda K_{\lambda\mu} s_\lambda(\mathbf{x}),$$

*or, equivalently,*

$$W_\mu \cong \bigoplus_\lambda V_\lambda^{\oplus K_{\lambda\mu}}.$$

This can be proven by identifying $\mathbb{C} \cdot W_\mu$ with the induced representation $\mathbb{C} \uparrow_{S_\mu}^{S_n}$ and using Frobenius reciprocity.

1.4. $\mathrm{GL}_n$ **representations.** Now we consider finite dimensional $\mathrm{GL}_n$ representations. Here we restrict ourselves to **rational** representations; that is, a representation $V$ determines a map $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$, and we require that we can find polynomials $f_{ij}$ (in $n^2$ variables) such that for a matrix $g$, $\rho(g)$ is the matrix $1/\det(g)^N \left[f_{ij}(g_{11}, \cdots, g_{nn})\right]$ for some nonnegative integer $N$. If such polynomials exist with $N \leq 0$, then $V$ is a **polynomial** representation.

The 1-dimensional trivial representation is polynomial, with $\rho(g) = [1]$, and the $n$-dimensional defining representation is also polynomial, since $\rho(g) = g$. For a rational (resp. polynomial) representation $V$, there are naturally defined rational (resp. polynomial) representations on $V^{\otimes k}$, $\bigwedge^k V$ and $\mathrm{Sym}^k V$.

Both $\bigwedge^k$ and $\mathrm{Sym}^k$ can be considered as operations which construct new representations from existing ones. They have generalizations, one for each partition $\lambda$, called **Schur functors**, and denoted $\mathcal{S}^\lambda$. Given a representation $V$, $\mathcal{S}^\lambda V$ is defined as follows.

For any $l$, there is the natural inclusion $\bigwedge^l V \to V^{\otimes l}$ given by $v_1 \wedge \cdots \wedge v_l \mapsto \sum_{\sigma \in S_l} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(l)}$, and the natural surjection $V^{\otimes l} \to \mathrm{Sym}^l V$ given by $v_1 \otimes \cdots \otimes v_n \mapsto v_1 \cdots v_n$. Note that these maps respect the $\mathrm{GL}_n$ action, so they are maps not only of vector spaces but also of $\mathrm{GL}_n$ representations. Let $T = \bigotimes_{(i,j)\in\lambda} V^{(i,j)}$, where each $V^{(i,j)}$ is an isomorphic copy of $V$, so that $T \cong V^{\otimes|\lambda|}$.

Now we define the map $i : \bigwedge^{\lambda'_1} V \otimes \cdots \otimes \bigwedge^{\lambda'_{\lambda_1}} V \to T$ by using the natural inclusion given above to map the tensor factor $\bigwedge^{\lambda'_k}$ to $\bigotimes_{j=1}^{\lambda'_k} V^{(j,k)}$. Then define the map $\pi : T \to \mathrm{Sym}^{\lambda_1} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{l(\lambda)}} V$ by using the natural surjection to map $\bigotimes_{i=1}^{\lambda_k} V^{(k,i)}$ to $\mathrm{Sym}^{\lambda_k} V$, and let $\phi = \pi \circ i$. Finally, $\mathcal{S}^\lambda V$ is defined to be $\mathrm{im}\, \phi$. In particular, for $\lambda = (k)$, $\mathcal{S}^{(k)} = \mathrm{Sym}^k$, and for $\lambda = (1^k)$, $\mathcal{S}^{(1^k)} = \bigwedge^k$. Since, assuming $V$ is a rational (resp. polynomial) representation, $\phi$ is a map of rational (resp. polynomial) $\mathrm{GL}_n$ representations, $\mathcal{S}^\lambda V$ is also a rational (resp. polynomial) representation.

For the remainder of this section let $V$ be the $n$-dimensional defining representation, and let $V^\lambda := \mathcal{S}^\lambda V$.

**Theorem 2.** (1) *The representation $V^\lambda$ is irreducible, and every irreducible polynomial representation of $\mathrm{GL}_n$ is $V^\lambda$ for some $\lambda$.*

(2) *Let $T \subseteq \mathrm{GL}_n$ denote the subgroup of (invertible) diagonal matrices, and let $g(\mathbf{x}) := \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} \in T$. Then $\mathrm{tr}(V^\lambda, g(\mathbf{x})) = s_\lambda(\mathbf{x})$. Equivalently, there is a decomposition of $V^\lambda$, considered as a representation of $T$, into $V^\lambda = \bigoplus_\mu (V^\lambda)_\mu$, where $g(\mathbf{x})$ acts on $(V^\lambda)_\mu$ by multiplication by $\mathbf{x}^\mu$, and $\dim(V^\lambda)_\mu = K_{\lambda\mu}$.*

The most basic examples are $\lambda = (k)$, in which case $\mathrm{tr}(\mathrm{Sym}^k V, g(\mathbf{x})) = h_k(\mathbf{x})$, and $\lambda = (1^k)$ for $k \le n$, for which $\mathrm{tr}(\bigwedge^k V, g(\mathbf{x})) = e_k(\mathbf{x})$.

1.5. **The $q$-analog $K_{\lambda\mu}(q)$.** The aim of this section is to describe a $q$-analog of $K_{\lambda\mu}$ known as $\widetilde{K}_{\lambda\mu}(q)$ and make some brief remarks about its properties. Here (algebraic) geometry makes its appearance. For each partition $\mu$, we will define a variety $Y_\mu$ whose cohomology ring will have a natural action of $S_n$. Then we will define $\widetilde{K}_{\lambda\mu}(q)$ as the graded Frobenius characteristic of this cohomology ring.

Let $\mathcal{N}$ be the set of nilpotent $n \times n$ matrices. This set can be given the structure of an algebraic variety; nilpotent matrices are precisely the matrices $X$ for which $X^n = 0$, and the entries of $X^n$ are polynomials in the entries of $X$, so $\mathcal{N}$ is defined as an affine variety in $\mathbb{C}^{n\times n}$ by the vanishing of these polynomials. The variety $\mathcal{N}$ is singular, so we would like to understand it better by studying a smooth variety similar to it. More precisely, we would like a **resolution of singularities** for $\mathcal{N}$, that is, a variety $\mathcal{Z}$ along with a map $\pi : \mathcal{Z} \to \mathcal{N}$, with the properties that $\mathcal{Z}$ is smooth, and $\pi$ is both projective and birational. (Birational means that $\pi$ is an isomorphism on an open dense set, and projective means that $\pi$ can be factored as some inclusion $i : \mathcal{Z} \to \mathcal{N} \times \mathbf{P}^k$ (for some $k$) followed by the usual projection $\mathcal{N} \times \mathbf{P}^k \to \mathcal{N}$.)

To construct $\mathcal{Z}$, we need the **flag variety**. A **flag** is a sequence of vector subspaces of $\mathbb{C}^n$ denoted $F_\bullet = 0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{n-1} \subseteq \mathbb{C}^n$, satisfying $\dim F_i = i$. The flag variety contains, as a set, all flags in $\mathbb{C}^n$; as a variety or manifold it is the quotient $G/B$ where $G = \mathrm{GL}_n$ and $B$ consists of the upper triangular matrices. Now we can let

$$\mathcal{Z} = \{(X, F_\bullet) \in \mathcal{N} \times G/B : XF_i \subseteq F_{i-1} \text{ for all } i\},$$

with the map $\pi$ being the projection onto the first factor.

Now we show that $\mathcal{Z}$ is smooth. Let $\psi : \mathcal{Z} \to G/B$ be the projection onto the second factor, and let $E_\bullet$ be the standard flag, that is, the flag with $E_i = \mathbb{C} \cdot \{e_1, \ldots, e_i\}$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$. The fiber $\psi^{-1}(E_\bullet)$ is given by $\psi^{-1}(E_\bullet) = \{(X, E_\bullet) : X \text{ is upper triangular}\}$, so $\psi^{-1}(E_\bullet)$ is a $\binom{n}{2}$-dimensional vector space. Moreover, for any flag $F_\bullet$, $F_\bullet = gE_\bullet$ for some $g \in G$, and $\psi^{-1}(F_\bullet) = \{(gXg^{-1}, F_\bullet) : X \text{ is upper triangular}\}$, also a vector space of dimension $\binom{n}{2}$. This makes $\mathcal{Z}$ into a vector bundle over $G/B$; since $G/B$ is smooth, $\mathcal{Z}$ must also be smooth. (Technically we also need to check that $\mathcal{Z}$ is locally trivial over $G/B$, but this is also easy to check using the group action.)

The map $\pi$ is projective because $G/B$ is a projective variety. Also, for any $X$ whose Jordan form has only one Jordan block, $\pi^{-1}(X)$ consists of a single flag, so, as these matrices $X$ form an open dense subset of $\mathcal{N}$, $\pi$ is birational.

Now let $G$ act on $\mathcal{N}$ by conjugation; that is, $g \cdot X := gXg^{-1}$ for $g \in G$ and $X \in \mathcal{N}$. Let $\mu$ be a partition. Let $M_\mu$ be the nilpotent Jordan matrix with Jordan blocks of size $\mu_1, \mu_2, \cdots, \mu_k$, and $O_\mu = \mathrm{GL}_n \cdot M_\mu$. These orbits cover all of $\mathcal{N}$, since every matrix has a Jordan form and we can conjugate by permutation matrices to rearrange the Jordan blocks so that their sizes are in non-increasing order. We have a corresponding action on $\mathcal{Z}$ by $g \cdot (X, F_\bullet) := (gXg^{-1}, gF_\bullet)$, so the fibers of $\pi$ over points in the same $G$ orbit are isomorphic. Let $Y_\mu = \pi^{-1}(P)$ for some point $P \in O_\mu$. (We will only be interested in isomorphism invariants of $Y_\mu$, so the choice of point is irrelevant.)

For example, for $\mu = (n)$, $Y_{(n)}$ is a single point, as already stated above. At the other extreme, when $X$ is the zero matrix, $(X, F_\bullet) \in \mathcal{Z}$ for every $F_\bullet$, so for $\mu = (1^n)$, $Y_{(1^n)} \cong G/B$.

The following theorem allows what we will consider the definition of $K_{\lambda\mu}(q)$.

**Theorem 3.** (1) *The natural map $H^*(G/B, \mathbb{C}) \to H^*(Y_\mu, \mathbb{C})$ is surjective.*
(2) *There are geometrically defined $S_n$ actions on $H^*(G/B, \mathbb{C})$ and $H^*(Y_\mu, \mathbb{C})$ such that the above map is $S_n$-equivariant.*
(3) *$H^*(Y_\mu, \mathbb{C}) \cong \mathbb{C} \cdot W_\mu \cong \bigoplus_\lambda V_\lambda^{\oplus K_{\lambda\mu}}$ as $S_n$-representations.*

Now we define $\widetilde{K}_{\lambda\mu}(q)$ by

$$\widetilde{K}_{\lambda\mu}(q) = \sum_i K_{\lambda\mu}^{(i)} q^i,$$

where $K^{(i)}_{\lambda\mu}$ is defined by

$$H^{2i}(Y_\mu, \mathbb{C}) \cong_{S_n} \bigoplus V_\lambda^{\oplus K^{(i)}_{\lambda\mu}}.$$

(The original definition of $K_{\lambda\mu}(q)$ is related to $\widetilde{K}_{\lambda\mu}(q)$ by $K_{\lambda\mu}(q) = q^N \widetilde{K}_{\lambda\mu}(q^{-1})$, where $N$ is a positive integer depending on $\mu$. The $\widetilde{K}_{\lambda\mu}(q)$ appear to be somewhat more natural so we will be using this form throughout the lectures.)

From the definition and part 3 of the theorem, it is clear that $\widetilde{K}_{\lambda\mu}(1) = K_{\lambda\mu}$, and that $\widetilde{K}_{\lambda\mu}(q)$ is a polynomial with positive integer coefficients, but it is not so clear how to compute $\widetilde{K}_{\lambda\mu}(q)$. We will see later a formula of Shoji and Lusztig for $\widetilde{K}_{\lambda\mu}(q)$, but it will be a rational expression from which it is not obvious that $\widetilde{K}_{\lambda\mu}$ is a polynomial, much less one with positive coefficients.

However, there is a combinatorial definition due to Lascoux and Schützenberger which gives $\widetilde{K}_{\lambda\mu}(q) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} q^{\mathrm{cc}(T)}$, where the **co-charge** $\mathrm{cc}(T)$ is a certain rather subtle combinatorial statistic on tableaux. Somewhat unsatisfactorily, the proofs that the two definitions are equivalent rely on showing that they both satisfy initial conditions and recurrence relations which are sufficient to determine $\widetilde{K}_{\lambda\mu}(q)$. A better proof would explain this equivalence by explicitly finding a basis of $H^*(Y_\mu, \mathbb{C})$ indexed by tableaux whose co-charge is equal to the cohomology degree of the basis element, with the $S_n$ action on the cohomology ring closely related to the $S_n$ action on the corresponding tableaux. However, no such conceptually satisfactory proof is yet known.

1.6. **Exercises.**

(1) Prove Lemma 1.
(2) Define $h_\mu(\mathbf{x}) := h_{\mu_1}(\mathbf{x}) h_{\mu_2}(\mathbf{x}) \cdots h_{\mu_{l(\mu)}}(\mathbf{x})$. Show that $F_{\mathbb{C}\cdot W_\mu}(\mathbf{x}) = h_\mu(\mathbf{x})$. Deduce that $h_\mu(\mathbf{x}) = \sum_\lambda K_{\lambda\mu} s_\lambda$.
(3) Find a basis and weight space decomposition of $V^\lambda$ (the $\mathrm{GL}_n$ representation) for $\lambda = (2, 1^{k-2})$.
(4) Let $V = \mathbb{C}^n = \mathbb{C} \cdot \{e_1, \ldots, e_n\}$ be the defining representation of $S_n$, that is, with the action $w \cdot e_i = e_{w(i)}$. Decompose $V$ into irreducibles and $F_V(\mathbf{x})$ into Schur functions, corresponding to your decomposition of $V$.

## 2. Catalan numbers, Trees, Lagrange Inversion, and their $q$-analogs

2.1. **Catalan numbers.** The Catalan numbers $C_n$ are known to count many different combinatorial objects, but for the sake of brevity we will only mention a small number which will be important for these lectures.

Let $w$ be a string consisting of $n$ left parentheses "(" and $n$ right parentheses ")". The string $w$ is **proper** if it makes sense as a parenthesization of something, that is, if, reading from left to right, one has encountered at every point at least as many left parentheses as right parentheses.

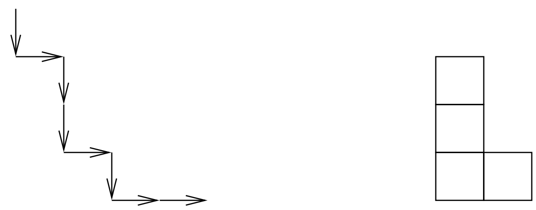

FIGURE 3. Dyck path and partition corresponding to ()((()()))

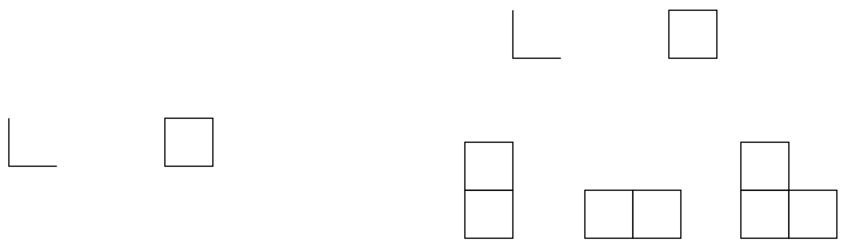

FIGURE 4. Partitions counted by $C_2$ and $C_3$

To every proper parentheses string we can associate a **Dyck path**, that is, a lattice path from $(0, n)$ to $(n, 0)$ (using Cartesian coordinates) which always remains below (or on) the line defined by $x + y = n$. We do this by starting at $(0, n)$ and, as we read a string $w$ from left to right, moving down by $(0, -1)$ every time we encounter a "(" and moving to the right by $(1, 0)$ every time we encounter a ")". By considering the Dyck path as the boundary of the diagram of a partition, the set of Dyck paths is also equivalent to the set of partitions $\lambda \subseteq \delta(n)$, where $\mu \subseteq \nu$ for partitions $\mu$ and $\nu$ means that the diagram of $\mu$ fits inside the diagram of $\nu$ (that is, $\mu_i \leq \nu_i$ for all $i$), and $\delta(n)$ is the partition $(n-1, n-2, \ldots, 1, 0)$.

For example, the above bijections associate the word "()((()()))" with the Dyck path in Figure 3 and the partition $(2, 1, 1)$.

The **Catalan numbers** $C_n$ can then be defined as the number of proper parentheses strings (of $n$ left and $n$ right parentheses), or equivalently the number of Dyck paths from $(0, n)$ to $(n, 0)$, or equivalently the number of partitions inside $\delta(n)$. We have $C_0 = C_1 = 1$, $C_2 = 2$, and $C_3 = 5$, as demonstrated by the Figure 4.

As is frequently useful in combinatorics, we can try to calculate or get a formula for $C_n$ by using a **generating function**. In this case, this means a power series $C(x)$ defined by $C(x) := \sum_n C_n x^n$.

Given a proper parentheses string, the initial "(" matches with some ")", and between those parentheses is a proper parenthesization of some length $k$, while after the specified ")" is another proper parentheses string of length $n - 1 - k$. In other words, a non-empty proper parentheses string looks like $(A)B$, where $A$ and $B$ are respectively parentheses strings of length $k$ and $n - 1 - k$. Therefore, the Catalan numbers satisfy the recurrence

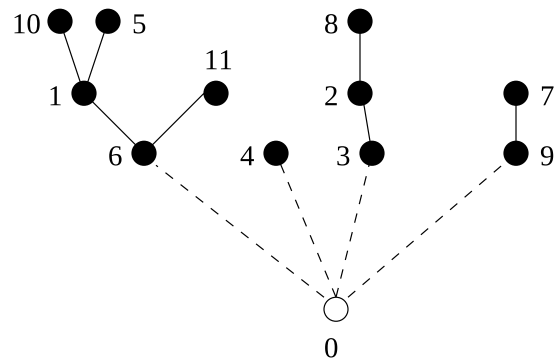


FIGURE 5. Construction of a rooted tree from a rooted forest

$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$. In terms of the generating function, we have

$$C(x) = 1 + xC(x)^2.$$

We can get a formula for $C_n$ by solving for $C(x)$ and using the binomial theorem, but we will instead get one by using Lagrange inversion later in this lecture. For now, note that our equation can be rewritten as

$$xC(x)(1 - xC(x)) = x,$$

or equivalently $F_1(x) \circ (xC(x)) = x$, where $\circ$ denotes functional composition and $F_1(x) = x(1-x)$. In other words, $F_1(x)$ and $xC(x)$ are **compositional inverses**.

2.2. **Rooted trees.** A **tree** is a connected graph with no cycles, and a **labelled tree** is a tree whose vertices are assigned distinct labels. A **rooted tree** is a tree in which one vertex is distinguished and designated as the root. Let $t_n$ be the number of labelled rooted trees with vertex set $\{1, 2, \ldots, n\}$, with $t_0 = 0$ by convention. A **rooted forest** is a graph with labelled vertices and no cycles where each connected component has a vertex designated as the root. Note that the number of rooted forests on $n$ vertices is the same as the number of unrooted trees with vertex set $\{0, \ldots, n\}$, which is $t_{n+1}/(n+1)$; this is because, as illustrated in Figure 5, for any rooted forest we can construct a tree by creating a new vertex labelled 0 and adding an edge between 0 and each root, and conversely given a tree with vertex set $\{0, \ldots, n\}$ we can construct a labelled rooted forest by removing the vertex 0 and calling each vertex formerly attached to 0 the root of its connected component.

As with Catalan numbers, we can form a generating series, but in this case it will be more convenient to form the **exponential generating series** $T(x) = \sum_n t_n x^n/n!$. This allows us to use the Exponential Formula [22, Cor. 5.1.6] to relate the generating series for the number of rooted trees and the number of rooted forests, so that, if $h_n = t_{n+1}/(n+1)$ is the number of rooted labelled forests and $H(x) = \sum_n h_n x^n/n!$, we have $H(x) = e^{T(x)}$. Therefore,

$$e^{T(x)} = \sum_n \frac{t_{n+1}}{n+1}\frac{x^n}{n!} = \sum_n t_{n+1}\frac{x^n}{(n+1)!} = \frac{T(x)}{x}.$$

Then $T(x)e^{-T(x)} = x$ so $F_2(x) = xe^{-x}$ is the compositional inverse of $T(x)$.

2.3. **The Lagrange inversion formula.** Given these examples, it would be nice to have a formula which, given a power series, calculates the coefficients of its compositional inverse. The Lagrange inversion formula exactly fulfills this need.

**Theorem 4.** *Let $E(x) = \sum_n e_n x^n$ and $K(x) = \sum_n k_n x^n$ be power series, with $e_0 = k_0 = 1$. Then*

$$\frac{x}{E(x)} \circ (xK(x)) = x$$

*if and only if*

$$k_n = \frac{1}{n+1}\,[x^n]\,E(x)^{n+1}.$$

Here and below the symbol $[x^n]$ denotes the coefficient of $x^n$ in the quantity that follows. We will later prove Theorem 4 as a special case of a $q$-Lagrange inversion theorem. A direct proof can be found, for example, in [22, Thm 5.4.2].

Let us use this theorem to calculate explicit formulas for $C_n$ and $t_n$.

To solve for $C_n$, let $E(x) = 1/(1-x)$. Then $x/E(x) = F_1(x)$, so

$$\frac{x}{E(x)} \circ (xC(x)) = x.$$

Applying the Lagrange inversion theorem,

$$C_n = \frac{1}{n-1}\,[x^n]\,\frac{1}{(1-x)^{n+1}} = \frac{1}{n-1}\left(\!\!\binom{n+1}{n}\!\!\right) = \frac{1}{n-1}\binom{2n}{n}.$$

It turns out to be slightly easier to solve for $h_n$, the number of rooted forests. If we let $E(x) = e^x$, then $x/E(x) = F_2(x)$, and

$$\frac{x}{E(x)} \circ (xH(x)) = x.$$

Once again applying Lagrange inversion,

$$\frac{h_n}{n!} = \frac{1}{n+1}\,[x^n]\,e^{(n+1)x} = \frac{1}{n+1}\frac{(n+1)^n}{n!},$$

so $h_n = (n+1)^{n-1}$, and $t_n = n^{n-1}$.

2.4. $q$**-analogs.** The Catalan numbers have two $q$-analogs, but we will only be concerned with the one originally defined by Carlitz and Riordan [3], defined by $C_n(q) = \sum_{\lambda\subseteq\delta(n)} q^{\binom{n}{2}-|\lambda|}$. This $q$-analog satisfies a recurrence as follows. We can separate all partitions $\lambda \subseteq \delta(n)$ into classes $\mathcal{C}_n^{(k)}$ for $0 \le k < n-1$ by putting $\lambda$ in $\mathcal{C}_n^{(k)}$ if $k$ is the smallest number such that $\lambda_{n-k-1} = k+1 = \delta(n)_{n-k-1}$, and $k = n-1$ if no such number exists. For example, as illustrated in Figure 6, the partition $(6,4,4,1) \subseteq \delta(7)$ belongs in $\mathcal{C}_7^{(3)}$. Now, for $\lambda \in \mathcal{C}_n^{(k)}$, let $\nu$ be the partition defined by $\nu_i = \lambda_{n-k-1+i}$, and let $\rho$ be the partition defined

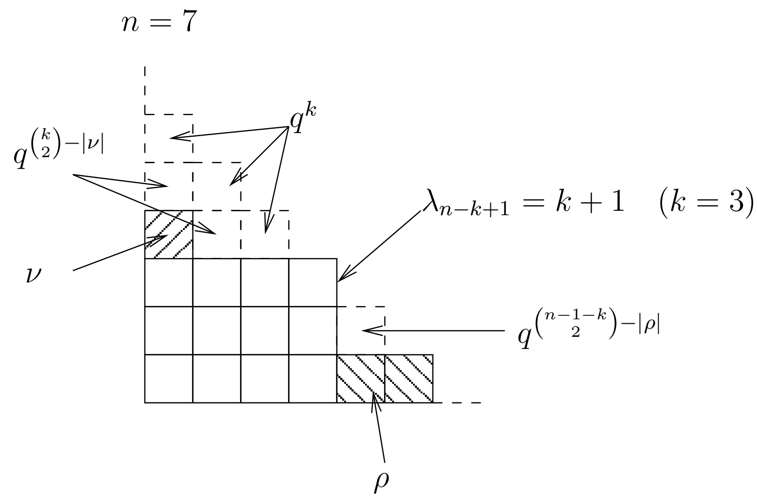


FIGURE 6. The $q$-Catalan recurrence illustrated for $\lambda = (6, 4, 4, 1)$.

by $\rho_i = \lambda_i - k - 1$, for $i$, $1 \leq i \leq n-k-1$, as illustrated in Figure 6. Notice that $\nu \subseteq \delta_k$, and $\rho \subseteq \delta_{n-1-k}$. Furthermore,

$$\binom{n}{2} - |\lambda| = (k + \binom{k}{2} - |\nu|) + (\binom{n-1-k}{2} - |\rho|),$$

so we have the recurrence

$$C_n(q) = \sum_{k=0}^{n-1} q^k C_k(q) C_{n-1-k}(q).$$

Now we turn to a $q$-analog of $h_n = (n+1)^{n-1}$. It is possible to give a statistic on rooted labelled forests that produces this $q$-analog, but it will be more convenient for us to define this $q$-analog later using parking functions. A **parking function** is a function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $\# f^{-1}(\{1, \ldots, k\}) \geq k$ for all $k \in \{1, \ldots, n\}$. (The name comes from the following description. Suppose we have $n$ parking spaces on a one way street, labelled in order, and $n$ cars. The cars arrive at the street in order, and each car $k$ immediately goes to its preferred parking space $f(k)$. If it is already filled by a previous car, then it keeps going and parks in the first empty space. The condition above is then satisfied if and only if every car successfully finds a parking space without having to enter the street a second time.) Denote the set of parking functions for $n$ cars by $\mathrm{PF}(n)$ The symmetric group $S_n$ acts on $\mathrm{PF}(n)$ by $w \cdot f = f \circ w^{-1}$ for $w \in S_n$ and $f \in \mathrm{PF}(n)$.

We can represent a parking function by a tableaux of **skew shape** $(\lambda + (1^n))/\lambda$ for some partition $\lambda$, that is a filling of the boxes in $\lambda + (1^n)$ but not in $\lambda$, strictly increasing in columns and weakly increasing in rows (although in this case there are no relevant rows) as usual. Let $\widetilde{f}$ be $f$ sorted into non-increasing order; in other words, we want $\widetilde{f} = w \cdot f$ for some $w$ such that $\widetilde{f}(i) \geq \widetilde{f}(i+1)$ for all $i$, $1 \leq i \leq n-1$. Now we specify $\lambda$ by requiring $\lambda_i = \widetilde{f}(i) - 1$.

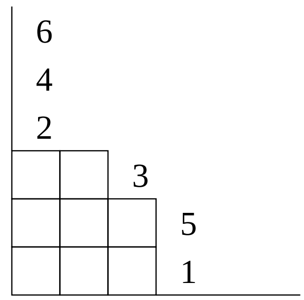


FIGURE 7. Tableau associated with the parking function $f(2) = f(4) = f(6) = 1, f(3) = 3, f(1) = f(5) = 4$

Note that the requirement that $f$ (or $\widetilde{f}$) be a parking function is equivalent to requiring that $\lambda \subseteq \delta(n)$. Now the $j$-th column in $(\lambda + (1^n))/\lambda$ will have $f^{-1}(j)$ many open boxes to fill, and we fill them with the elements of $f^{-1}(j)$ in increasing order. Figure 7 shows the tableau associated with the parking function $f(2) = f(4) = f(6) = 1, f(3) = 3, f(1) = f(5) = 4$. The content of this tableaux is always $(1^n)$.

Note that $\binom{n}{2} - |\lambda| = \sum_{i=1}^{n} i - \sum_{i=1}^{n} f(i)$, and we will denote this quantity as $\mathrm{wt}(f)$. (This quantity is sometimes known as the "frustration factor" of the parking function since it counts the sum total of how far drivers park from their preferred space.) Let $P_n(q) := \sum_{f \in \mathrm{PF}(n)} q^{\mathrm{wt}(f)}$. Counting parking functions according to the partition representing them, we get that

$$P_n(q) = \sum_{\lambda \subseteq \delta(n)} q^{\binom{n}{2} - |\lambda|} \binom{n}{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}},$$

where $\alpha_i$ is the number of parts of $\lambda$ of size $i$ (adding parts of 0 if necessary so that $\lambda$ has exactly $n$ parts; that is, $\alpha_0 = n - \sum_{i=1}^{n-1} \alpha_i$).

2.5. **$q$-Lagrange inversion.** To understand the above $q$-analogs better, we will give a $q$-analog of Lagrange inversion. Of course, for $q$-Lagrange inversion to make sense, we first have to define a $q$-analog of functional composition. The relevant definition is due to Garsia and Gessel independently [8, 9].

**Definition 1.** *Let $F(x) = \sum_n f_n x^n$ and $G(x) = \sum_n g_n x^n$ be power series with $g_0 = 0$. Then the **$q$-composition** of $F(x)$ and $G(x)$, denoted $F(x) \circ_q G(x)$, is defined to be $\sum_n f_n G(q^{n-1}x) G(q^{n-2}x) \cdots G(qx) G(x)$.*

Note that, if we substitute $q = 1$, we have $F(x) \circ G(x) = \sum_n f_n G(x)^n$, which is just ordinary composition of functions. Now we have a theorem, due to Garsia [8], which states that this setting gives a good $q$-analog of compositional inverses.

**Theorem 5.** *We have*

$$F(x) \circ_q G(x) = x \text{ if and only if } G(x) \circ_{q^{-1}} F(x) = x.$$

*Furthermore, when the above holds,*

$$(\Psi(x) \circ_q G(x)) \circ_{q^{-1}} F(x) = (\Psi(x) \circ_{q^{-1}} F(x)) \circ_q G(x) = \Psi(x) \textit{ for all } \Psi(x).$$

*Proof.* Suppose that $F(x) \circ_q G(x) = x$. We will show that for any power series $\Psi(x)$, $(\Psi(x) \circ_{q^{-1}} F(x)) \circ_q G(x) = \Psi(x)$. In other words, we will show that, if we define maps $\pi, \phi : \mathbb{C}(q)[[x]] \to \mathbb{C}(q)[[x]]$ by $\pi : \Psi \mapsto \Psi \circ_{q^{-1}} F$ and $\phi : \Psi \mapsto \Psi \circ_q G$, $\phi \circ \pi$ is the identity map. Now, two power series are equal if and only if they are equal modulo $x^N$ for every $N$, so we can view $\pi$ and $\psi$ as countable sequences of maps of finite-dimensional vector spaces. Therefore, if $\phi \circ \pi$ is the identity, $\pi \circ \phi$ is the identity, so, once we show that $(\Psi(x) \circ_{q^{-1}} F(x)) \circ_q G(x) = \Psi(x)$, we will have proven that $(\Psi(x) \circ_q G(x)) \circ_{q^{-1}} F(x) = \Psi(x)$. Then the forward direction of the first statement follows by $\Psi(x) = x$, which shows $G(x) \circ_{q^{-1}} F(x) = x$, and the reverse direction follows by substituting $q^{-1}$ for $q$.

Now we show that $(\Psi(x) \circ_{q^{-1}} F(x)) \circ_q G(x) = \Psi(x)$. First we need the following lemma whose proof is straightforward from the definition of $q$-composition and left as an exercise.

**Lemma 2.**

$$(xF(x)) \circ_q G(x) = G(x)(F(x) \circ_q G(qx)) = G(x)(F(x) \circ_q G(x))_{x \mapsto qx}.$$

Now, if $F(x) \circ_q G(x) = x$, by applying the lemma we have $(xF(x)) \circ_q G(x) = G(x)qx$, by applying the lemma again we have $(x^2 F(x)) \circ_q G(x) = G(x)G(qx)q^2 x$, and by applying the lemma repeatedly we have

$$(x^r F(x)) \circ_q G(x) = G(x)G(qx) \cdots G(q^{r-1}x)q^r x.$$

Therefore, for all power series $\Phi(x) = \sum_n \phi_n x^n$,

$$\begin{aligned}(\Phi(x)F(x)) \circ_q G(x) &= \sum_n (\phi_n x^n F(x)) \circ_q G(x) \\ &= \sum_n \phi_n x q^n G(q^{n-1}x) G(q^{n-2}x) \cdots G(qx)G(x) \\ &= (\Phi(qx) \circ_q G(x))x.\end{aligned}$$

Apply the above equation with $\Phi(x) = F(q^{-1}x)$ to get

$$(F(q^{-1}x)F(x)) \circ_q G(x) = x^2.$$

Then apply the equation with $\Phi(x) = F(q^{-2}x)F(q^{-1}x)$, and the last equation, to get

$$(F(q^{-2}x)F(q^{-1}x)F(x)) \circ_q G(x) = x^3.$$

By induction, we have

$$(F(q^{-(n-1)}x) \cdots F(q^{-1}x)F(x)) \circ_q G(x) = x^n.$$

Therefore, for any power series $\Psi(x) = \sum_n \psi_n x^n$,

$$\begin{aligned}(\Psi(x) \circ_{q^{-1}} F(x)) \circ_q G(x) &= \sum_n \psi_n (F(q^{-(n-1)}x) \cdots F(qx)F(x)) \circ_q G(x) \\ &= \sum_n \psi_n x^n \\ &= \Psi(x),\end{aligned}$$

as desired. □

For usual functional composition, it turned out that it was easier to get the explicit Lagrange inversion formula for the modified form

$$\frac{x}{E(x)} \circ xK(x) = x,$$

or equivalently,

$$K(x) = E(x) \circ xK(x),$$

was easier to solve for the coefficients. (The equivalence is obvious once one stops using the $\circ$ notation.) Similarly, for $q$-composition, it is easier to state the $q$-Lagrange inversion formula for the following forms, whose equivalence is left as a (not so trivial) exercise.

**Proposition 2.**

$$\frac{x}{E(x)} \circ_q xK(qx) = x$$

*if and only if*

$$K(x) = E(x) \circ_q xK(x).$$

Now we are ready state the $q$-Lagrange inversion formula. It will not have a simple algebraic form, but will instead be a combinatorial sum that relates to the $q$-analogs described in section 2.4.

**Theorem 6.** *Let $E(x) = \sum_n e_n x^n$ and $K(x) = \sum_n k_n(q) x^n$ be power series, with $e_0 = k_0(q) = 1$. Then*

$$\frac{x}{E(x)} \circ_q (xK(qx)) = x$$

*if and only if*

$$k_n(q) = \sum_{\lambda \in \delta(n)} q^{\binom{n}{2} - |\lambda|} e_{\alpha_0(\lambda)} e_{\alpha_1(\lambda)} \cdots e_{\alpha_{n-1}(\lambda)},$$

*where $\alpha_i(\lambda)$ is the number of parts of $\lambda$ having size $i$, and $\alpha_0 = n - \sum_{i=1}^{n-1} \alpha_i$. (For example, if $n = 4$ and $\lambda = (3, 1, 1)$, then $\alpha_1 = 2$, $\alpha_0 = \alpha_3 = 1$, and $\alpha_2 = 0$.)*

*Proof.* By Proposition 2,

$$\frac{x}{E(x)} \circ_q xK(qx) = x$$

if and only if

$$K(x) = E(x) \circ_q xK(x),$$

and, expanding the second equation, we have the recurrence

$$\begin{aligned} k_n(q) &= [x^n]\left(1+\sum_{r>0} e_r q^{r-1} xK(q^{r-1}x)\cdots qxK(qx)xK(x)\right) \\ &= [x^n]\sum_{r>0} e_r q^{\binom{r}{2}} x^r K(q^{r-1}x)\cdots K(qx)K(x) \\ &= \sum_{r>0} e_r q^{\binom{r}{2}} \left[x^{n-r}\right] K(q^{r-1}x)\cdots K(qx)K(x) \\ &= \sum_{r>0} e_r q^{\binom{r}{2}} \sum_{m_1+\cdots+m_r=n-r} \prod_i [x_i^m]\, K(q^{r-i}x) \\ &= \sum_{r>0} e_r q^{\binom{r}{2}} \sum_{m_1+\cdots+m_r=n-r} \prod_i q^{(r-i)m_i} k_{m_i}(q) \\ &= \sum_{r>0} e_r \sum_{m_1+\cdots+m_r=n-r} q^{\sum_i (m_i+1)(r-i)} \prod_i k_{m_i}(q) \end{aligned}$$

It is clear that this recurrence has a unique solution (given the initial condition $k_0(q)=1$), so we need to show that

$$k_n(q) = \sum_{\lambda\in\delta(n)} q^{\binom{n}{2}-|\lambda|} e_{\alpha_0(\lambda)} e_{\alpha_1(\lambda)}\cdots e_{\alpha_{n-1}(\lambda)}$$

satisfies this recurrence.

As with the recurrence for the $q$-Catalan numbers, we will show this recurrence holds by dividing the set of partitions $\lambda\in\delta(n)$ into classes. First put $\lambda$ into the class $\mathcal{K}^{(r)}$ where $r=n-l(\lambda)$. Now we further subdivide each class $\mathcal{K}^{(r)}$ into classes $\mathcal{K}^{(r)}_{(m_1,\ldots,m_r)}$, one for each composition $m_1,\ldots,m_r$ of $n-r$. For each $\lambda\in\mathcal{K}^{(r)}$ and each $i$ with $0\le i\le r-1$, let $n_i(\lambda)$ be the largest number less than or equal to $n-r$ such that $\lambda_{n_i} > n-n_i-r+i$. (Recall that the $n_i$-th part of $\delta(n)$ has size $n-n_i$, so $n_i$ is the highest row, not including the top $r$ rows, with fewer than $r-i$ entries in $\delta(n)-\lambda$.) Notice that, by definition, $\lambda_{n-r} > 0 = n-(n-r)-r+0$, so $n_0 = n-r$, and we set $n_r = 0$ by convention. Now let $m_i(\lambda) = n_{i-1}(\lambda)-n_i(\lambda)$, and place $\lambda$ into $\mathcal{K}^{(r)}_{(m_1,\ldots,m_r)}$ accordingly; it is clear that $m_1,\ldots,m_r$ will be a composition of $n-r$. Figure 8 shows that $(13,10,7,7,6,2,2,1)$ is in $\mathcal{K}^{(6)}_{(3,0,3,1,0,1)}$ (for $n=14$).

For each partition $\lambda$ in $\mathcal{K}^{(r)}_{(m_1,\ldots,m_r)}$, and each $i$ with $1\le i\le r$, we define partitions $\nu^{(i)}(\lambda)$ by letting $\nu^{(i)}_j(\lambda) = \lambda_{n_i(\lambda)+j} - \lambda_{n_{i-1}(\lambda)}$ for $j$ such that $1\le j\le m_i$. Now Figure 8 shows that

$$\begin{aligned} q^{\binom{n}{2}-|\lambda|} e_{\alpha_0(\lambda)} e_{\alpha_1(\lambda)}\cdots e_{\alpha_{n-1}(\lambda)} &= q^{\sum_i (m_i+1)(r-i)+\sum_i \binom{m_i}{2}-\left|\nu^{(i)}(\lambda)\right|} e_{\alpha_0(\lambda)} \prod_{i=1}^{r} \prod_{j=\lambda_{n_{i-1}}}^{\lambda_{n_i-1}} e_{\alpha_j(\lambda)} \\ &= e_r q^{\sum_i (m_i+1)(r-i)} \prod_i q^{\binom{m_i}{2}-\left|\nu^{(i)}\right|} e_{\alpha_0(\nu^{(i)})}\cdots e_{\alpha_{m_i-1}(\nu^{(i)})}. \end{aligned}$$

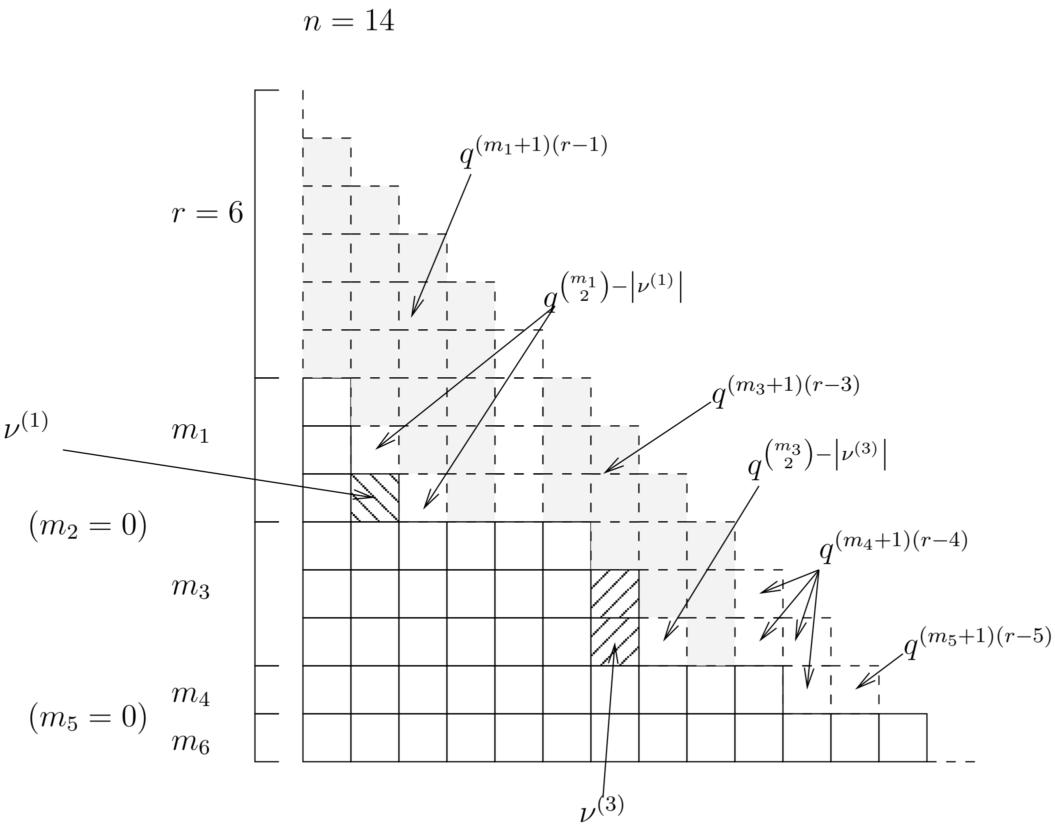


FIGURE 8. The $q$-Lagrange inversion recurrence illustrated for $\lambda = (13, 10, 7, 7, 6, 2, 2, 1)$.

Therefore,

$$\begin{aligned} k_n(q) &= \sum_{\lambda \in \delta(n)} q^{\binom{n}{2}-|\lambda|} e_{\alpha_0(\lambda)} e_{\alpha_1(\lambda)} \cdots e_{\alpha_{n-1}(\lambda)} \\ &= \sum_r e_r \sum_{m_1+\cdots+m_r=n-r} q^{\sum_i (m_i+1)(r-i)} \sum_{\lambda \in \mathcal{K}^{(r)}_{(m_1,\ldots,m_r)}} \prod_i q^{\binom{m_i}{2}-|\nu^{(i)}|} e_{\alpha_0(\nu^{(i)})} \cdots e_{\alpha_{m_i-1}(\nu^{(i)})} \\ &= \sum_{r>0} e_r \sum_{m_1+\cdots+m_r=n-r} q^{\sum_i (m_i+1)(r-i)} \prod_i k_{m_i}(q), \end{aligned}$$

which is the desired recurrence. □

We now relate $q$-Lagrange inversion to $q$-Catalan numbers and to parking functions counted by frustration factor. Let $E(x) = 1/(1-x)$; then $e_n = 1$ for all $n$. We see that, by the above theorem,

$$C(x;q) := \sum_n C_n(q)x^n = \sum_n \left( \sum_{\lambda \in \delta(n)} q^{\binom{n}{2}-|\lambda|} \right) x^n$$

is the specified solution to the $q$-Lagrange inversion problem $\frac{x}{E(x)} \circ_q xK(qx) = x$, so we have

$$x(1-x) \circ_q xC(qx;q) = x.$$

As for parking functions, let $E(x) = e^x$; then $e_n = \frac{1}{n!}$. Now,

$$n!k_n(q) = \sum_{\lambda \in \delta(n)} q^{\binom{n}{2}-|\lambda|} \binom{n}{\alpha_0, \alpha_1, \ldots, \alpha_n} = P_n(q),$$

so the exponential generating function $P(x;q) = \sum_n P_n(q)x^n/n!$ solves $xe^{-x} \circ_q xP(qx;q) = x$. In particular, this shows, by setting $q = 1$, that $P_n(1) = (n+1)^{n-1}$, or that the number of parking functions for $n$ cars is the same as the number of rooted forests on $n$ vertices.

### 2.6. **Exercises.**

(1) Prove Lemma 2.
(2) Prove Proposition 2.
(3) Use Theorem 6 to prove Theorem 4 by setting $q = 1$. (Hint: First show that, if $(\alpha_0, \ldots, \alpha_n) \in \mathbb{N}$ satisfy $\alpha_0 + \cdots + \alpha_n = n$, the sequence $(\alpha_0, \ldots, \alpha_n)$ has a unique rotation $(\beta_0, \ldots, \beta_n)$ such that there is a partition $\lambda \subseteq \delta(n)$ with $\alpha_i(\lambda) = \beta_i$ for all $i$.)
(4) Prove directly that there are $(n+1)^{n-1}$ parking functions on $\{1, \ldots, n\}$.
(5) Let $S_n$ act on $\mathrm{PF}(n)$ as previously stated, and view $\mathbb{C} \cdot \mathrm{PF}(n)$ as an $S_n$ representation graded by $\mathrm{wt}(f)$. Show that $\mathbb{C} \cdot \mathrm{PF}(n)$ is a direct sum of induced representations $\mathbb{C} \uparrow_{S_\mu}^{S_n}$ (which are respectively isomorphic to the representations $\mathbb{C} \cdot W_\mu$ introduced in Lecture 1) in which the generating function for the multiplicity of $\mathbb{C} \uparrow_{S_\mu}^{S_n}$ in the graded degrees is equal to the coefficient of $e_{\mu_1} \cdots e_{\mu_k}$ in $k_n(q)$.

## 3. Macdonald polynomials

The Macdonald polynomials are a basis for the ring of symmetric functions over the base field $\mathbb{Q}(q,t)$. This basis has a number of useful and interesting properties, but, unfortunately, the polynomials are difficult to write out explicitly; indeed we will only have space to give an abstract definition and a number of their important properties, mostly without proof. These statements will require some notation and machinery, as well as motivation, from the general theory of symmetric functions, which we will now proceed to explain in the first part of this lecture.

Throughout this lecture, $\Lambda_k$ denotes the ring of symmetric functions over the base field (or occasionally base ring) $k$.

3.1. **Symmetric function bases and the involution $\omega$.** In the first lecture, we saw two bases for the ring $\Lambda_{\mathbb{Q}}$, namely the monomial symmetric functions and the Schur functions. We now proceed to define three more bases.

The **complete homogeneous symmetric function** $h_n$ is defined by $h_n := \sum_{|u|=n} m_\mu = s_{(n)}$. In other words, $h_n$ is the sum of all the monomials of degree $n$. We can then define $h_\mu$ for all partitions $\mu$ by $h_\mu := h_{\mu_1} h_{\mu_2} \cdots h_{\mu_k}$. By Exercise 1.6(2), we have that $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$.

The **elementary symmetric function** $e_n$ is defined by $e_n := m_{(1^n)} = s_{(1^n)}$; it is the sum of all square-free monomials of degree $n$. We define $e_\mu$ similarly by $e_\mu := e_{\mu_1} \cdots e_{\mu_k}$.

Finally, the **power sum symmetric function** $p_n$ is defined by $p_n := m_{(n)} = \sum_i x_i^n$, with $p_\mu$ defined by $p_\mu := p_{\mu_1} \cdots p_{\mu_k}$.

As $\mu$ ranges over all partitions, the sets $\{h_\mu\}$, $\{e_\mu\}$, and $\{p_\mu\}$ are all bases of $\Lambda_{\mathbb{Q}}$.

Let $\omega\colon \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ be the ring homomorphism sending $e_n$ to $h_n$; since the $e_n$ are algebraically independent and generate $\Lambda_{\mathbb{Q}}$, this map $\omega$ exists and is unique. Let $\lambda'$ denote the partition **conjugate** to $\lambda$, that is, the partition whose diagram is the transpose of the diagram of $\lambda$. We will see later that $\omega$ is in fact an involution, that $\omega s_\lambda = s_{\lambda'}$, and that $\omega p_k = (-1)^k p_k$. In terms of the representation theory of $S_n$, $\omega$ corresponds to tensoring by the sign representation.

3.2. **Plethystic substitution.** Let $R$ be a ring, and designate some (possibly infinite) set $\{a_1, a_2, \ldots\}$ of elements of $R$ as **indeterminates** with the property that, for each $k \in \mathbb{Z}_{\geq 0}$, there exists a ring homomorphism $\phi_k : R \to R$ with $\phi_k(a_i) = a_i^k$. Given $A \in R$ and some symmetric function $f$, we define $f[A]$, the **plethystic substitution** of $A$ into $f$, as follows. First define $p_k[A] := \phi_k(A)$. Then let $p_\mu[A] := p_{\mu_1}[A] p_{\mu_2}[A] \cdots p_{\mu_l}[A]$. Finally, since the power sum symmetric functions are a basis for the ring of symmetric functions, we extend linearly to all symmetric functions.

The most trivial example is as follows. Let $R = \mathbb{C}[x_1, x_2, \ldots]$, and all $x_i$ be indeterminates. Then for any symmetric function $f$, if $X = \sum_i x_i$, $f[X] = f(\mathbf{x})$. Less trivially, note that $p_k[-X] = -p_k(\mathbf{x}) = (-1)^k \omega p_k(\mathbf{x})$. Therefore, for any symmetric function $f$ homogeneous of degree $d$, $f[-X] = (-1)^d \omega f(\mathbf{x})$.

It is customary to neglect to specify $R$ and the set of indeterminates, and allow the set of indeterminates to be all the variables appearing in the expression. The ring $R$ then will be the polynomial ring in the indeterminates, or the field of rational functions in the indeterminates, or some other similar object such as the formal power series ring or Laurent series ring in the indeterminates, subject to any relations we have imposed. If we do impose any relations, we must be careful not to impose relations which make some $\phi_k$ no longer well-defined; that is, if we impose some relation $x(a_1, a_2, \ldots) = y(a_1, a_2, \ldots)$, we must take care that $\phi_k(x(a_1, a_2, \ldots)) = \phi_k(y(a_1, a_2, \ldots))$ for every $k$.

For example, if $t$ is an indeterminate, $X$ is any expression in $R$, and $f$ a symmetric function homogeneous of degree $d$, $f[tX] = t^d f[X]$. However, $f[-X] = (-1)^d \omega f[X] \neq (-1)^d f[X]$. This is because $t = -1$ is not an allowable relation for an indeterminate $t$, since $\phi_k(-1) = -1$ but $\phi_k(t) = t^d \neq -1$ for $k$ even.

However, if we take $X = x_1 + x_2 + \cdots$ for an infinite set of variables, then the $p_k[X]$ are algebraically independent, so any plethystic equation which holds for $X$ holds identically for any expression in place of $X$. The same is true if we have several independent infinite sets of variables $X$, $Y$, $Z$, and so on.

3.3. **The Cauchy Kernel and Hall Inner Product.** Let $\Omega$ denote the **Cauchy kernel**, which is the symmetric power series $\Omega := \sum_{n\geq 0} h_n$, so for $X = x_1 + x_2 + \cdots$, $\Omega[X] = \sum_{n\geq 0} h_n[X] = \prod_i 1/(1-x_i)$. Notice that, if we have $Y = y_1 + y_2 + \cdots$ as well, $\Omega[X+Y] = \Omega[X]\Omega[Y]$, and since this identity holds with $X$ and $Y$ both sums of infinite sets of variables, it holds for any $X$ and $Y$. In particular, $\Omega[X]\Omega[-X] = \Omega[0] = 1$, so $\Omega[-X] = \prod_i(1-x_i) = \sum_n(-1)^n e_n(\mathbf{x})$. Taking the degree $n$ piece of this identity, we have $h_n[-X] = (-1)^n e_n[X]$, which shows that, for $f$ a homogeneous symmetric function of degree $d$, $\omega f[X] = (-1)^d f[-X]$.

Now we define the **Hall inner product** $\langle\cdot,\cdot\rangle$ on symmetric functions by declaring that $\langle h_\lambda, m_\mu\rangle = 0$ if $\lambda \neq \mu$, and $\langle h_\mu, m_\mu\rangle = 1$. This inner product has the following interpretation in terms of $\Omega$.

**Proposition 3.** *Two bases $\{u_\lambda\}$ and $\{v_\lambda\}$ are dual (with respect to the Hall inner product) if and only if $\Omega[XY] = \sum_\lambda u_\lambda[X]v_\lambda[Y]$.*

*Proof.* First we note that $\Omega[XY] = \prod_{i,j} 1/(1-x_iy_j) = \prod_i \Omega[x_iY] = \prod_i \sum_n x_i^n h_n[Y] = \sum_\lambda m_\lambda[X]h_\lambda[Y]$, and, by symmetry, $\Omega[XY] = \sum_\lambda h_\lambda[X]m_\lambda[Y]$. (This is known as the first **Cauchy formula**.)

Let $\langle\cdot,\cdot\rangle_{\mathbf{x}}$ denote the Hall inner product with respect to the $x$ variables only. Then $\langle m_\lambda[X], \Omega[XY]\rangle_{\mathbf{x}} = \langle m_\lambda[X], h_\lambda[X]m_\lambda[Y]\rangle_{\mathbf{x}} = m_\lambda[Y]$, so by linearity, $\langle f[X], \Omega[XY]\rangle_{\mathbf{x}} = f[Y]$ for all $f$. If $\Omega[XY] = \sum_\lambda u_\lambda[X]v_\lambda[Y]$, then $\langle v_\lambda[X], \sum_\lambda u_\lambda[X]v_\lambda[Y]\rangle_{\mathbf{x}} = v_\lambda[Y]$, so $\{u_\lambda\}$ and $\{v_\lambda\}$ are dual bases. Since the Hall inner product is non-degenerate, the only way to have $\langle v_\lambda[X], g[XY]\rangle_{\mathbf{x}} = v_\lambda[X]$ for all $\lambda$ is to have $g = \Omega$, which proves the reverse direction. $\square$

It can be shown, for example by using the Robinson-Schensted-Knuth correspondence, that $\Omega[XY] = \sum_\lambda s_\lambda[X]s_\lambda[Y]$, so the Schur functions are an orthonormal basis for $\Lambda_{\mathbb{Q}}$ under this inner product. Therefore, in terms of the representation theory of $S_n$, we therefore have that $\dim(\mathrm{Hom}^{S_n}(V,W)) = \langle F_V, F_W\rangle$ for any two representations $V$ and $W$.

Finally, note that $\omega$ is an isometry with respect to the Hall inner product. In other words, $\langle \omega f, \omega g\rangle = \langle f, g\rangle$ for any symmetric functions $f$ and $g$.

3.4. **Dominance Ordering.** The final definition we need is a partial order on partitions known as **dominance order**. Being the only order relation we will use on partitions, we will simply denote it by $\leq$. We say $\lambda \leq \mu$ if $\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k$ for every positive integer $k$.

**Proposition 4.** *If $\mu \not\leq \lambda$, then $K_{\lambda\mu} = 0$.*

*Proof.* Let $T$ be a SSYT of shape $\lambda$. Note that, for any $k$, any box of $T$ with a label $i < k$ must occur in one of the first $k$ rows. Therefore, if $\mu$ is the content of $T$, $\mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k$. Since this holds for any $k$, we have that, if $K_{\lambda\mu} \neq 0$, then $\mu \leq \lambda$, proving the proposition. $\square$

We will also need the following proposition, whose (not entirely trivial) proof is left as an exercise.

**Proposition 5.** *$\lambda \leq \mu$ iff $\lambda' \leq \mu'$.*

3.5. **Definition of Macdonald polynomials.** Now we are ready to define the Macdonald polynomials and their predecessors, the Hall-Littlewood polynomials.

**Theorem-Definition 1.** *The ring $\Lambda_{\mathbb{Q}(t)}$ has a unique basis $\widetilde{H}_\mu(\mathbf{x};t)$ characterized by*

1. $\widetilde{H}_\mu(\mathbf{x};t) \in \mathbf{Q}(t)\{s_\lambda | \lambda \geq \mu\}$
2. $\widetilde{H}_\mu[X(1-t);t] \in \mathbf{Q}(t)\{s_\lambda | \lambda \geq \mu'\}$
3. $\widetilde{H}_\mu[1;t] = 1$.

*These polynomials are known as the* **Hall-Littlewood polynomials**. *Moreover, $\widetilde{H}_\mu(\mathbf{x};t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(t) s_\lambda(\mathbf{x})$, where the $\widetilde{K}_{\lambda\mu}(t)$ are as in Lecture 1.*

**Theorem-Definition 2.** *The ring $\Lambda_{\mathbb{Q}(q,t)}$ has a unique basis $\widetilde{H}_\mu(\mathbf{x};q,t)$ characterized by*

1. $\widetilde{H}_\mu[X(1-q);q,t] \in \mathbf{Q}(q,t)\{s_\lambda | \lambda \geq \mu\}$
2. $\widetilde{H}_\mu[X(1-t);q,t] \in \mathbf{Q}(q,t)\{s_\lambda | \lambda \geq \mu'\}$
3. $\widetilde{H}_\mu[1;q,t] = 1$.

*These polynomials are known as the* **Macdonald polynomials**.

Now we can define a two variable $q,t$-analog of the Kostka numbers.

**Definition 2.** *Since the Schur functions are a basis for $\Lambda_{\mathbb{Q}(q,t)}$,*

$$\widetilde{H}_\mu(\mathbf{x};q,t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q,t) s_\lambda(\mathbf{x})$$

*for some rational functions $\widetilde{K}_{\lambda\mu}(q,t)$. These are the* **$q,t$-Kostka numbers**.

We do not have time to give a complete proof of these theorems. We will however prove uniqueness and give an indication of the main ingredients in the existence proof. Details can be found in [15, 21].

Pick an arbitrary degree $n$, and suppose $\left\{\widetilde{H}_\mu(\mathbf{x};q,t)\right\}_{|\mu|=n}$ and $\left\{\widetilde{H}'_\mu(\mathbf{x};q,t)\right\}_{|\mu|=n}$ are two bases of the degree $n$ part of $\Lambda_{\mathbb{Q}(q,t)}$ characterized by the three given conditions. Order these bases by choosing the same refinement of dominance order for both. Then there will be a transition matrix, which we denote $T$, which tells us how to write elements of one basis with respect to the other. Our goal is to show that $T$ must be the identity matrix.

Define the operator $\Pi_{(1-q)}$ on $\Lambda_{\mathbb{Q}(q,t)}$ by $\Pi_{(1-q)} f = f[X(1-q)]$. Also, define $\Pi_{1/(1-q)}$ by $\Pi_{1/(1-q)} f = f[X/(1-q)]$, and similarly define $\Pi_{(1-t)}$ and $\Pi_{1/(1-t)}$. By checking for $f = p_n$, we see that $\Pi_{(1-q)}$ and $\Pi_{1/(1-q)}$ are clearly inverse to each other, as are $\Pi_{(1-t)}$ and $\Pi_{1/(1-t)}$.

Now let $P$ and $P'$ be the transition matrices respectively expressing $\left\{\widetilde{H}_\mu(\mathbf{x};q,t)\right\}_{|\mu|=n}$ and $\left\{\widetilde{H}'_\mu(\mathbf{x};q,t)\right\}_{|\mu|=n}$ in terms of the basis $\{s_\lambda[X/(1-q);q,t]\}_{|\lambda|=n}$. By applying $\Pi_{1/(1-q)}$

to condition (1), we see that both $P$ and $P'$ are upper triangular. Therefore, $T = P^{-1}P'$ is upper triangular. Rewriting condition (2) as $\widetilde{H}_\mu[X(1-q);q,t] \in \mathbf{Q}(q,t)\{s_{\lambda'}|\lambda \leq \mu\}$ and applying $\Pi_{1/(1-t)}$, we have that the transition matrices expressing $\left\{\widetilde{H}_\mu(\mathbf{x};q,t)\right\}_{|\mu|=n}$ and $\left\{\widetilde{H}'_\mu(\mathbf{x};q,t)\right\}_{|\mu|=n}$ in terms of the basis $\{s_{\lambda'}[X/(1-t);q,t]\}_{|\lambda|=n}$ are lower triangular, so $T$ is also lower triangular, and therefore diagonal. Condition (3) then forces $T$ to be the identity.

Now we outline the ideas behind the existence proof. Let $X = x_1 + x_2 + \cdots$ as usual. Define a linear operator $\Delta_0$ on $\Lambda_{\mathbb{Q}(q,t)}$ by

$$\Delta_0 f = [u^0](f[X + (1-q)(1-t)u^{-1}]\Omega[-uX]),$$

and define the linear operator $\Delta$ by

$$\Delta f = \frac{f - \Delta_0 f}{(1-q)(1-t)}.$$

The operator $\Delta$ is known as the **Macdonald operator**, and the existence of Macdonald polynomials is proved by showing that $\Pi_{(1-q)}\Delta\Pi_{1/(1-q)}$ is upper triangular with respect to $\{s_\lambda\}_{|\lambda|=n}$, with diagonal entries $B_\lambda(q,t) := \sum_{(i,j)\in\lambda} t^i q^j = \sum_i t^{i-1}(1-q^{\lambda_i})/(1-q)$. (Note the convention is that powers of $q$ increase as one moves to the right and powers of $t$ increase as one moves up.) Therefore, eigenfunctions for $\Delta$ must satisfy (1). Furthermore, $\Delta$ and $B_\lambda$ are symmetric with respect to simultaneously exchanging $q$ and $t$ and exchanging $\mu$ and $\mu'$, so these eigenfunctions must satisfy (2). Condition (3) is just a scalar normalization factor. Note that, in particular,

$$\Delta\widetilde{H}_\mu(\mathbf{x};q,t) = B_\mu(q,t)\widetilde{H}_\mu(\mathbf{x};q,t).$$

Some properties of Macdonald polynomials are easy to see from the definition and theorem. First, $\widetilde{H}_\mu(\mathbf{x};0,t) = \widetilde{H}_\mu(\mathbf{x};t)$, and $\widetilde{K}_{\lambda\mu}(0,t) = \widetilde{K}_{\lambda\mu}(t)$. In other words, setting $q = 0$ in a Macdonald polynomial recovers the corresponding Hall-Littlewood polynomial. Also, the definition looks the same when we both swap $q$ and $t$ and swap $\mu$ and $\mu'$, so by uniqueness, $\widetilde{H}_\mu(\mathbf{x};q,t) = \widetilde{H}_{\mu'}(\mathbf{x};t,q)$. In particular, if $\mu = \mu'$, then $\widetilde{H}_\mu(\mathbf{x};q,t)$ is symmetric under switching $q$ and $t$.

From the definition it is possible to compute $\widetilde{H}_{(n)}(\mathbf{x};q,t)$. Every partition dominates $(1^n) = (n)'$, so the second condition is vacuous. The first condition states that $\widetilde{H}_{(n)}[X(1-q);q,t] = fh_n(\mathbf{x})$ for some $f \in \mathbb{Q}(q,t)$, or, equivalently, that $\widetilde{H}_{(n)}(\mathbf{x};q,t) = fh_n[X/(1-q);q,t]$ for some $f$. Now we use the third condition to solve for $f$; namely $f = \widetilde{H}_{(n)}[1;q,t]/h_n[1/(1-q)]$. Note that

$$h_n[1/(1-q)] = h_n(1,q,q^2,\ldots) = \sum_{l(\lambda)\leq n} q^{|\lambda|} = \sum_{\lambda_1 \leq n} q^{|\lambda|} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Therefore,

$$\widetilde{H}_{(n)}(\mathbf{x};q,t) = (1-q)\cdots(1-q^n)h_n\left[\frac{X}{1-q}\right].$$

Next we compute $\widetilde{H}_\mu(\mathbf{x};q,1)$ for all $\mu$. First, note that $\Delta\mid_{t=1}$ is a derivation on $\Lambda_{\mathbb{Q}(q)}$; that is, for any $f,g\in\Lambda_{\mathbb{Q}(q)}$, $\Delta(fg)\mid_{t=1}= f(\Delta(g)\mid_{t=1})+(\Delta(f)\mid_{t=1})g$. Since $\Delta\mid_{t=1}$ is linear on $\Lambda_{\mathbb{Q}(q)}$, this statement can be proven by showing that it holds when $f=p_\mu$ and $g=p_\nu$, and this is left as an exercise. Now note that $\widetilde{H}_{(n)}(\mathbf{x};q,t)=\widetilde{H}_{(n)}(\mathbf{x};q,1)$, so we have that

$$\Delta\widetilde{H}_{(n)}(\mathbf{x};q,1)=B_{(n)}(q,1)\widetilde{H}_{(n)}(\mathbf{x};q,1)=(1-q^n)/(1-q)\widetilde{H}_{(n)}(\mathbf{x};q,1).$$

Using that $\Delta\mid_{t=1}$ is a derivation on $\Lambda_{\mathbb{Q}(q)}$,

$$\Delta\mid_{t=1}(\widetilde{H}_{\mu_1}(\mathbf{x};q,1)\cdots\widetilde{H}_{\mu_k}(\mathbf{x};q,1))=B_\mu(q,1)(\widetilde{H}_{\mu_1}(\mathbf{x};q,1)\cdots\widetilde{H}_{\mu_k}(\mathbf{x};q,1)).$$

Now the uniqueness of Macdonald polynomials tells us that

$$\widetilde{H}_\mu(\mathbf{x};q,1)=\prod_i\widetilde{H}_{\mu_i}(\mathbf{x};q,1).$$

Finally, we compute $\widetilde{H}_\mu(\mathbf{x};1,1)$ for all $\mu$. First,

$$\widetilde{H}_{(1)}(\mathbf{x};q,1)=(1-q)h_1[\frac{X}{1-q};q,1]=h_1(\mathbf{x}),$$

so

$$\widetilde{H}_{(1)}(\mathbf{x};1,1)=h_1(\mathbf{x}).$$

In particular, it follows from the previous paragraph, specialized at $q=1$, that

$$\widetilde{H}_{(1^n)}(\mathbf{x};1,1)=\widetilde{H}_{(1)}(\mathbf{x};1,1)^n=h_{(1^n)}(\mathbf{x}).$$

Now, since

$$\widetilde{H}_{(n)}(\mathbf{x};q,1)=\widetilde{H}_{(1^n)}(\mathbf{x};1,q),$$

we get that

$$\widetilde{H}_\mu(\mathbf{x};1,1)=\prod_i\widetilde{H}_{(\mu_i)}(\mathbf{x};1,1)=h_{(1^{|\mu|})}(\mathbf{x}).$$

Note in particular this does not depend on the partition $\mu$ as long as $|\mu|=n$.

3.6. **More properties of Macdonald polynomials.** The theory of Macdonald polynomials is a large subject fit for another series of lectures, so we will not be able to cover most of it. Instead we merely state here, without proof, a few facts which we will need in subsequent lectures.

To help in understanding these properties, we provide one small example. It can be calculated that $\widetilde{H}_{(22)}(\mathbf{x};q,t)=s_{(4)}(\mathbf{x})+(q+t+qt)s_{(31)}(\mathbf{x})+(q^2+t^2)s_{(22)}(\mathbf{x})+(q^2t+qt^2+qt)s_{(211)}(\mathbf{x})+q^2t^2s_{(1111)}(\mathbf{x})$. We can draw this as a picture as in Figure 9.

First we describe the action of $\omega$. For a partition $\mu$, let $n(\mu)=\sum_i(i-1)\mu_i$. Then

$$\omega\widetilde{H}_\mu(\mathbf{x};q,t)=t^{n(\mu)}q^{n(\mu')}\widetilde{H}_\mu(\mathbf{x};q^{-1},t^{-1}).$$

Here $n(\mu)$ and $n(\mu')$ appear because they are respectively the top degrees of $t$ and $q$ found in $\widetilde{H}_\mu(\mathbf{x};q,t)$, so the multiplication by $t^{n(\mu)}q^{n(\mu')}$ normalizes the right hand side to be a polynomial in $q$ and $t$ with nonzero constant term.

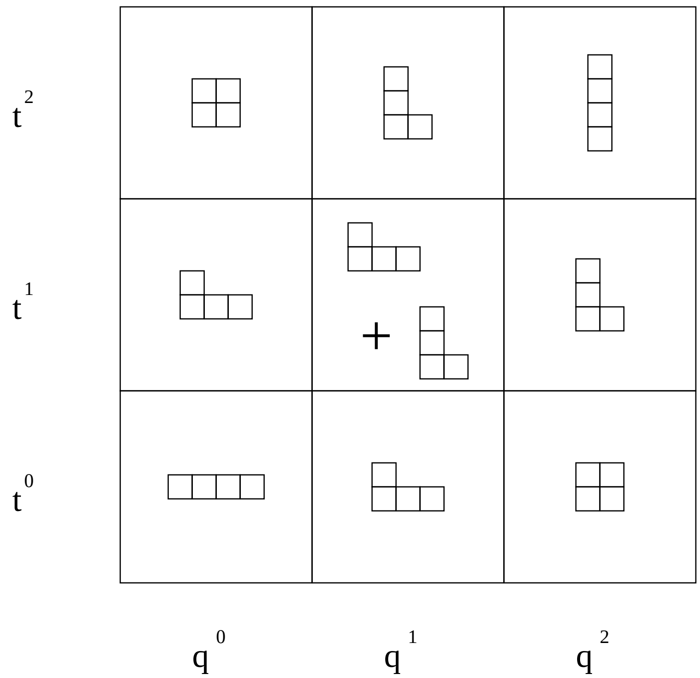


FIGURE 9. $\widetilde{H}_{(22)}(\mathbf{x};q,t)$

There is also the **Macdonald specialization formula**, which states that

$$\widetilde{H}_\mu[1-u;q,t] = \Omega[-uB_u] = \prod_{(i,j)\in\mu}(1-uq^jt^i).$$

We can use this formula to derive $\widetilde{K}_{\lambda\mu}$ when $\lambda$ a hook shape, that is, if $\lambda=(n-r,1^r)$ for some $r$. Specifically,

$$\widetilde{K}_{(n-r,1^r),\mu} = e_r[B_\mu - 1].$$

Finally, we describe a $q,t$-analog of the Hall inner product and give a corresponding Cauchy formula for Macdonald polynomials. Define

$$\langle f,g\rangle_* := \langle f[X(1-q);q,t], \omega g[X(1-t);q,t]\rangle,$$

where the inner product on the right is the usual Hall inner product (with respect to the $x$ variables). Then $\langle \widetilde{H}_\lambda(\mathbf{x};q,t), \widetilde{H}_\mu(\mathbf{x};q,t)\rangle_* = \langle \widetilde{H}_\lambda[X(1-q);q,t], \omega\widetilde{H}_\mu[X(1-t);q,t]\rangle$, and, expanding both parts of the inner product in terms of the orthonormal basis of Schur functions, we see that $\langle \widetilde{H}_\lambda(\mathbf{x};q,t), \widetilde{H}_\mu(\mathbf{x};q,t)\rangle_* \neq 0$ iff $\{\nu : \nu \geq \lambda\} \cap \{\nu : \nu' \geq \mu'\} \neq \emptyset$ iff $\lambda \leq \mu$. By symmetry of the inner product (which follows from $\omega$ being an isometry and $\Pi_{(1-q)}$ and $\Pi_{1/(1-q)}$ being adjoint), we also have $\lambda \geq \mu$, so $\langle \widetilde{H}_\lambda(\mathbf{x};q,t), \widetilde{H}_\mu(\mathbf{x};q,t)\rangle_* = 0$ if $\lambda \neq \mu$.

Let $c$ be a cell in the diagram of some partition $\lambda$. The **arm** and **leg** of $c$, respectively denoted $a(c)$ and $l(c)$, are the number of boxes strictly to the right of, and respectively the

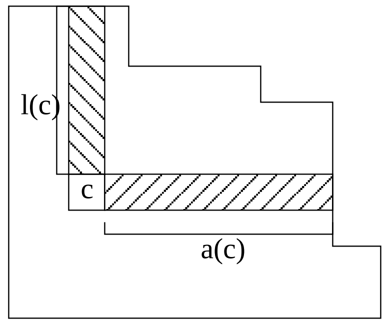


FIGURE 10. Arm and Leg of a cell $c \in \lambda$.

number of boxes strictly above, the box $c$ in the diagram of $\lambda$, as illustrated in Figure 10. It turns out that

$$\langle \widetilde{H}_\mu(\mathbf{x}; q, t), \widetilde{H}_\mu(\mathbf{x}; q, t)\rangle_* = t^{n(\mu)} q^{n(\mu')} \prod_{c\in\mu} (1 - t^{l(c)+1} q^{-a(c)})(1 - t^{-l(c)} q^{a(c)+1}).$$

Therefore, we have

$$\Omega[XY] = \sum_\mu \frac{t^{-n(\mu)} q^{-n(\mu')} \widetilde{H}_\mu[X(1-q); q, t] \omega \widetilde{H}_\mu[Y(1-t); q, t]}{\prod_{c\in\mu}(1 - t^{l(c)+1} q^{-a(c)})(1 - t^{-l(c)} q^{a(c)+1})},$$

or, after substituting $X/(1-q)$ for $X$ and $-Y/(1-t)$ for $Y$, taking the degree $n$ piece, and multiplying both sides by $(-1)^n$,

$$e_n\left[\frac{XY}{(1-q)(1-t)}\right] = \sum_{|\mu|=n} \frac{t^{-n(\mu)} q^{-n(\mu')} \widetilde{H}_\mu(\mathbf{x}; q, t) \widetilde{H}_\mu(\mathbf{y}; q, t)}{\prod_{c\in\mu}(1 - t^{l(c)+1} q^{-a(c)})(1 - t^{-l(c)} q^{a(c)+1})}.$$

3.7. **Exercises.**

(1) Let $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$. Express $e_n[X - Y]$ in terms of symmetric functions separately in the $x$ and $y$ variables.
(2) Show that the graded Frobenius series of $\mathbb{C}[x_1, \cdots, x_n]$ as an $S_n$ representation, that is, $\sum_d t^d F_{\mathbb{C}[x_1,\cdots,x_n]_d}$ (where $\mathbb{C}[x_1, \cdots, x_n]_d$ denotes the polynomials of degree $d$), is $h_n[X/(1-t)]$.
(3) Prove Proposition 5.
(4) Show that $\Delta \mid_{t=1}$ is a derivation on $\Lambda_{\mathbb{Q}(q)}$ by showing that
$$\Delta(p_\mu p_\nu) \mid_{t=1} = \Delta(p_\mu) \mid_{t=1} p_\nu + p_\mu \Delta(p_\nu) \mid_{t=1}.$$
(5) Show that
$$\omega \widetilde{H}_\mu(\mathbf{x}; q, t) = t^{n(\mu)} q^{n(\mu')} \widetilde{H}_\mu(\mathbf{x}; q^{-1}, t^{-1}).$$
(You will need to use the Macdonald specialization formula.)
(6) Use the Macdonald specialization formula to show that $\widetilde{K}_{\lambda\mu}(q, t) = e_r[B_\mu - 1]$ when $\lambda = (n - r, 1^r)$.

(7) (a) Prove that for any expression $A$

$$\frac{e_n[(1-u)A]}{1-u}\,|_{u=1} = (-1)^{n-1}p_n[A].$$

(b) For the Macdonald operator $\Delta$, show that

$$\Delta\left((-1)^{n-1}p_n\left[\frac{X}{(1-q)(1-t)}\right]\right) = \frac{e_n[X]}{(1-q)(1-t)}.$$

(c) Let $\Pi_\mu(q,t) = \prod_{(i,j)\in\mu\setminus(0,0)}(1-q^jt^i)$. Now use parts (a) and (b), the Macdonald specialization, and the Cauchy formula to prove that

$$e_n(\mathbf{x}) = \sum_{|\mu=n|} \frac{t^{-n(\mu)}q^{-n(\mu')}(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\widetilde{H}_\mu(\mathbf{x};q,t)}{\prod_{c\in\mu}(1-t^{l(c)+1}q^{-a(c)})(1-t^{-l(c)}q^{a(c)+1})}.$$

## 4. Connecting Macdonald polynomials and $q$-Lagrange inversion; $(q,t)$-analogs

In this lecture we will take expressions which at first appear to be relatively unmotivated symmetric functions and show that in fact they are a $(q,t)$-analog of the $k_n(q)$ which solved the $q$-Lagrange inversion problem in Lecture 2. Most of the lecture will be devoted to this proof which includes some complicated calculations. They have been included because they reflect many of the calculational techniques which are important in this subject. The main general reference for this section is [7].

4.1. **The operator $\nabla$ and a $(q,t)$-analog of $k_n(q)$.** Recall from Lecture 3 and specifically Exercise 3.7(7) that

$$e_n(\mathbf{x}) = \sum_{|\mu=n|} \frac{t^{-n(\mu)}q^{-n(\mu')}(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\widetilde{H}_\mu(\mathbf{x};q,t)}{\prod_{c\in\mu}(1-t^{l(c)+1}q^{-a(c)})(1-t^{-l(c)}q^{a(c)+1})},$$

where $B_\mu(q,t) = \sum_{(i,j)\in\lambda} t^iq^j$, $\Pi_\mu(q,t) = \Omega[1-B_\mu] = \prod_{(i,j)\in\mu\setminus(0,0)}(1-t^iq^j)$, and, for $c$ a cell in the diagram of $\mu$, $a(c)$ and $l(c)$ denote respectively the arm and leg of $c$.

Define an operator $\nabla$ on $\Lambda_{\mathbb{Q}(q,t)}$ by letting

$$\nabla\widetilde{H}_\mu := t^{n(\mu)}q^{n(\mu')}\widetilde{H}_\mu$$

and extending by linearity. Applying this operator to the above expansion of $e_n$ gives

$$\nabla e_n = \sum_{|\mu=n|} \frac{(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\widetilde{H}_\mu(\mathbf{x};q,t)}{\prod_{c\in\mu}(1-t^{l(c)+1}q^{-a(c)})(1-t^{-l(c)}q^{a(c)+1})}.$$

Now we calculate $\langle \nabla e_n, e_n \rangle$. Notice that

$$\begin{aligned}\langle \widetilde{H}_\mu(\mathbf{x};q,t), e_n\rangle &= \langle \widetilde{H}_\mu(\mathbf{x};q,t), s_{(1^n)}\rangle \\ &= \widetilde{K}_{(1^n),\mu} \\ &= e_{n-1}[B_\mu - 1] \\ &= e_{n-1}[\sum_{(i,j)\in\mu\setminus(0,0)} t^i q^j] = \prod_{(i,j)\in\mu\setminus(0,0)} t^i q^j = t^{n(\mu)} q^{n(\mu')},\end{aligned}$$

where the third equality comes from the Macdonald specialization formula as discussed in Lecture 3. Therefore,

$$\langle \nabla e_n, e_n \rangle = \sum_{|\mu=n|} \frac{t^{n(\mu)} q^{n(\mu')}(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)}{\prod_{c\in\mu}(1-t^{l(c)+1}q^{-a(c)})(1-t^{-l(c)}q^{a(c)+1})}.$$

Define $C_n(q,t)$ to be this rational function $\langle \nabla e_n, e_n\rangle$. It turns out that $C_n(q,t)$ is a polynomial with positive integer coefficients, and that $C_n(q,1) = C_n(q)$, the $q$-analog of the Catalan numbers discussed in Lecture 2. Furthermore, $C_n(q,t)$ is symmetric under exchanging $q$ and $t$; that is, $C_n(q,t) = C_n(t,q)$. For example, $C_3(q,t) = q^3+q^2t+qt+qt^2+t^3$, and specializing to $t=1$ gives $C_3(q) = q^3+q^2+2q+1$ which is what we had earlier. Therefore, it makes sense to think of $C_n(q,t)$ as a $(q,t)$-analog of the Catalan numbers.

Now notice that $C_n(q) = k_n(q)|_{e_k\mapsto 1}$, as we saw at the end of Lecture 2. Since $h_n = \sum_{|\mu|=n} m_\mu$ and $\{h_\mu\}$ and $\{m_\mu\}$ are dual bases, $\langle h_\mu, h_n\rangle = 1$ for all $\mu$, and consequently, since $\omega$ is an isometry with respect to the Hall inner product, $\langle e_\mu, e_n\rangle = 1$ for all $\mu$. Therefore, if we pretend that the $e_k$ in $k_n(q)$ actually stand for elementary symmetric functions, then $C_n(q) = \langle k_n(q), e_n\rangle$.

Comparing the equations $C_n(q) = \langle k_n(q), e_n\rangle$ and $C_n(q,t) = \langle \nabla e_n, e_n\rangle$ hints at a possible connection between $k_n(q)$ and $\nabla e_n$. It turns out that there is indeed a connection given by the following theorem, which we will spend most of the remainder of this lecture proving.

**Theorem 7.** *Interpreting the $e_k$ in $k_n(q)$ as elementary symmetric functions, we have that*

$$\nabla e_n|_{t=1} = k_n(q).$$

Before we go into the proof, let us mention two corollaries giving $(q,t)$-analogs of our main examples from Lecture 2. The first corollary follows from the discussion above. To prove the second, recall that $h_{(1^n)} = \sum_{|\mu|=n} \binom{n}{\mu_1,\ldots,\mu_l} m_\mu$, so $\langle e_\mu, e_{(1^n)}\rangle = \binom{n}{\mu_1,\ldots,\mu_l}$.

**Corollary 1.** *Define $C_n(q,t)$, as above, by $C_n(q,t) = \langle \nabla e_n, e_n\rangle$. Then*

$$C_n(q,1) = C_n(q).$$

**Corollary 2.** *Define $P_n(q,t)$ by $P_n(q,t) := \langle \nabla e_n, e_{(1^n)}\rangle$. Then*

$$P_n(q,1) = P_n(q) = \sum_{\lambda\in\delta(n)} q^{\binom{n}{2}-|\lambda|} \binom{n}{\alpha_0,\alpha_1,\ldots,\alpha_n}.$$

4.2. **Proof of Theorem 7.** Let $K(z) = \sum_n k_n(q)z^n$, and $E(z) = \sum_n e_n z^n$. Identifying the $e_n$ with the elementary symmetric functions $e_n(\mathbf{x})$, we have $E(z) = \omega\Omega[zX] = \sum_n e_n[X]z^n$. For convenience, let us define $E := \omega\Omega$ as a symmetric power series, so $E(z) = E[zX]$. Notice that $E[zX] = \prod_i(1 + zx_i)$, so $E[z(A + B)] = E[zA]E[zB]$ for any expressions $A$ and $B$. Consequently, since $1 = E[0] = E[z(A - A)] = E[zA]E[-zA]$, we have that $E[-zA] = 1/E[zA]$.

Now recall that $K(z)$ is in fact the solution to the $q$-Lagrange inversion problem $z/E[zX] \circ_q zK(qz) = z$, or, equivalently by Theorem 5, $z = zK(qz) \circ_{q^{-1}} z/E[zX]$. For convenience, let $g_n$ be the coefficient of $z^n$ in $zK(qz)$, so $g_n = q^{n-1}k_{n-1}(q)$. We can now calculate that

$$\begin{aligned} z &= zK(qz) \circ_{q^{-1}} z/E[zX] \\ &= \sum_n g_n \frac{z}{E[zX]} \frac{q^{-1}z}{E[q^{-1}zX]} \cdots \frac{q^{-(n-1)}z}{E[q^{-(n-1)}zX]} \\ &= \sum_n g_n z^n q^{-\binom{n}{2}} \frac{1}{E[z(1 + q^{-1} + \cdots + q^{-(n-1)})X]} \\ &= \sum_n g_n z^n q^{-\binom{n}{2}} \frac{1}{E[z(1 - q^{-n})X/(1 - q^{-1})]} \\ &= \sum_n g_n z^n q^{-\binom{n}{2}} \frac{E[zq^{-n}X/(1 - q^{-1})]}{E[zX/(1 - q^{-1})]}. \end{aligned}$$

Hence

$$\sum_n g_n z^n q^{-\binom{n}{2}} E\left[\frac{q^{-n}zX}{1 - q^{-1}}\right] = zE\left[\frac{zX}{1 - q^{-1}}\right]. \tag{1}$$

For any series $\Psi(z) = \sum_n \Psi_n z^n$, define ${}^\vee\Psi(z) = \sum_n \Psi_n q^{\binom{n}{2}} z^n = \Psi(z) \circ_q z$. Now we need a lemma about the behavior of ${}^\vee$.

**Lemma 3.** *We have the identities*

(1) ${}^\vee(z^n q^{-\binom{n}{2}} \Psi(q^{-n}z)) = z^n\,{}^\vee\Psi(qz)$
(2) ${}^\vee(z\Psi(z)) = z\,{}^\vee\Psi(qz)$.

*Proof.* By linearity, it suffices to prove this for $\Psi(z) = z^r$ (for all $r$) in both cases. We see that

$$\begin{aligned} {}^\vee(z^n q^{-\binom{n}{2}} (q^{-n}z)^r) &= q^{\binom{n+r}{2}} q^{-\binom{n}{2} - nr} z^{n+r} \\ &= q^{\binom{r}{2}} z^{n+r} \\ &= z^n\,{}^\vee z^r. \end{aligned}$$

Also,

$$\begin{aligned} {}^{\vee}(z^{r+1}) &= q^{\binom{r+1}{2}} z^{r+1} \\ &= z q^{\binom{r}{2}} q^r z^r \\ &= z^{\vee}((qz)^r). \end{aligned}$$

□

Apply the operator $^{\vee}$ to both sides of equation 1. Using the first part of the lemma on the left hand side and the second part on the right hand side, we get

$$\sum_n g_n z^{n\vee} E\left[\frac{zX}{1-q^{-1}}\right] = z^{\vee} E\left[\frac{qzX}{1-q^{-1}}\right].$$

Hence,

$$zK(qz) = G(z) = \sum_n g_n z^n = \frac{z^{\vee}E\left[qzX/(1-q^{-1})\right]}{{}^{\vee}E\left[zX/(1-q^{-1})\right]},$$

and, substituting $q^{-1}z$ for $z$,

$$K(z) = \sum_n k_n(q) z^n = \frac{{}^{\vee}E\left[zX/(1-q^{-1})\right]}{{}^{\vee}E\left[q^{-1}zX/(1-q^{-1})\right]}.$$

Specializing to $q = 1$ appropriately here gives us $k_n = [z^n]\frac{E(z)^{n+1}}{n+1}$, that is, the last formula is actually a $q$-analog of the classical formula in Theorem 4.

To understand $K(z)$ more explicitly, we make two further definitions; neither is strictly necessary but they will both make our notation significantly more compact. First, for each partition $\mu$, define the symmetric functions $f_\mu(\mathbf{x})$ (sometimes known as the **forgotten symmetric functions**) by the identity

$$e_n[XY] := \sum_{|\mu|=n} h_\mu[X] f_\mu[Y].$$

Equivalently, we could also define $f_\mu$ as the dual basis to $\{e_\mu\}$ under the Hall inner product, or by letting $f_\mu := \omega m_\mu$. Secondly, we introduce a fictitious alphabet $A$ such that

$$h_n[A] := q^{\binom{n}{2}} h_n[X/(1-q)].$$

Now we produce the following identity to simplify our expression for $K(z)$:

$$\begin{aligned}
{}^{\vee}E\left[\frac{q^{-1}zX}{1-q^{-1}}\right] &= {}^{\vee}E\left[\frac{-zX}{1-q}\right] \\
&= {}^{\vee}\sum_n (-1)^n \omega e_n[X/(1-q)]z^n \\
&= {}^{\vee}\sum_n h_n[X/(1-q)](-z)^n \\
&= \sum_n q^{\binom{n}{2}} h_n[X/(1-q)](-z)^n \\
&= \sum_n h_n[A](-z)^n = \sum_n e_n[-zA] = \sum_n E[-zA] = 1/E[zA].
\end{aligned}$$

From this identity, our previous equation for $K(z)$ reduces to

$$K(z) = \frac{E[zA]}{E[qzA]} = E[z(1-q)A] = \sum_\mu h_\mu[A] f_\mu[z(1-q)],$$

the last equality coming from our definition of $f_\mu$. By our definition of $A$,

$$K(z) = \sum_\mu \left( \prod_{i=1}^{l(\mu)} q^{\binom{\mu_i}{2}} h_{\mu_i}\left[X/(1-q)\right] \right) f_\mu[z(1-q)].$$

Extracting on both sides the coefficient of $z^n$, we end up with

$$k_n(q) = \sum_{|\mu|=n} q^{n(\mu')} h_\mu[X/(1-q)] f_\mu[1-q].$$

Finally, recall from Lecture 3 that

$$\widetilde{H}_\mu(\mathbf{x};q,1) = \prod_i \widetilde{H}_{\mu_i}(\mathbf{x};q,1) = \left( \prod_i (1-q)\cdots(1-q^{\mu_i}) \right) h_\mu\left[X/(1-q)\right],$$

which means that

$$\nabla|_{t=1} h_\mu[X/(1-q)] = q^{n(\mu')} h_\mu[X/(1-q)].$$

Hence,

$$k_n(q) = \nabla|_{t=1} \left( \sum_{|\mu|=n} h_\mu[X/(1-q)] f_\mu[1-q] \right) = \nabla e_n(\mathbf{x})|_{t=1},$$

as desired. □

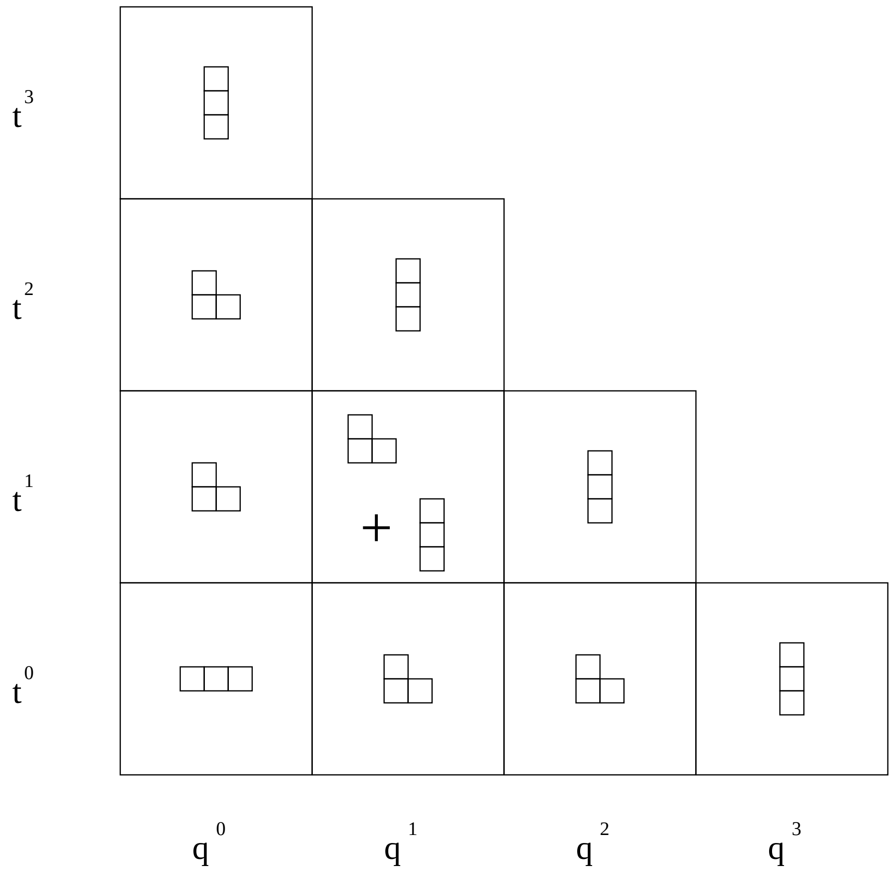


FIGURE 11. $\nabla e_3$

4.3. **First Remarks on Positivity.** Making some calculations, we see that $\nabla e_3 = s_{(3)} + (q+t+q^2+qt+t^2)s_{(21)} + (qt+q^3+q^2t+qt^2+t^3)s_{(111)}$. This is pictured in Figure 11.

Looking at the $s_{(111)} = e_3$ part gives us $C_3(q,t) = qt+q^3+q^2t+qt^2+t^3$, while taking the Hall inner product with $e_{(111)}$ gives us $P_3(q,t) = 1+2q+2t+2q^2+3qt+2t^2+q^3+q^2t+qt^2+t^3$, since $\langle s_{(3)}, e_{(111)}\rangle = \langle s_{(111)}, e_{(111)}\rangle = 1$, while $\langle s_{(21)}, e_{(111)}\rangle = 2$.

Notice that $P_3(q,t)$ and $C_3(q,t)$ are both polynomials in $q$ and $t$ with positive integer coefficients. This in turn follows from the coefficients of $s_\lambda$ in the Schur function expansion of $\nabla e_3$ all being polynomials with positive integer coefficients. This and further calculations suggest that $\langle \nabla e_n, s_\mu\rangle$ should always be a polynomial with positive integer coefficients. One can hope to prove this positivity in two ways. First, one can hope that $\nabla e_n$ has a combinatorial interpretation under which one can calculate $\langle \nabla e_n, s_\lambda\rangle$ by counting some set of objects (associated with the partition $\lambda$) with appropriate weights. More precisely, there should be combinatorially defined sets $\mathcal{S}_\lambda$ and functions $\mathrm{qwt}, \mathrm{twt} : \mathcal{S}_\lambda \to \mathbb{N}$ such that

$\langle \nabla e_n, s_\lambda \rangle = \sum_{s\in\mathcal{S}_\lambda} q^{\mathrm{qwt}(s)} t^{\mathrm{twt}(s)}$. Secondly, one can hope $\nabla e_n$ has a representation theoretic interpretation by which $\nabla e_n$ is the bi-graded Frobenius characteristic $F_{V_n}^{(q,t)}$ for some naturally defined family of bi-graded $S_n$ representations $V_n$.

Since the Macdonald polynomials $\widetilde{H}_\mu(\mathbf{x}; q, t)$ are also Schur-positive, that is, have only positive integer polynomial coefficients in their Schur function expansions, there should also be similar interpretations of the Macdonald polynomials.

At present, there are known interpretations of the Macdonald polynomials and of $\nabla e_n$ in terms of $S_n$-representation theory. Both $\nabla e_n$ and $\widetilde{H}_\mu$ turn out to be the Frobenius characteristics of certain finite dimensional quotients of the rings $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ which we will describe in the last lecture. Although these quotient rings can be defined in an elementary way, the existing proofs of these theorems require some fairly sophisticated algebraic geometry involving the Hilbert scheme of points in the plane [16, 17].

As for combinatorial interpretations, those relating to $\nabla e_n$ are known and proved only for $C_n(q,t)$ and some related specializations. Some recent conjectures have, however, shed further light on this subject. These will be the main topic of the final lecture.

### 4.4. **Exercises.**

(1) Prove that $\nabla e_n|_{t=0} = \widetilde{H}_{(n)}$, and that therefore $P_n(q,0) = \langle \widetilde{H}_{(n)}, e_{(1^n)} \rangle = [n]_q!$, where by definition $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ and $[k]_q = \frac{q^k-1}{q-1}$.

## 5. Positivity and Combinatorics?

### 5.1. **Representation theory of $\widetilde{H}_\mu(\mathbf{x}; q, t)$.** 

Recall the Frobenius characteristic of an $S_n$ representation $V$ is defined as

$$F_V(\mathbf{x}) = \sum_{|\mu|=n} (\dim V^{S_\mu}) m_\mu(\mathbf{x}).$$

Recall also that

$$F_{V_\lambda} = s_\lambda(\mathbf{x})$$

for the irreducible representation $V_\lambda$ and that Frobenius characteristic is additive on direct sums (of representations).

If $V$ is **graded**, that is, $V = \oplus_{i\in\mathbb{N}} V_i$ where each $V_i$ is an $S_n$ representation, then we can define

$$F_V(\mathbf{x}; q) = \sum_{i\in\mathbb{N}} F_{V_i}(\mathbf{x}) q^i.$$

Similarly, if $V$ is **bi-graded** with $V = \oplus_{i,j\in\mathbb{N}} V_{i,j}$, we can define

$$F_V(\mathbf{x}; q, t) = \sum_{i\in\mathbb{N}} F_{V_{i,j}}(\mathbf{x}) q^i t^j.$$

By construction, $\langle F_V(\mathbf{x}; q, t), s_\lambda \rangle \in \mathbb{N}[q,t]$ for every $\lambda$. Therefore, one method for showing that a symmetric function $f \in \Lambda_{\mathbb{Q}(q,t)}$ has the property that $\langle f, s_\lambda \rangle \in \mathbb{N}[q,t]$ for every $\lambda$ is to show that $f = F_V(\mathbf{x}; q, t)$ for some bi-graded representation $V$.

In this section we will construct this representation $V$ for $f = \widetilde{H}_\mu(\mathbf{x}; q, t)$, which shows that $\widetilde{K}_{\lambda\mu} \in \mathbb{N}[q,t]$. In the next section we will do the same for $f = \nabla e_n$. Although we will be able to explicitly describe these representations, the proof that they have the right Frobenius characteristic involves fairly sophisticated algebraic geometry involving the Hilbert scheme of points in the plane, and would require another entire series of lectures to present. No elementary proof that these representations have the right Frobenius characteristic is known.

Given a partition $\mu$ with $|\mu| = n$, let $\{(p_1, q_1), \ldots, (p_n, q_n)\}$ be the coordinates of the boxes in its diagram. Now define

$$\Delta_\mu(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = \det \left[ x_i^{p_j} y_i^{q_j} \right].$$

For example, for $\mu = (3, 2)$,

$$\Delta_{(3,2)}(\mathbf{x}, \mathbf{y}) = \det \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2y_2 & y_2^2 \\ 1 & x_3 & y_3 & x_3y_3 & y_3^2 \\ 1 & x_4 & y_4 & x_4y_4 & y_4^2 \\ 1 & x_5 & y_5 & x_5y_5 & y_5^2 \end{bmatrix}.$$

For $\mu = (1^n)$, only the $x$ variables are involved, and $\Delta_{(1^n)}(\mathbf{x}, \mathbf{y})$ is just the classical Vandermonde determinant

$$\Delta(\mathbf{x}) = \det \left[ x_i^{j-1} \right]_{i,j=1}^n = \prod_{i>j} (x_i - x_j).$$

(Note that the convention is for powers of $x$ to increase along the vertical axis in the partition diagram and for powers of $y$ to increase along the horizontal axis, contrary to the usual expectation for Cartesian coordinates. Our peculiar convention has become established in the literature because the rings $R_\mu$ we will soon define were first studied in the case of Hall-Littlewood polynomials, and these are conventionally written in terms of $t$ and the $x$-variables, setting $q$ and the $y$-variables to 0.)

Now let $S$ denote the ring $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, bi-graded so that its $(i,j)$-th graded piece consists of polynomials homogeneous of degree $j$ in the $x$ variables and degree $i$ in the $y$ variables. (In the lectures and in a number of places in the literature, $\mathbb{Q}$ is used instead of $\mathbb{C}$ here. This is an irrelevant difference since the representation theory of $S_n$ is exactly the same over the two fields. We have reverted to using $\mathbb{C}$ since that is more consistent with earlier lectures and the general study of representation theory.) Now for each partition $\mu$ with $|\mu| = n$, define an ideal $J_\mu$ of $S$ by

$$J_\mu = \left\{ f : f\left(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}\right) \Delta_\mu(\mathbf{x}, \mathbf{y}) = 0 \right\}.$$

In other words, $J_\mu$ consists of all polynomials that, when considered as partial differentiation operators, annihilate $\Delta_\mu$. Now let $R_\mu = S/J_\mu$.

The simplest example is $\mu = (1^n)$. As mentioned before, $\Delta_{(1^n)}$ is the classical Vandermonde determinant, and $J_{(1^n)} = \langle y_1, \ldots, y_n, e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})\rangle$. Therefore,

$$R_{(1^n)} = \mathbb{C}[\mathbf{x}]/\langle \mathbb{C}[\mathbf{x}]_+^{S_n}\rangle,$$

or, in words, the polynomial ring in the $x$ variables modulo the ideal generated by all homogeneous non-constant symmetric functions. This ring is known as the ring of **covariants**, and it is a classical theorem that $R_{(1^n)} \cong_{S_n} \mathbb{C}\cdot S_n \cong \mathbb{C}\uparrow_1^{S^n}$, and, furthermore, that

$$F_{R_{(1^n)}}(\mathbf{x}; q, t) = (1-t)(1-t^2)\cdots(1-t^n)h_n\left[X/(1-t)\right] = \widetilde{H}_{(1^n)}(\mathbf{x}; q, t).$$

Generalizing this, we have the following theorem.

**Theorem 8** ([16]). *There holds the identity*

$$F_{R_\mu}(\mathbf{x}; q, t) = \widetilde{H}_\mu(\mathbf{x}; q, t).$$

Since, as computed in Lecture 3, $\widetilde{H}_\mu(\mathbf{x}; 1, 1) = h_{(1^n)}(\mathbf{x})$, this means that $R_\mu \cong \mathbb{C}\cdot S_n$ as $S_n$ representations. In particular, $\dim(R_\mu) = n!$. This was the "$n!$ conjecture," which turned out to be the most difficult point in the proof of Theorem 8.

5.2. **Representation theory of $\nabla e_n$.** A different quotient of the ring $S$ gives a module whose Frobenius series is $\nabla e_n$. Let $J_n$ be the ideal of $S$ generated $\mathbb{C}[\mathbf{x},\mathbf{y}]_+^{S_n}$, that is, the ideal generated by all homogeneous non-constant functions symmetric with respect to the diagonal action of $S_n$ on the $x$ and $y$ variables. One set of generators for $J_n$ is the **polarized elementary symmetric functions**, defined by

$$e_{a,b}(\mathbf{x},\mathbf{y}) = \sum_{\substack{I,J\in\{1,\ldots,n\}\\ I\cap J=\emptyset\\ \#I=a,\#J=b}} \prod_{i\in I} x_i \prod_{j\in J} y_j.$$

Now let $R_n = \mathbb{C}/J_n$, the coinvariant ring for the diagonal action. We have the following theorem.

**Theorem 9** ([17]). *There holds the identity*

$$F_{R_n}(\mathbf{x}; q, t) = \nabla e_n.$$

**Corollary 3.** (1) $\nabla e_n \in \mathbb{N}[q,t]\cdot\{s_\lambda : |\lambda| = n\}$.
(2) $C_n(q,t) = \langle \nabla e_n, e_n\rangle = \sum_{i,j}\dim(R_n^\epsilon)_{i,j} t^i q^j \in \mathbb{N}[q,t]$.
(3) $P_n(q,t) = \langle \nabla e_n, e_{(1^n)}\rangle = \sum_{i,j}\dim(R_n)_{i,j} t^i q^j \in \mathbb{N}[q,t]$.

*Proof.* (1) holds because the Frobenius series of any (positively graded) $S_n$-module is in $\mathbb{N}[q,t]\cdot\{s_\lambda : |\lambda| = n\}$. Since $\langle f, e_n\rangle$ picks out the coefficient of $e_n = s_{(1^n)}$ in the expansion of $f$ in the Schur function basis, if $f = F_V$ for some $S_n$ representation $V$, $\langle f, e_n\rangle$ gives the multiplicity of the sign representation in $V$. Since the sign representation is 1-dimensional, (2) follows. Finally, for any $S_n$ representation $V$, $\langle F_V, e_{(1^n)}\rangle = \langle F_V, h_{(1^n)}\rangle$, which is the coefficient of $m_{(1^n)}$ in the monomial expansion of $F_V$. By definition, this is the dimension of the subspace of $V$ fixed by the trivial group, which is all of $V$, giving (3). □

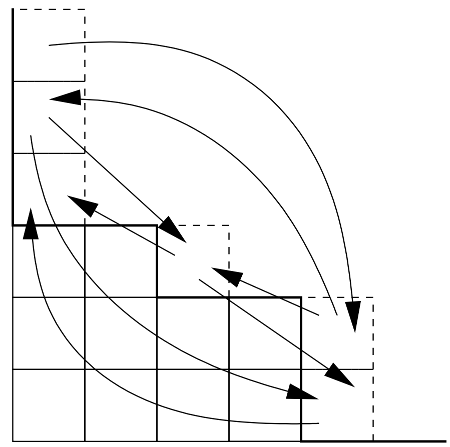

FIGURE 12. Attacking pairs of cells for $\lambda = (4, 4, 2)$ (and $n = 6$)

5.3. **Combinatorics of $\nabla e_n$.** In this section we discuss a combinatorial interpretation of $\nabla e_n$ in terms of the tableaux used to represent parking functions, although we will allow tableaux of any content instead of just content $(1^n)$. We will give two functions $\mathrm{qwt}_n, \mathrm{twt}_n : \bigcup_{\lambda \subseteq \delta(n)} \mathrm{SSYT}(\lambda + (1^n)/\lambda) \to \mathbb{N}$ such that, conjecturally, $\nabla e_n = \sum_T q^{\mathrm{qwt}_n(T)} t^{\mathrm{twt}_n(T)} x^T$. It will not be obvious at first glance that these are indeed symmetric functions, but that has been proven in [13]. Note that this is not exactly the desired combinatorial interpretation, as it gives an expansion of $\nabla e_n$ in terms of monomial symmetric functions rather than Schur functions, but it may be a useful first step.

Since setting $t = 1$ should give us $k_n(q)$, the desired function $\mathrm{qwt}_n$ should simply be $T \mapsto \binom{n}{2} - |\lambda|$, where $T$ is a tableau of skew shape $\lambda + (1^n)/\lambda$. The appropriate function $\mathrm{twt}_n$ is much more subtle. We will describe this function first in terms of a combinatorial interpretation of the $(q, t)$-Catalan numbers $C_n(q, t)$.

Let $\lambda \subseteq \delta(n)$, and $\widehat{\lambda} = \lambda + (1^n)/\lambda$. We say that a cell $(i, j) \in \widehat{\lambda}$ **attacks** $(i', j') \in \widehat{\lambda}$ if either $i + j = i' + j'$ and $j < j'$, or $i + j = i' + j' + 1$ and $j > j'$. More pictorially, a cell $c$ attacks $c'$ if either $c$ and $c'$ are on the same diagonal with $c$ to the left of $c'$, or $c$ is one diagonal above and strictly to the right of $c'$. Now simply let $\mathrm{twt}_n(\lambda) = \#\{(c, c') | c, c' \in \widehat{\lambda} \text{ and } c \text{ attacks } c'\}$. Figure 12 shows that $\mathrm{twt}_6((4, 4, 2)) = 9$.

Now we have the following theorem.

**Theorem 10** ([6, 13])**.** *There holds the identity*

$$C_n(q, t) = \sum_{\lambda \in \delta(n)} q^{\mathrm{qwt}_n(\lambda)} t^{\mathrm{twt}_n(\lambda)}.$$

Although $C_n(q, t)$ is invariant under switching $q$ and $t$, it is still an open problem to find an involution on partitions which would combinatorially explain this symmetry. More precisely, there should be a combinatorially defined involution $I$ such that $\mathrm{qwt}_n(I(\lambda)) = \mathrm{twt}_n(\lambda)$ and $\mathrm{twt}_n(I(\lambda)) = \mathrm{qwt}_n(\lambda)$, but no such involution is known.

Now we come to the conjectured combinatorial description of $\nabla e_n$. As stated earlier, let $T$ be a tableau of shape $\widehat{\lambda}$, and let $\mathrm{qwt}(T) = \mathrm{qwt}_n(\lambda) = \binom{n}{2} - |\lambda|$. Now use the notion of attack defined earlier to define $\mathrm{twt}(T) = \#\{(c, c')|c, c' \in \widehat{\lambda}, T(c) > T(c'), \text{ and } c \text{ attacks } c'\}$.

**Theorem 11.** *For each $\lambda \subseteq \delta(n)$,*

$$D_\lambda(\mathbf{x}; t) = \sum_{T \in \mathrm{SSYT}(\widehat{\lambda})} t^{\mathrm{twt}(T)} x^T$$

*is a symmetric function, and $D_\lambda(\mathbf{x}; t) \in \mathbb{N}[t] \cdot \{s_\lambda\}$.*

In fact, $D_\lambda(\mathbf{x}; t)$ is shown in [13] to be an example of an LLT polynomial, as defined by Lascoux, Leclerc and Thibon in [19].

**Conjecture 1.**

$$\nabla e_n = \sum_{\lambda \subseteq \delta(n)} q^{\mathrm{qwt}_n(\lambda)} D_\lambda(\mathbf{x}; t) = \sum_{\substack{\lambda \subseteq \delta(n) \\ T \in \mathrm{SSYT}(\widehat{\lambda})}} q^{\mathrm{qwt}(T)} t^{\mathrm{twt}(T)} x^T.$$

This conjecture, if true, would have the following corollary; recall that a tableau of skew shape $\widehat{\lambda}$ and content $(1^n)$ corresponds directly to a parking function.

**Corollary 4** (to Conjecture 1)**.**

$$P_n(q, t) = \langle \nabla e_n, h_{(1^n)} \rangle = \sum_{\substack{\lambda \subseteq \delta(n) \\ T \in \mathrm{SSYT}(\widehat{\lambda}, (1^n))}} q^{\mathrm{qwt}(T)} t^{\mathrm{twt}(T)}.$$

It is also mysterious why this should be symmetric under switching $q$ and $t$, and what connection these combinatorics may have with the ring $R_n$ described above.

It can at least be shown that insofar as $C_n(q, t)$ is concerned, Conjecture 1 agrees with Theorem 10. First of all, in keeping with how $\omega$ usually acts on objects indexed by tableaux, it can be shown that

$$\omega D_\lambda(\mathbf{x}; t) = \sum_{T \in \mathrm{SSYT}_-(\widehat{\lambda})} t^{\mathrm{twt}(T)} x^T,$$

where $\mathrm{SSYT}_-(\widehat{\lambda})$ denotes the set of **imaginary** tableaux $T$ of shape $\widehat{\lambda}$, whose "imaginary" entries increase weakly along columns and strictly along rows (the requirement on rows is irrelevant in the case of the shapes $\widehat{\lambda}$ occurring in the above formula). For imaginary tableaux, $\mathrm{twt}(T)$ is redefined to allow a contribution from a pair of cells $(c, c')$ if $c$ attacks $c$ and $T(c) \geq T(c')$ (instead of requiring $T(c) > T(c')$).

For each $\lambda \subseteq \delta(n)$, there is a unique imaginary tableau $T$ of shape $\widehat{\lambda}$ with all entries being $\overline{1}$, and for this imaginary tableau, $\mathrm{twt}(T) = \mathrm{twt}_n(\lambda)$. Therefore, $\langle \sum_{\lambda \in \delta(n)} q^{\mathrm{qwt}_n(\lambda)} \omega D_\lambda(\mathbf{x}; t), h_n \rangle = C_n(q, t)$. Since $\langle \sum_{\lambda \subseteq \delta(n)} q^{\mathrm{qwt}_n(\lambda)} D_\lambda(\mathbf{x}; t), e_n \rangle = \langle \sum_{\lambda \subseteq \delta(n)} q^{\mathrm{qwt}_n(\lambda)} \omega D_\lambda(\mathbf{x}; t), h_n \rangle$, the theorem for $C_n(q, t)$ agrees with the conjecture.

5.4. **Combinatorics of $\widetilde{H}_\mu(\mathbf{x};q,t)$.** This topic was addressed, not in these lectures, but in a satellite lecture by Jim Haglund. We will comment briefly on the latest developments. Haglund conjectured, and discussed in his lecture, a combinatorial formula analogous to Conjecture 1 for the monomial expansion of $\widehat{H}_\mu(\mathbf{x};q,t)$. Like Conjecture 1, Haglund's formula can be expressed as a $q$-weighted sum of LLT polynomials in the parameter $t$, which shows in particular that it is in fact a symmetric function. (This also shows, subject to a general Schur-positivity conjecture for LLT polynomials, that Haglund's formula is Schur-positive. The special case of the LLT positivity conjecture required for Schur-positivity of the formula in Conjecture 1 is known to hold.) Between the the PCMI meeting and the preparation of the final version of these notes, Haglund's conjecture has been proven by Haglund, Haiman and Loehr, who verify directly that Haglund's formula satisfies the defining axioms for Macdonald polynomials in Theorem-Definition 2. For details, see [12].

5.5. **Exercises.** Show that Conjecture 1 gives the correct predictions for the following.

(1) $\nabla e_n \mid_{t=1} = k_n(q)$
(2) $\nabla e_n \mid_{q=0} = (1-t)(1-t^2)\cdots(1-t^n)h_n\left[X/(1-t)\right] = \widetilde{H}_{(1^n)}(\mathbf{x};q,t)$
(3) $\nabla e_n \mid_{t=0}$ (This one is trickier.)

Proving that the conjecture gives the correct prediction for $\nabla e_n \mid_{q=1}$ is an open problem. Using the first exercise, this is presumably a special case for showing combinatorially that the conjecture gives a function symmetric under switching $q$ and $t$.

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