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RESEARCH ARTICLE

A Shuffle Theorem for Paths Under Any Line

Jonah Blasiak[1], Mark Haiman[2], Jennifer Morse[3], Anna Pun[4] and George H. Seelinger[5]

[1]Dept. of Mathematics, Drexel University, Philadelphia, PA; E-mail: jblasiak@gmail.com.
[2]Dept. of Mathematics, University of California, Berkeley, CA; E-mail: mhaiman@math.berkeley.edu.
[3]Dept. of Mathematics, University of Virginia, Charlottesville, VA; E-mail: morsej@virginia.edu.
[4]Dept. of Mathematics, CUNY-Baruch College, New York, NY; E-mail: anna.pun@baruch.cuny.edu.
[5]Dept. of Mathematics, University of Michigan, Ann Arbor, MI; E-mail: ghseeli@umich.edu.
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Abstract
We generalize the Shuffle Theorem and its (km, kn) version, as conjectured by Haglund et al. and Bergeron et al., and proven by Carlsson and Mellit, and Mellit, respectively. In our version the (km, kn) Dyck paths on the combinatorial side are replaced by lattice paths lying under a line segment whose x and y intercepts need not be integers, and the algebraic side is given either by a Schiffmann algebra operator formula or an equivalent explicit raising operator formula. We derive our combinatorial identity as the polynomial truncation of an identity of infinite series of GL_l characters, expressed in terms of infinite series versions of LLT polynomials. The series identity in question follows from a Cauchy identity for non-symmetric Hall-Littlewood polynomials.

Contents

1. Introduction

1.1. Overview

The *Shuffle Theorem*, conjectured by Haglund et al. [16] and proven by Carlsson and Mellit [7], is a combinatorial formula for the symmetric polynomial ∇e_k as a sum over LLT polynomials indexed by Dyck paths—that is, lattice paths from $(0,k)$ to $(k,0)$ that lie weakly below the line segment connecting these two points. Here e_k is the k-th elementary symmetric function, and ∇ is an operator on symmetric functions with coefficients in $\mathbb{Q}(q,t)$ that arises in the theory of Macdonald polynomials [3].

The polynomial ∇e_k is significant because it describes the character of the ring R_k of diagonal coinvariants for the symmetric group S_k [19, Proposition 3.5]. The character of R_k had been conjectured in the early 1990s to have surprising connections with the enumeration of combinatorial objects such as trees and Dyck paths—for instance, its dimension is equal to the number $(k+1)^{k-1}$ of rooted trees on $k+1$ labelled vertices, and the multiplicity of the sign character is equal to the Catalan number C_k. A summary of these early conjectures, contributed by a number of people, can be found in [20]. More precisely, ∇e_k describes the character of R_k as a doubly graded S_k module. The double grading in R_k thus gives rise to (q,t)-analogs of the numbers that enumerate the associated combinatorial objects. The conjectures connect specializations of these (q,t)-analogs with previously known q-analogs in combinatorics.

Using results in [11], the whole suite of earlier combinatorial conjectures follows from the character formula ∇e_k and the Shuffle Theorem.

Along with the formula for ∇e_k given by the Shuffle Theorem, Haglund et al. conjectured a combinatorial formula for $\nabla^m e_k$ as a sum over LLT polynomials indexed by lattice paths below the line segment from $(0,k)$ to $(km,0)$. Extending this, Bergeron et al. [4] conjectured, and Mellit [28] proved, an identity giving the symmetric polynomial $e_k[-MX^{m,n}]\cdot 1$ as a sum over LLT polynomials indexed by lattice paths below the segment from $(0,kn)$ to $(km,0)$, for any pair of positive integers expressed in the form (km,kn) with m,n coprime. Here $e_k[-MX^{m,n}]$, where $M=(1-q)(1-t)$, is our notation for a certain element of the elliptic Hall algebra $\mathcal{E}$ of Burban and Schiffmann [6], which acts on symmetric functions in such a way that for $n=1$ we have $e_k[-MX^{m,1}]\cdot 1=\nabla^m e_k$, as explained in §3.

In this paper we prove an even more general version of the Shuffle Theorem, involving a sum over LLT polynomials indexed by lattice paths lying under the line segment between arbitrary points $(0, s)$ and $(r, 0)$ on the positive real axes, and reducing to the theorem of Bergeron et al. and Mellit when $(r, s) = (km, kn)$ are integers.

Our generalized Shuffle Theorem (Theorem 5.5.1) is an identity

$$D_{\mathbf{b}} \cdot 1 = \sum_{\lambda} t^{a(\lambda)}\, q^{\mathrm{dinv}_p(\lambda)}\, \omega(\mathcal{G}_{\nu(\lambda)}(X; q^{-1})), \tag{1}$$

whose ingredients we now summarize briefly, deferring full details to later parts of the paper.

The sum on the right hand side of (1) is over lattice paths λ from $(0, \lfloor s \rfloor)$ to $(\lfloor r \rfloor, 0)$ that lie below the line segment from $(0, s)$ to $(r, 0)$. The quantity $a(\lambda)$ is the number of lattice squares enclosed between the path λ and the highest such path δ (see Example 5.5.3 and Figure 5). We set $p = s/r$ and define $\mathrm{dinv}_p(\lambda)$ to be the number of 'p-balanced' hooks in the (French style) Young diagram enclosed by λ and the x and y axes. A hook is p-balanced if a line of slope $-p$ passes through the segments at the top of its leg and the end of its arm (Definition 5.4.1 and Figure 2).

In the remaining factor $\omega(\mathcal{G}_{\nu(\lambda)}(X; q^{-1}))$, the function $\mathcal{G}_\nu(X; q)$ is an 'attacking inversions' LLT polynomial (Definition 4.1.2), and $\omega(h_\mu) = e_\mu$ is the standard involution on symmetric functions. The index $\nu(\lambda)$ is a tuple of one-row skew Young diagrams of the same lengths as runs of contiguous south steps in λ, arranged so that the reading order on boxes of $\nu(\lambda)$ corresponds to the ordering on south steps in λ given by decreasing values of $y + px$ at the upper endpoint of each step.

The operator $D_{\mathbf{b}} = D_{b_1,\ldots,b_l}$ on the left hand side of (1) is one of a family of special elements defined by Negut [30] in the Schiffmann algebra $\mathcal{E}$. Letting δ again denote the highest path under the given line segment, the index $\mathbf{b}$ is defined by taking b_i to be the number of south steps in δ on the line $x = i - 1$.

To recover the previously known cases of the theorem, we take $s = kn$ and r slightly larger than km, so $p = n/m - \epsilon$ for a small $\epsilon > 0$. The segment from $(0, s)$ to $(r, 0)$ has the same lattice paths below it as the segment from $(0, kn)$ to $(km, 0)$, and $\mathrm{dinv}_p(\lambda)$ reduces to the version of $\mathrm{dinv}(\lambda)$ in the original conjectures. The element $D_{\mathbf{b}}$ associated to the highest path δ below the segment from $(0, kn)$ to $(km, 0)$ is equal to $e_k[-MX^{m,n}]$. Hence, (1) reduces to the (km, kn) Shuffle Theorem.

1.2. Preview of the proof

We prove our generalized Shuffle Theorem by a remarkably simple method, which we now outline to help orient the reader in following the details. In §3 we will see that the left hand side of (1), after applying ω and evaluating in $l = \lfloor r \rfloor + 1$ variables $x_1, \ldots, x_l$, becomes the polynomial part

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \ldots, x_l) = \mathcal{H}_{\mathbf{b}}(x)_{\mathrm{pol}} \tag{2}$$

of an explicit infinite series of GL_l characters

$$\mathcal{H}_{\mathbf{b}}(x) = \sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1<j} (1 - q\, t\, x_i/x_j)}{\prod_{i<j} \big((1 - x_j/x_i)(1 - q\, x_i/x_j)(1 - t\, x_i/x_j) \big)} \right). \tag{3}$$

When ν is a tuple of one-row skew shapes $(\beta_i)/(\alpha_i)$, the LLT polynomial $\mathcal{G}_\nu(x; q^{-1})$ is equal, up to a factor of the form q^d, to the polynomial part of an infinite GL_l character series

$$q^d\, \mathcal{G}_\nu(x; q^{-1}) = \mathcal{L}_{\beta/\alpha}(x; q)_{\mathrm{pol}} \tag{4}$$

defined by Grojnowski and the second author [14]. In Theorem 5.3.1 we establish an identity of infinite series

$$\mathcal{H}_{\mathbf{b}}(x) = \sum_{a_1,\ldots,a_{l-1}\geq 0} t^{|\mathbf{a}|}\mathcal{L}^{\sigma}_{((b_l,\ldots,b_1)+(0;\mathbf{a}))/(\mathbf{a};0)}(x;q), \tag{5}$$

where $\mathcal{L}^{\sigma}_{\beta/\alpha}(x;q)$ is a 'twisted' variant of $\mathcal{L}_{\beta/\alpha}(x;q)$ (see §4). Then (1) follows once we see that the polynomial part of the right hand side of (5) is equal to the right hand side of (1) with the ω omitted.

In fact, (4) holds when $\alpha_i \leq \beta_i$ for all i, and otherwise $\mathcal{L}_{\beta/\alpha}(x;q)_{\mathrm{pol}} = 0$. When we take the polynomial part in (5), this leaves a non-vanishing term $t^{|\mathbf{a}|}q^d\mathcal{G}_{\nu(\lambda)}(x;q^{-1})$ for each path λ under the given line segment, with $\mathbf{a}$ giving the number of lattice squares in each column between λ and the highest path δ, so $t^{|\mathbf{a}|} = t^{a(\lambda)}$. The factor q^d from (4) turns out to be precisely $q^{\mathrm{dinv}_p(\lambda)}$, yielding (1).

Finally, the infinite series identity (5) from Theorem 5.3.1 is essentially a corollary to a Cauchy identity for non-symmetric Hall-Littlewood polynomials, Theorem 5.1.1. This Cauchy formula is quite general and can be applied in other situations, some of which we will take up elsewhere.

1.3. Further remarks

(i) The conjectures in [4, 16] and proofs in [7, 28] use a version of $\mathrm{dinv}(\lambda)$ that coincides with $\mathrm{dinv}_p(\lambda)$ for $p = n/m - \epsilon$. Alternatively, one can tilt the segment from $(0, kn)$ to $(km, 0)$ in the other direction, taking $r = km$ and s slightly larger than kn, to get a version of the original conjectures with a variant of $\mathrm{dinv}(\lambda)$ that coincides with $\mathrm{dinv}_p(\lambda)$ for $p = n/m+\epsilon$. Our theorem implies this version as well.

(ii) Carlsson and Mellit [7, 28] prove more general 'compositional' versions of the classical and (km, kn) Shuffle Theorems, as conjectured by Haglund, Zabrocki and the third author [17] in the classical case, and Bergeron et al. [4] in the (km, kn) case. Although our results here do not cover the compositional versions, Mellit has pointed out to us privately that [28, Theorem 5.11] may contain clues to a possible compositional extension of our generalized Shuffle Theorem.

(iii) By [16, Proposition 5.3.1], the LLT polynomials $\mathcal{G}_{\nu(\lambda)}(x;q)$ in (1) are q Schur positive, meaning that their coefficients in the basis of Schur functions belong to $\mathbb{N}[q]$. The right hand side of (1) is therefore q, t Schur positive. In the cases corresponding to the (km, kn) Shuffle Theorem for $k = 1$, this can also be seen from the representation theoretic interpretation of the right hand side given by Hikita [21].

Identity (1) therefore implies that the expression $D_{\mathbf{b}} \cdot 1$ on the left hand side is q, t Schur positive. In the cases where the left hand side coincides with $\nabla^m e_k$, this can be explained using the representation theoretic interpretations of ∇e_k in [19] and $\nabla^m e_k$ in [16, Proposition 6.1.1]. In §7 we conjecture a more general condition for $D_{\mathbf{b}} \cdot 1$ to be q, t Schur positive.

(iv) The algebraic left hand side of (1) is manifestly symmetric in q and t. Hence, the combinatorial right hand side is also symmetric in q and t. No direct combinatorial proof of this symmetry is currently known.

2. Symmetric functions and GL_l characters

This section serves to fix notation and terminology for standard notions concerning symmetric functions and characters of the general linear groups GL_l.

2.1. Symmetric functions

Integer partitions are written $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$, possibly with trailing zeroes. We set $|\lambda| = \lambda_1 + \cdots + \lambda_l$ and let $\ell(\lambda)$ be the number of non-zero parts. We also define

$$n(\lambda) = \sum_i (i-1)\lambda_i. \tag{6}$$

The transpose of λ is denoted λ^*.

The partitions of a given integer n are partially ordered by

$$\lambda \leq \mu \quad \text{if} \quad \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \text{ for all } k, \tag{7}$$

where the sums include trailing zeroes for $k > \ell(\lambda)$ or $k > \ell(\mu)$.

The (French style) Young or Ferrers diagram of a partition λ is the set of lattice points $\{(i,j) \mid 1 \leq j \leq \ell(\lambda),\ 1 \leq i \leq \lambda_j\}$. We often identify λ and its diagram with the set of lattice squares, or *boxes*, with northeast corner at a point $(i,j) \in \lambda$. A *skew diagram* (or *skew shape*) λ/μ is the difference between the diagram of a partition λ and that of a partition $\mu \subseteq \lambda$ contained in it. The *diagram generator* of λ is the polynomial

$$B_\lambda(q,t) = \sum_{(i,j)\in\lambda} q^{i-1}\, t^{j-1}. \tag{8}$$

Let $\Lambda = \Lambda_{\mathbf{k}}(X)$ be the algebra of symmetric functions in an infinite alphabet of variables $X = x_1, x_2, \ldots$, with coefficients in the field $\mathbf{k} = \mathbb{Q}(q,t)$. We follow Macdonald's notation [26] for various graded bases of Λ, such as the elementary symmetric functions $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$, complete homogeneous symmetric functions $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$, power-sums $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$, monomial symmetric functions m_λ and Schur functions s_λ. The involutory $\mathbf{k}$-algebra automorphism $\omega\colon \Lambda \to \Lambda$ mentioned in the introduction may be defined by any of the formulas

$$\omega\, e_k = h_k, \quad \omega\, h_k = e_k, \quad \omega\, p_k = (-1)^{k-1} p_k, \quad \omega\, s_\lambda = s_{\lambda^*}. \tag{9}$$

We also need the symmetric bilinear inner product $\langle -, - \rangle$ on Λ defined by any of

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}, \qquad \langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}, \qquad \langle p_\lambda, p_\mu \rangle = z_\lambda\, \delta_{\lambda,\mu}, \tag{10}$$

where $z_\lambda = \prod_i r_i!\, i^{r_i}$ if $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$.

We write $f^\bullet$ for the operator of multiplication by a function f. Otherwise, the custom of writing f for both the operator and the function would make it hard to distinguish between operator expressions such as $(\omega f)^\bullet$ and $\omega \cdot f^\bullet$. When f is a symmetric function, we write $f^\perp$ for the $\langle -, - \rangle$ adjoint of $f^\bullet$.

2.2. Plethystic evaluation

We briefly recall the device of *plethystic evaluation.* If A is an expression in terms of indeterminates, such as a polynomial, rational function, or formal series, we define $p_k[A]$ to be the result of substituting a^k for every indeterminate a occurring in A. We define $f[A]$ for any $f \in \Lambda$ by substituting $p_k[A]$ for p_k in the expression for f as a polynomial in the power-sums p_k, so that $f \mapsto f[A]$ is a homomorphism.

The variables q, t from our ground field $\mathbf{k}$ count as indeterminates.

As a simple example, the plethystic evaluation $f[x_1 + \cdots + x_l]$ is just the ordinary evaluation $f(x_1, \ldots, x_l)$, since $p_k[x_1 + \cdots + x_l] = x_1^k + \cdots + x_l^k$. This also works in infinitely many variables.

When $X = x_1, x_2, \ldots$ is the name of an infinite alphabet of variables, we use $f(X)$, with round brackets, as an abbreviation for $f(x_1, x_2, \ldots) \in \Lambda(X)$. In this situation we also make the convention that *when X appears inside plethystic brackets, it means $X = x_1 + x_2 + \cdots$.* With this convention, $f[X]$ is another way of writing $f(X)$.

As a second example and caution to the reader, the formula in (9) for $\omega\, p_k$ implies the identity $\omega f(X) = f[-zX]|_{z=-1}$. Note that $f[-zX]|_{z=-1}$ does not reduce to $f(X)$, as it might at first appear, since specializing the indeterminate z to a number does not commute with plethystic evaluation.

Plethystic evaluation of a symmetric infinite series is allowed if the result converges as a formal series. The series

$$\Omega = 1 + \sum_{k>0} h_k = \exp \sum_{k>0} \frac{p_k}{k}, \quad \text{or} \quad \Omega[a_1 + a_2 + \cdots - b_1 - b_2 - \cdots] = \frac{\prod_i (1 - b_i)}{\prod_i (1 - a_i)} \tag{11}$$

is particularly useful. The classical Cauchy identity can be written using this notation as

$$\Omega[XY] = \sum_\lambda s_\lambda[X] s_\lambda[Y]. \tag{12}$$

Taking the inner product with $f(X)$ in (12) yields $f[A] = \langle \Omega[AX], f(X) \rangle$, which implies

$$\langle \Omega[AX]\Omega[BX], f(X) \rangle = f[A + B] = \langle \Omega[BX], f[X + A] \rangle. \tag{13}$$

As B is arbitrary, $\Omega[BX]$ is in effect a general symmetric function, so (13) implies

$$\Omega[AX]^\perp f(X) = f[X + A]. \tag{14}$$

Note that although $\Omega[AX]^\perp = \sum_k h_k[AX]^\perp$ is an infinite series, it converges formally as an operator applied to any $f \in \Lambda(X)$, since $h_k[AX]^\perp$ has degree $-k$, and so kills f for $k \gg 0$.

Identifying Λ with a polynomial ring in the power-sums p_k, we have

$$p_k^\perp = k \frac{\partial}{\partial p_k}. \tag{15}$$

In fact, taking $A = z$ and $f = p_k$ in (14) shows that $\exp(\sum (p_k^\perp z^k)/k)$ is the operator that substitutes $p_k + z^k$ for p_k in any polynomial $f(p_1, p_2, \ldots)$. This operator can also be written $\exp(\sum z^k \frac{\partial}{\partial p_k})$, giving (15).

Another consequence of (14) is the operator identity

$$\Omega[AX]^\perp\, \Omega[BX]^\bullet = \Omega[AB]\, \Omega[BX]^\bullet\, \Omega[AX]^\perp \tag{16}$$

with notation $\Omega[BX]^\bullet$ as in §2.1.

2.3. GL_l characters

The weight lattice of GL_l is $X = \mathbb{Z}^l$, with Weyl group $W = S_l$ permuting the coordinates. Letting $\varepsilon_1, \ldots, \varepsilon_l$ be unit vectors, the positive roots are $\varepsilon_i - \varepsilon_j$ for $i < j$, with simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, l-1$. The standard pairing on $\mathbb{Z}^l$ in which the ε_i are orthonormal identifies the dual lattice X^* with X. Under this identification, the coroots coincide with the roots, and the simple coroots $\alpha_i^\vee$ with the simple roots. A weight $\lambda \in \mathbb{Z}^l$ is dominant if $\lambda_1 \geq \cdots \geq \lambda_l$; a weight is regular (has trivial stabilizer in S_l) if $\lambda_1, \ldots, \lambda_l$ are distinct.

A *polynomial weight* is a dominant weight λ such that $\lambda_l \geq 0$. In other words, polynomial weights of GL_l are integer partitions of length at most l.

The algebra of virtual GL_l characters $(\mathbf{k}X)^W$ can be identified with the algebra of symmetric Laurent polynomials $\mathbf{k}[x_1^{\pm 1}, \dots, x_l^{\pm 1}]^{S_l}$. If λ is a polynomial weight, the irreducible character χ_λ is equal to the Schur function $s_\lambda(x_1, \dots, x_l)$. Given a virtual GL_l character $f(x) = f(x_1, \dots, x_l) = \sum_\lambda c_\lambda \chi_\lambda$, we denote the partial sum over polynomial weights λ by $f(x)_{\mathrm{pol}}$ (this is different from taking the polynomial terms of $f(x)$ considered as a Laurent polynomial). Thus, $f(x)_{\mathrm{pol}}$ is a symmetric polynomial in l variables. We also use this notation for infinite formal sums $f(x)$ of irreducible GL_l characters, in which case $f(x)_{\mathrm{pol}}$ is a symmetric formal power series.

The Weyl symmetrization operator for GL_l is

$$\boldsymbol{\sigma}\, f(x_1, \dots, x_l) = \sum_{w \in S_l} w \left(\frac{f(x)}{\prod_{i<j} (1 - x_j/x_i)} \right). \tag{17}$$

For dominant weights λ, the Weyl character formula can be written $\chi_\lambda = \boldsymbol{\sigma}(x^\lambda)$.

Fix a weight ρ such that $\langle \alpha_i^\vee, \rho \rangle = 1$ for each simple coroot $\alpha_i^\vee$, e.g., $\rho = (l-1, \dots, 1, 0)$. Although ρ is only unique up to adding a constant vector, all formulas in which ρ appears will be independent of the choice. Let μ be any weight, not necessarily dominant. If $\mu + \rho$ is not regular, then $\boldsymbol{\sigma}(x^\mu) = 0$. Otherwise, if $w \in S_l$ is the unique permutation such that $w(\mu + \rho) = \lambda + \rho$ for λ dominant,

$$\boldsymbol{\sigma}(x^\mu) = (-1)^{\ell(w)} \chi_\lambda. \tag{18}$$

The following identities are useful for working with the Weyl symmetrization operator. Here and after, $\langle - \rangle\, \Psi$ stands for a coefficient in an expression Ψ, relative to an expansion which will be clear from the context. Thus, in (20), $\langle \chi_\lambda \rangle$ denotes the coefficient of an irreducible GL_n character, while $\langle x^0 \rangle$ denotes the constant term in the variables x_i.

Lemma 2.3.1. *For any GL_l weights λ, μ and Laurent polynomial $\phi(x) = \phi(x_1, \dots, x_l)$, we have*

$$\overline{\chi_\lambda} \prod_{i<j} (1 - x_i/x_j) = \sum_{w \in S_l} (-1)^{\ell(w)} x^{-w(\lambda+\rho)+\rho}, \tag{19}$$

$$\langle \chi_\lambda \rangle\, \boldsymbol{\sigma}(\phi(x)) = \langle x^0 \rangle\, \overline{\chi_\lambda}\, \phi(x) \prod_{i<j} (1 - x_i/x_j), \tag{20}$$

$$\boldsymbol{\sigma}(x^\mu)_{\mathrm{pol}} = \langle z^{-\mu} \rangle \Omega[\overline{Z} X] \prod_{i<j} (1 - z_i/z_j) \tag{21}$$

in alphabets $X = x_1 + \cdots + x_l$ and $Z = z_1 + \cdots + z_l$, where $\overline{Z} = z_1^{-1} + \cdots + z_l^{-1}$.

Proof. Identity (19) follows directly from the Weyl character formula. To prove (20), by linearity it suffices to verify the formula when $\phi(x) = x^\mu$ is any Laurent monomial. Then using (19), the right side becomes $\langle x^{-\mu} \rangle\, \overline{\chi_\lambda} \prod_{i<j} (1 - x_i/x_j) = \langle x^{-\mu} \rangle \sum_{w \in S_l} (-1)^{\ell(w)} x^{-w(\lambda+\rho)+\rho}$. This is $(-1)^{\ell(w)}$ if $\mu + \rho = w(\lambda + \rho)$, or zero if there is no such w, which agrees with $\langle \chi_\lambda \rangle\, \boldsymbol{\sigma}(x^\mu)$.

The last identity is then proved from the Cauchy identity (12) followed by (20) (applied with z in place of x on the right):

$$\begin{aligned} \langle z^{-\mu} \rangle \Omega[\overline{Z} X] \prod_{i<j} (1 - z_i/z_j) &= \sum_\lambda s_\lambda(X) \cdot \langle z^{-\mu} \rangle s_\lambda[\overline{Z}] \prod_{i<j} (1 - z_i/z_j) \\ &= \sum_\lambda s_\lambda(X) \cdot \langle \chi_\lambda \rangle \boldsymbol{\sigma}(x^\mu) = \boldsymbol{\sigma}(x^\mu)_{\mathrm{pol}}. \end{aligned} \tag{22}$$

□

2.4. Hall-Littlewood symmetrization

Given a Laurent polynomial $\phi(x_1, \ldots, x_l)$, we define

$$\mathbf{H}_q(\phi(x)) = \boldsymbol{\sigma}\Big(\frac{\phi(x)}{\prod_{i<j}(1 - q\, x_i/x_j)}\Big) = \sum_{w \in S_l} w \left(\frac{\phi(x)}{\prod_{i<j}((1 - x_j/x_i)(1 - q\, x_i/x_j))} \right). \tag{23}$$

Here and in similar raising operator formulas elsewhere, the factors $1/(1 - q\, x_i/x_j)$ are to be understood as geometric series, making $\mathbf{H}_q(\phi(x))$ an infinite formal sum of irreducible GL_l characters with coefficients in $\mathbf{k}$. Since $1/(1 - q\, x_i/x_j)$ is a power series in q, if we expand the coefficients of $\phi(x)$ as formal Laurent series in q, then $\mathbf{H}_q(\phi(x))$ becomes a formal Laurent series in q over virtual GL_l characters. This is how the last formula in (23) should be interpreted.

Remark 2.4.1. The dual Hall-Littlewood polynomials, defined by $H_\mu(X; q) = \sum_\lambda K_{\lambda,\mu}(q) s_\lambda$, where $K_{\lambda,\mu}(q)$ are the q-Kostka coefficients, are given in l variables by $H_\mu(x_1, \ldots, x_l; q) = \mathbf{H}_q(x^\mu)_{\mathrm{pol}}$. This explains our terminology.

3. The Schiffmann algebra

3.1. Overview

We recall here some definitions and results from [6, 8, 30, 31, 32] concerning the elliptic Hall algebra $\mathcal{E}$ of Burban and Schiffmann [6] (or *Schiffmann algebra*, for short) and its action on the algebra of symmetric functions Λ, constructed by Feigin and Tsymbaliuk [8] and Schiffmann and Vasserot [32].

From a certain point of view, this material is unnecessary: two of the three quantities equated by (1) and (2) are defined without reference to the Schiffmann algebra, and we could take "Shuffle Theorem" to mean the identity between these two, namely

$$\mathcal{H}_{\mathbf{b}}(x)_{\mathrm{pol}} = \sum_\lambda t^{a(\lambda)}\, q^{\mathrm{dinv}_p(\lambda)}\, \mathcal{G}_{\nu(\lambda)}(x_1, \ldots, x_l; q^{-1}), \tag{24}$$

with $\mathcal{H}_{\mathbf{b}}(x)$ as in (3). This identity still has the virtue of equating the combinatorial right hand side, involving Dyck paths and LLT polynomials, with a simple algebraic left hand side that is manifestly symmetric in q and t. Furthermore, our proof of (1) in Theorem 5.5.1 proceeds by combining (2) with a proof of (24) (via Theorem 5.3.1) that makes no use of the Schiffmann algebra.

What we need the Schiffmann algebra for is to provide the link between our Shuffle Theorem and the classical and (km, kn) versions. Indeed, the very definition of the algebraic side in the (km, kn) Shuffle Theorem is $e_k[-MX^{m,n}] \cdot 1$ for a certain operator $e_k[-MX^{m,n}] \in \mathcal{E}$, while the classical Shuffle Theorems refer implicitly to the same operators through the identity $\nabla^m e_k = e_k[-MX^{m,1}] \cdot 1$.

In this section, we review what is needed to relate the symmetric functions $\nabla^m e_k$ and $(\mathcal{H}_{\mathbf{b}})_{\mathrm{pol}}$ to the action of the elements $e_k[-MX^{m,n}]$ and $D_{\mathbf{b}}$ on Λ. For ease of reference, we have collected most of the statements that will be used elsewhere in the paper in §3.7, after the technical development in §§3.2–3.6.

3.2. The algebra $\mathcal{E}$

Let $\mathbf{k} = \mathbb{Q}(q, t)$. The Schiffmann algebra $\mathcal{E}$ is generated by subalgebras $\Lambda(X^{m,n})$ isomorphic to the algebra $\Lambda_{\mathbf{k}}$ of symmetric functions, one for each pair of coprime integers (m, n), and a central Laurent polynomial subalgebra $\mathbf{k}[c_1^{\pm 1}, c_2^{\pm 1}]$. A full presentation of $\mathcal{E}$ in our notation can be found in [5, §3.2]. Here we only use a few of the defining relations, invoking them where needed.

For purposes of comparison with [6, 31, 32], our notation (on the left hand side of each formula) is related as follows to that in [6, Definition 6.4] (on the right hand side). Note that our indices $(m,n) \in \mathbb{Z}^2$ correspond to transposed indices (n,m) in [6].

$$\begin{gathered} q = \sigma^{-1}, \quad t = \bar{\sigma}^{-1}, \\ c_1^m c_2^n = \kappa_{n,m}^{-2}, \\ \omega p_k(X^{m,n}) = \kappa_{kn,km}^{\varepsilon_{n,m}} u_{kn,km}, \\ e_k[-\widehat{M} X^{m,n}] = \kappa_{kn,km}^{\varepsilon_{n,m}} \theta_{kn,km}, \end{gathered} \tag{25}$$

where $\varepsilon_{n,m}$, which is equal to $(1-\epsilon_{n,m})/2$ in the notation of [6], is given by

$$\varepsilon_{n,m} = \begin{cases} 1 & n < 0 \text{ or } (m,n) = (-1,0) \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

The expression $e_k[-\widehat{M} X^{m,n}]$ in (25) uses plethystic substitution (§2.2) with

$$\widehat{M} = (1 - \frac{1}{q\,t}) M \quad \text{where} \quad M = (1-q)(1-t). \tag{27}$$

The quantity M will be referred to repeatedly.

3.3. Action of $\mathcal{E}$ on Λ

A natural action of $\mathcal{E}$ by operators on $\Lambda(X)$ has been constructed in [8, 32]. Actually these references give several different normalizations of essentially the same action. The action we use is a slight variation of the action in [32, Theorem 3.1].

To write it down we need to recall some notions from the theory of Macdonald polynomials. Let $\tilde{H}_\mu(X;q,t)$ denote the modified Macdonald polynomials [11], which can be defined in terms of the integral form Macdonald polynomials $J_\mu(X;q,t)$ of [26] by

$$\tilde{H}_\mu(X;q,t) = t^{n(\mu)} J_\mu[\frac{X}{1-t^{-1}}; q, t^{-1}], \tag{28}$$

with $n(\mu)$ as in (6). For any symmetric function $f \in \Lambda$, let $f[\mathbf{B}]$, $f[\overline{\mathbf{B}}]$ denote the eigenoperators on the basis $\{\tilde{H}_\mu\}$ of Λ such that

$$f[\mathbf{B}]\, \tilde{H}_\mu = f[B_\mu(q,t)]\, \tilde{H}_\mu, \quad f[\overline{\mathbf{B}}]\, \tilde{H}_\mu = f[\overline{B_\mu(q,t)}]\, \tilde{H}_\mu \tag{29}$$

with $B_\mu(q,t)$ as in (8) and $\overline{B_\mu(q,t)} = B_\mu(q^{-1},t^{-1})$. More generally we use the overbar to signify inverting the variables in any expression, for example

$$\overline{M} = (1-q^{-1})(1-t^{-1}). \tag{30}$$

The next proposition essentially restates the contents of [32, Theorem 3.1 and Proposition 4.10] in our notation. To be more precise, since these two theorems refer to different actions of $\mathcal{E}$ on Λ, one must first use the plethystic transformation in [32, §4.4] to express [32, Proposition 4.10] in terms of the action in [32, Theorem 3.1]. Rescaling the generators $u_{\pm 1,l}$ then yields the following.

Proposition 3.3.1. *There is an action of $\mathcal{E}$ on Λ characterized uniquely by the following properties.*
(i) The central parameters c_1, c_2 act as scalars

$$c_1 \mapsto 1, \quad c_2 \mapsto (q\,t)^{-1}. \tag{31}$$

(ii) The subalgebras $\Lambda(X^{\pm 1,0})$ act as

$$f(X^{1,0}) \mapsto (\omega f)[\mathbf{B} - 1/M], \quad f(X^{-1,0}) \mapsto (\omega f)[\overline{1/M - \mathbf{B}}]. \tag{32}$$

(iii) The subalgebras $\Lambda(X^{0,\pm 1})$ act as

$$f(X^{0,1}) \mapsto f[-X/M]^{\bullet}, \quad f(X^{0,-1}) \mapsto f(X)^{\perp}, \tag{33}$$

using the notation in §2.1.

Remark 3.3.2. The subalgebras $\Lambda(X^{m,n})$ and $\Lambda(X^{-m,n})$ satisfy Heisenberg relations that depend on the central element $c_1^m c_2^n$. If $c_1^m c_2^n = 1$, the Heisenberg relation degenerates and $\Lambda(X^{\pm(m,n)})$ commute. In particular, the value $c_1 \mapsto 1$ in (31) makes $\Lambda(X^{\pm 1,0})$ commute, consistent with (32). The value $c_2 \mapsto 1/(qt)$ makes $\Lambda(X^{0,\pm 1})$ satisfy Heisenberg relations consistent with (33).

We will show in Proposition 3.3.4, below, that the elements $p_1[-MX^{1,a}] \in \mathcal{E}$ act on Λ as operators D_a given by the coefficients $D_a = \langle z^{-a}\rangle D(z)$ of a generating series

$$D(z) = \sum_{a\in\mathbb{Z}} D_a z^{-a} \tag{34}$$

defined by either of the equivalent formulas

$$D(-z) = \Omega[-z^{-1}X]^{\bullet}\,\Omega[zMX]^{\perp} \quad \text{or} \quad D(z) = (\omega\Omega[z^{-1}X])^{\bullet}\,(\omega\Omega[-zMX])^{\perp}, \tag{35}$$

using the operator notation from §2.1. These operators D_a differ by a sign $(-1)^a$ from those studied in [3, 12], and by a plethystic transformation from operators previously introduced by Jing [23].

Lemma 3.3.3. *We have the identities*

$$[\,(\omega p_k[-X/M])^{\bullet},\, D_a\,] = -D_{a+k}, \quad [\,(\omega p_k(X))^{\perp},\, D_a\,] = D_{a-k}. \tag{36}$$

Proof. We start with the second identity, which is equivalent to

$$[\,(\omega p_k(X))^{\perp},\, D(z)\,] = z^{-k} D(z). \tag{37}$$

Since all operators of the form $f(X)^{\perp}$ commute with each other, (37) follows from the definition of $D(z)$ and

$$[\,(\omega p_k(X))^{\perp},\, (\omega\Omega[z^{-1}X])^{\bullet}\,] = z^{-k}(\omega\Omega[z^{-1}X])^{\bullet}. \tag{38}$$

To verify the latter identity, note first that (15) and $\Omega[z^{-1}X] = \exp\sum_{k>0} p_k z^{-k}/k$ imply

$$[\,p_k(X)^{\perp},\, \Omega[z^{-1}X]^{\bullet}\,] = z^{-k}\,\Omega[z^{-1}X]^{\bullet}. \tag{39}$$

Conjugating both sides by ω and using $(\omega f)^{\bullet} = \omega\cdot f^{\bullet}\cdot\omega$ and $(\omega f)^{\perp} = \omega\cdot f^{\perp}\cdot\omega$ gives (38).

For the first identity in (36), consider the modified inner product defined by

$$\langle f, g\rangle' = \langle f[-MX], g\rangle = \langle f, g[-MX]\rangle. \tag{40}$$

The second equality here, which shows that $\langle -,-\rangle'$ is symmetric, follows from orthogonality of the power-sums p_λ. For any f, the operators $f^\perp$ and $f[-X/M]^\bullet$ are adjoint with respect to $\langle -,-\rangle'$. Using this and the definition of $D(z)$, we see that $D(z^{-1})$ is the $\langle -,-\rangle'$ adjoint of $D(z)$, hence D_{-a} is adjoint to D_a. Taking adjoints on both sides of the second identity in (36) now implies the first. □

Proposition 3.3.4. *In the action of $\mathcal{E}$ on Λ given by Proposition 3.3.1, the element $p_1[-MX^{1,a}] = -Mp_1(X^{1,a}) \in \mathcal{E}$ acts as the operator D_a defined by* (34).

Proof. It is known [18, Proposition 2.4] that $D_0\tilde{H}_\mu = (1-MB_\mu)\tilde{H}_\mu$. From Proposition 3.3.1 (ii), we see that $p_1[-MX^{1,0}]$ acts by the same operator, giving the case $a=0$.

Among the defining relations of $\mathcal{E}$ are

$$[\,\omega p_k(X^{0,1}),\, p_1(X^{1,a})\,] = -p_1(X^{1,a+k}), \quad [\,\omega p_k(X^{0,-1}),\, p_1(X^{1,a})\,] = p_1(X^{1,a-k}). \tag{41}$$

Multiplying these by $-M$ and using Proposition 3.3.1 (iii) to compare with (36) reduces the general result to the case $a=0$. □

3.4. The operator ∇

As in [3], we define an eigenoperator ∇ on the Macdonald basis by

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu, \tag{42}$$

with $n(\mu)$ as in (6). Since $t^{n(\mu)}q^{n(\mu^*)} = e_n[B_\mu(q,t)]$ for $n=|\mu|$, we see that ∇ coincides in degree n with $e_n[\mathbf{B}]$. Although the operators $e_n[\mathbf{B}]$ belong to $\mathcal{E}$ acting on Λ, the operator ∇ does not. Its role, rather, is to internalize a symmetry of this action.

Lemma 3.4.1. *Conjugation by the operator ∇ provides a symmetry of the action of $\mathcal{E}$ on Λ, namely*

$$\nabla\, f(X^{m,n})\, \nabla^{-1} = f(X^{m+n,n}). \tag{43}$$

Proof. For $m=\pm 1$, $n=0$, this says that ∇ commutes with the other Macdonald eigenoperators, which is clear.

It is known from [6] that the group of $\mathbf{k}$-algebra automorphisms of $\mathcal{E}$ includes one which acts on the subalgebras $\Lambda(X^{m,n})$ by $f(X^{m,n}) \mapsto f(X^{m+n,n})$, and on the central subalgebra $\mathbf{k}[c_1^{\pm 1}, c_2^{\pm 1}]$ by an automorphism which fixes the central character in Proposition 3.3.1 (i).

The $\Lambda(X^{m,n})$ for $n>0$ are all contained in the subalgebra of $\mathcal{E}$ generated by the elements $p_1(X^{a,1})$. To prove (43) for $n>0$, it therefore suffices to verify the operator identity $\nabla\, p_1(X^{a,1})\, \nabla^{-1} = p_1(X^{a+1,1})$.

In $\mathcal{E}$ there are relations

$$[\,\omega p_k(X^{1,0}),\, p_1(X^{a,1})\,] = p_1(X^{a+k,1}), \quad [\,c_1^{-k}\omega p_k(X^{-1,0}),\, p_1(X^{a,1})\,] = -p_1(X^{a-k,1}). \tag{44}$$

Since ∇ commutes with the action of $\Lambda(X^{\pm 1,0})$, these relations reduce the problem to the case $a=0$, that is, to the identity $\nabla\, p_1(X^{0,1})\, \nabla^{-1} = p_1(X^{1,1})$. By Propositions 3.3.1 and 3.3.4, this is equivalent to the operator identity $\nabla \cdot p_1(X)^\bullet \cdot \nabla^{-1} = D_1$, which is [3, I.12 (iii)].

We leave the similar argument for the case $n<0$, using [3, I.12 (iv)], to the reader. □

3.5. Shuffle algebra

The operators of interest to us belong to the 'right half-plane' subalgebra $\mathcal{E}^+ \subseteq \mathcal{E}$ generated by the $\Lambda(X^{m,n})$ for $m>0$, or equivalently by the elements $p_1(X^{1,a})$. The subalgebra $\mathcal{E}^+$ acts on

Λ as the algebra generated by the operators D_a. It was shown in [32] that $\mathcal{E}^+$ is isomorphic to the *shuffle algebra* constructed in [8] and studied further in [30], whose definition we now recall.

We fix the rational function

$$\Gamma(x/y) = \frac{1 - q\,t\,x/y}{(1 - y/x)(1 - q\,x/y)(1 - t\,x/y)}, \tag{45}$$

and define, for each l, a q, t analog of the symmetrization operator $\mathbf{H}_q$ in (23) by

$$\begin{aligned}\mathbf{H}_{q,t}(\phi(x_1,\ldots,x_l)) &= \sum_{w\in S_l} w\Big(\phi(x)\cdot\prod_{i<j}\Gamma(\frac{x_i}{x_j})\Big)\\ &= \boldsymbol{\sigma}\left(\frac{\phi(x)\,\prod_{i<j}(1-q\,t\,x_i/x_j)}{\prod_{i<j}((1-q\,x_i/x_j)(1-t\,x_i/x_j))}\right).\end{aligned} \tag{46}$$

We write $\mathbf{H}^{(l)}_{q,t}$ when we want to make the number of variables explicit.

Let $T = T(\mathbf{k}[x^{\pm 1}])$ be the tensor algebra on the Laurent polynomial ring $\mathbf{k}[x^{\pm 1}]$ in one variable, that is, the non-commutative polynomial algebra with generators corresponding to the basis elements x^a of $\mathbf{k}[x^{\pm 1}]$ as a vector space. Identifying $T^k = T^k(\mathbf{k}[x^{\pm 1}])$ with $\mathbf{k}[x_1^{\pm 1},\ldots,x_k^{\pm 1}]$, the product in T is given by 'concatenation,'

$$f\cdot g = f(x_1,\ldots,x_k)g(x_{k+1},\ldots,x_{k+l}),\quad \text{for } f\in T^k,\ g\in T^l. \tag{47}$$

For each l, let $I^l \subseteq T^l$ be the kernel of the symmetrization operator $\mathbf{H}^{(l)}_{q,t}$. Since $\mathbf{H}^{(l)}_{q,t}$ factors through the operator $\mathbf{H}^{(k)}_{q,t}$ in any k consecutive variables $x_{i+1},\ldots,x_{i+k}$, the graded subspace $I = \bigoplus_l I^l \subseteq T$ is a two-sided ideal. The shuffle algebra is defined to be the quotient $S = T/I$. Note that S is generated by its tensor degree 1 component S^1 by definition. We will not use the second, larger, type of shuffle algebra that was also introduced in [8, 30].

Proposition 3.5.1 ([32, Theorem 10.1])**.** *There is an algebra isomorphism* $\psi\colon S \to \mathcal{E}^+$ *given on the generators by* $\psi(x^a) = p_1[-MX^{1,a}]$.

For clarity, we note that the factor $-M$ in $p_1[-MX^{1,a}] = -Mp_1(X^{1,a})$ makes no difference to the statement, but is a convenient normalization for us, since it makes $\psi(x_1^{a_1}\cdots x_l^{a_l})$ act on Λ as $D_{a_1}\cdots D_{a_l}$. We also note that our $\Gamma(x/y)$ differs by a factor symmetric in x, y from the function $g(y/x)$ in [32, (10.3)], which makes the product in our shuffle algebra S opposite to that of the algebra $\mathbf{S}$ in [32]. This is as it should be, since the isomorphism in [32] is from $\mathbf{S}$ to the 'upper half-plane' subalgebra of $\mathcal{E}$ generated by the elements $p_1(X^{a,1})$, and the symmetry $p_1(X^{a,1}) \leftrightarrow p_1(X^{1,a})$ is an antihomomorphism.

By construction, Laurent polynomials $\phi(x), \phi'(x)$ in variables $x_1,\ldots,x_l$ define the same element of S, or equivalently, map via ψ to the same element of $\mathcal{E}^+$, if and only if $\mathbf{H}_{q,t}(\phi) = \mathbf{H}_{q,t}(\phi')$. We can regard $\mathbf{H}_{q,t}(\phi)$ as an infinite formal sum of GL_l characters with coefficients in $\mathbf{k}$, in the same manner as for $\mathbf{H}_q$. Representing elements of S, or of $\mathcal{E}^+$, in this way leads to the following useful formula for describing their action on $1 \in \Lambda$.

Proposition 3.5.2. *Given a Laurent polynomial* $\phi = \phi(x_1,\ldots,x_l)$*, let* $\zeta = \psi(\phi) \in \mathcal{E}^+$ *be its image under the isomorphism in Proposition 3.5.1. Then with the action of* $\mathcal{E}$ *on* Λ *given by Proposition 3.3.1, we have*

$$\omega(\zeta\cdot 1)(x_1,\ldots,x_l) = \mathbf{H}_{q,t}(\phi)_{\mathrm{pol}}. \tag{48}$$

Moreover, the Schur function expansion of the symmetric function $\omega(\zeta\cdot 1)(X)$ *contains only terms* s_λ *with* $\ell(\lambda)\le l$*, so* (48) *determines* $\zeta\cdot 1$.

Proof. It suffices to consider the case when $\phi(x) = x^{\mathbf{a}} = x_1^{a_1} \cdots x_l^{a_l}$ and thus (by Proposition 3.3.4) ζ acts on Λ as the operator $D_{a_1} \cdots D_{a_l}$. To find $\zeta \cdot 1$, we use (16) to compute

$$D(z_1) \cdots D(z_l) = \big(\prod_{i<j} \Omega[-z_i/z_j M]\big)\, (\omega\Omega[\overline{Z}X])^{\bullet}\, (\omega\Omega[-ZMX])^{\perp}, \tag{49}$$

where $Z = z_1 + \cdots + z_l$. Acting on 1, applying ω, and taking the coefficient of $z^{-\mathbf{a}}$ gives

$$\begin{aligned} \omega(\zeta \cdot 1)(X) &= \langle z^{-\mathbf{a}} \rangle \big(\prod_{i<j} \Omega[-z_i/z_j M]\big)\, \Omega[\overline{Z}X] \\ &= \langle z^{0} \rangle \big(z^{\mathbf{a}} \prod_{i<j} \frac{1 - q\,t\,z_i/z_j}{(1 - q\,z_i/z_j)(1 - t\,z_i/z_j)}\big)\, \Omega[\overline{Z}X] \prod_{i<j}(1 - z_i/z_j). \end{aligned} \tag{50}$$

Since Z has l variables, this implies that all Schur functions s_λ in $\omega(\zeta \cdot 1)(X)$ have $\ell(\lambda) \leq l$. Identity (48) for $\phi(x) = x^{\mathbf{a}}$ follows by specializing X to $x_1 + \cdots + x_l$ and applying (21). □

3.6. Distinguished elements

Given a rational function $\phi(x_1, \ldots, x_l)$, it may happen that we have an identity of rational functions $\mathbf{H}_{q,t}(\phi) = \mathbf{H}_{q,t}(\eta)$ for some Laurent polynomial $\eta(x)$. In this case, $\mathbf{H}_{q,t}(\phi)$ is the representative of the image of η in S, or of $\psi(\eta) \in \mathcal{E}^+$, even though $\phi(x)$ is not necessarily a Laurent polynomial. For the shuffle algebra S under consideration here, Negut [30, Proposition 6.1] showed that this happens for

$$\phi(x) = \frac{x_1^{b_1} \cdots x_l^{b_l}}{\prod_{i=1}^{l-1}(1 - q\,t\,x_i/x_{i+1})}. \tag{51}$$

Accordingly, there are distinguished elements

$$D_{b_1,\ldots,b_l} = \psi(\eta) \in \mathcal{E}^+, \tag{52}$$

where $\psi\colon S \to \mathcal{E}^+$ is the isomorphism in Proposition 3.5.1 and $\eta(x)$ is any Laurent polynomial such that $\mathbf{H}_{q,t}(\phi) = \mathbf{H}_{q,t}(\eta)$ for the function ϕ in (51). Although there seems to be no particularly nice preferred choice for η, we can avoid working directly with η by using the element ϕ from (51) in (48).

Negut identified certain of the elements $D_{b_1,\ldots,b_l}$ as (in our notation) ribbon skew Schur functions $s_R[-MX^{m,n}]$. The following result is a special case.

Proposition 3.6.1 ([30, Proposition 6.7]). *Let m, k be positive integers and n any integer with m, n coprime.*

For $i = 1, \ldots, km$, let $b_i = \lceil in/m \rceil - \lceil (i-1)n/m \rceil$; if $n \geq 0$, this is the number of south steps at $x = i - 1$ in the highest south-east lattice path δ weakly below the line from $(0, kn)$ to $(km, 0)$. Then

$$e_k[-MX^{m,n}] = D_{b_1,\ldots,b_{km}} \tag{53}$$

Lemma 3.6.2. *For any indices $b_1, \ldots, b_l$ we have*

$$D_{b_1,\ldots,b_l,0} \cdot 1 = D_{b_1,\ldots,b_l} \cdot 1. \tag{54}$$

Proof. Using Proposition 3.5.2, each side of (54) is characterized by its evaluation

$$\omega(D_{b_1,\ldots,b_l}\cdot 1)(x_1,\ldots,x_l) = \mathbf{H}^{(l)}_{q,t}\Big(\frac{x_1^{b_1}\cdots x_l^{b_l}}{\prod_{i=1}^{l-1}(1-q\,t\,x_i/x_{i+1})}\Big)_{\mathrm{pol}} \tag{55}$$

$$\omega(D_{b_1,\ldots,b_l,0}\cdot 1)(x_1,\ldots,x_{l+1}) = \mathbf{H}^{(l+1)}_{q,t}\Big(\frac{x_1^{b_1}\cdots x_l^{b_l}}{(1-q\,t\,x_l/x_{l+1})\prod_{i=1}^{l-1}(1-q\,t\,x_i/x_{i+1})}\Big)_{\mathrm{pol}}. \tag{56}$$

Terms with a negative exponent of x_{l+1} inside the parenthesis in (56) contribute zero after we apply $\mathbf{H}_{q,t}(-)_{\mathrm{pol}}$. We can therefore drop all but the constant term of the geometric series factor $1/(1-q\,t\,x_l/x_{l+1})$, since the other factors are independent of x_{l+1}. This shows that the right hand side of (56) is the same as in (55), except that it has $\mathbf{H}^{(l+1)}_{q,t}$ in place of $\mathbf{H}^{(l)}_{q,t}$.

It follows that $\omega(D_{b_1,\ldots,b_l,0}\cdot 1)$ is a linear combination of Schur functions $s_\lambda(X)$ with $\ell(\lambda)\leq l$, and that $\omega(D_{b_1,\ldots,b_l,0}\cdot 1)$ and $\omega(D_{b_1,\ldots,b_l}\cdot 1)$ evaluate to the same symmetric function in l variables. Hence, they are identical. □

3.7. Summary

Most of what we use from this section in other parts of the paper can be summarized as follows.

Definition 3.7.1. Given $\mathbf{b}=(b_1,\ldots,b_l)\in\mathbb{Z}^l$, the infinite series of GL_l characters $\mathcal{H}_{\mathbf{b}}(x)=\mathcal{H}_{b_1,\ldots,b_l}(x_1,\ldots,x_l)$ is defined by

$$\mathcal{H}_{\mathbf{b}}(x) = \mathbf{H}_{q,t}\Big(\frac{x^{\mathbf{b}}}{\prod_{i=1}^{l-1}(1-q\,t\,x_i/x_{i+1})}\Big) = \mathbf{H}_q\Big(x^{\mathbf{b}}\,\frac{\prod_{i+1<j}(1-q\,t\,x_i/x_j)}{\prod_{i<j}(1-t\,x_i/x_j)}\Big), \tag{57}$$

where $\mathbf{H}_{q,t}$ is given by (46) and $\mathbf{H}_q$ by (23); or where $\mathcal{H}_{\mathbf{b}}(x)$ is given in fully expanded form by (3).

In terms of this definition, we have the following special case of Proposition 3.5.2, which was stated as identity (2) in the introduction.

Corollary 3.7.2. *For the Negut element $D_{\mathbf{b}}\in\mathcal{E}$ acting on Λ, the symmetric function $\omega(D_{\mathbf{b}}\cdot 1)$ evaluated in l variables is given by*

$$\omega(D_{\mathbf{b}}\cdot 1)(x_1,\ldots,x_l) = \mathcal{H}_{\mathbf{b}}(x)_{\mathrm{pol}}\,. \tag{58}$$

Moreover, all terms s_λ in the Schur expansion of $\omega(D_{\mathbf{b}}\cdot 1)(X)$ have $\ell(\lambda)\leq l$, so $\omega(D_{\mathbf{b}}\cdot 1)$ is determined by its evaluation in l variables.

In the special cases where the index $\mathbf{b}$ is the sequence of south runs in the highest (km,kn) Dyck path, $D_{\mathbf{b}}\cdot 1$ can also be expressed as follows.

Corollary 3.7.3. *For $i=1,\ldots,l=km+1$, let b_i be the number of south steps at $x=i-1$ in the highest south-east lattice path δ weakly below the line from $(0,kn)$ to $(km,0)$, including $b_l=0$. Then the Negut element $D_{b_1,\ldots,b_l}$ and the operator $e_k[-MX^{m,n}]$ agree when applied to $1\in\Lambda$, that is, we have*

$$D_{b_1,\ldots,b_l}\cdot 1 = e_k[-MX^{m,n}]\cdot 1. \tag{59}$$

Proof. Immediate from Proposition 3.6.1 and Lemma 3.6.2. □

Corollary 3.7.4 (also proven in [4]). *In the case $n=1$ of (59), we further have*

$$\nabla^m e_k(X) = e_k[-MX^{m,1}]\cdot 1. \tag{60}$$

Proof. By Proposition 3.3.1 (iii), $e_k[-MX^{0,1}] \cdot 1 = e_k(X)$. Since $\nabla(1) = 1$, the result now follows from Lemma 3.4.1. ☐

Remark 3.7.5. Equations (46), (54), (57), (58), (59), and (60) imply the raising operator formula

$$(\omega\nabla^m e_k)(x_1, \ldots, x_l) = \boldsymbol{\sigma}\left(\frac{x_1 x_{m+1} x_{2m+1} \cdots x_{(k-1)m+1} \prod_{i+1<j}(1 - q\,t\,x_i/x_j)}{\prod_{i<j}((1 - q\,x_i/x_j)(1 - t\,x_i/x_j))}\right)_{\mathrm{pol}}, \tag{61}$$

provided $l \geq (k-1)m + 1$.

4. LLT polynomials

In this section we review the definition of the combinatorial LLT polynomials $\mathcal{G}_\nu(X;q)$, using the attacking inversions formulation from [16], which is better suited to our purposes than the original ribbon tableau formulation in [24].

We also define and prove some results on the infinite LLT series $\mathcal{L}_{\beta/\alpha}(x;q)$ introduced in [14]. Since [14] is unpublished, due for revision, and doesn't cover the 'twisted' variants $\mathcal{L}^\sigma_{\beta/\alpha}(x;q)$, we give here a self-contained treatment of the material we need.

4.1. Combinatorial LLT polynomials

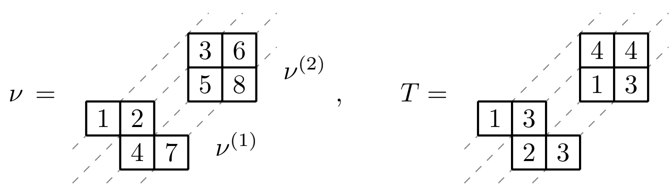
The *content* of a box $a = (x, y)$ in row y, column x of any skew diagram is $c(a) = x - y$.

Let $\nu = (\nu^{(1)}, \ldots, \nu^{(k)})$ be a tuple of skew diagrams. When referring to boxes of ν, we identify ν with the disjoint union of the $\nu^{(j)}$. Fix $\epsilon > 0$ small enough that $k\,\epsilon < 1$. The *adjusted content* of a box $a \in \nu^{(j)}$ is $\tilde{c}(a) = c(a) + j\,\epsilon$. A *reading order* is any total ordering of the boxes $a \in \nu$ on which $\tilde{c}(a)$ is increasing. In other words, the reading order is lexicographic, first by content, then by the index j for which $a \in \nu^{(j)}$, with boxes of the same content in the same component $\nu^{(j)}$ ordered arbitrarily.

Boxes $a, b \in \nu$ *attack* each other if $0 < |\tilde{c}(a) - \tilde{c}(b)| < 1$. If $a \in \nu^{(i)}$ precedes $b \in \nu^{(j)}$ in the reading order, the attacking condition means that either $c(a) = c(b)$ and $i < j$, or $c(b) = c(a)+1$ and $i > j$. We also say that a, b form an *attacking pair* in ν.

By a semistandard Young tableau on the tuple ν we mean a map $T\colon \nu \to \mathbb{Z}_{>0}$ which restricts to a semistandard tableau on each component $\nu^{(j)}$. We write $\mathrm{SSYT}(\nu)$ for the set of these. The *weight* of $T \in \mathrm{SSYT}(\nu)$ is $x^T = \prod_{a\in\nu} x_{T(a)}$. An *attacking inversion* in T is an attacking pair a, b such that $T(a) > T(b)$, where a precedes b in the reading order. We define $\mathrm{inv}(T)$ to be the number of attacking inversions in T.

Example 4.1.1. The picture below shows a tuple of skew diagrams $\nu = (\nu^{(1)}, \nu^{(2)})$, with dashed lines indicating boxes of equal content and boxes numbered in reading order, along with a semistandard tableau T on ν.



(62)

The tuple ν contains 7 attacking pairs (a, b): in the numbering shown, these are $(2, 3)$, $(3, 4)$, $(4, 5)$, $(4, 6)$, $(5, 7)$, $(6, 7)$, $(7, 8)$. The pairs numbered $(3, 4)$, $(4, 5)$, and $(6, 7)$ form inversions in T, with entries $(T(a), T(b))$ equal to $(4, 2)$, $(2, 1)$, and $(4, 3)$, respectively.

Definition 4.1.2. The *combinatorial LLT polynomial* indexed by a tuple of skew diagrams ν is the generating function

$$\mathcal{G}_\nu(X;q) = \sum_{T \in \mathrm{SSYT}(\nu)} q^{\mathrm{inv}(T)} x^T. \tag{63}$$

In [16] it was shown that $\mathcal{G}_\nu(X;q^{-1})$ coincides up to a factor q^e with a ribbon tableau LLT polynomial as defined in [24], and is therefore a symmetric function. A direct and more elementary proof that $\mathcal{G}_\nu(X;q)$ is symmetric was given in [15].

It is useful in working with the LLT polynomials $\mathcal{G}_\nu(X;q)$ to consider a more general combinatorial formalism, as in [15, §10]. Let $\mathcal{A} = \mathcal{A}_+ \coprod \mathcal{A}_-$ be a 'signed' alphabet with a *positive* letter $v \in \mathcal{A}_+$ and a *negative* letter $\overline{v} \in \mathcal{A}_-$ for each $v \in \mathbb{Z}_{>0}$, and an arbitrary total ordering on $\mathcal{A}$.

A *super tableau* on a tuple of skew shapes ν is a map $T\colon \nu \to \mathcal{A}$, weakly increasing along rows and columns, with positive letters increasing strictly on columns and negative letters increasing strictly on rows. A usual semistandard tableau is thus the same thing as a super tableau with all entries positive. Let $\mathrm{SSYT}_\pm(\nu)$ denote the set of super tableaux on ν.

An attacking inversion in a super tableau is an attacking pair a, b, with a preceding b in the reading order, such that either $T(a) > T(b)$ in the ordering on $\mathcal{A}$, or $T(a) = T(b) = \overline{v}$ with $\overline{v}$ negative. As before, $\mathrm{inv}(T)$ denotes the number of attacking inversions.

Lemma 4.1.3 ([15, (81–82) and Proposition 4.2])**.** *We have the identity*

$$\omega_Y \mathcal{G}_\nu[X+Y;q] = \sum_{T \in \mathrm{SSYT}_\pm(\nu)} q^{\mathrm{inv}(T)} x^{T_+} y^{T_-}, \tag{64}$$

where the weight is given by

$$x^{T_+} y^{T_-} = \prod_{a \in \nu} \begin{cases} x_i, & T(a) = i \in \mathcal{A}_+, \\ y_i, & T(a) = \overline{i} \in \mathcal{A}_-. \end{cases} \tag{65}$$

This holds for any choice of the ordering on the signed alphabet $\mathcal{A}$.

Corollary 4.1.4. *We have*

$$\omega\, \mathcal{G}_\nu(X;q) = \sum_{T \in \mathrm{SSYT}_-(\nu)} q^{\mathrm{inv}(T)} x^T, \tag{66}$$

where the sum is over super tableaux T with all entries negative, and we abbreviate x^{T_-} to x^T in this case.

Definition 4.1.5. Given a tuple of skew diagrams $\nu = (\nu^{(1)}, \dots, \nu^{(k)})$, ν^R denotes the tuple $((\nu^{(1)})^R, \dots, (\nu^{(k)})^R)$, where $(\nu^{(j)})^R$ is the $180°$ rotation of the transpose $(\nu^{(j)})^*$, positioned so that each box in ν^R has the same content as the corresponding box in ν.

Proposition 4.1.6. *With ν^R as in Definition 4.1.5, we have*

$$\omega\, \mathcal{G}_\nu(X;q) = q^{I(\nu)}\, \mathcal{G}_{\nu^R}(X;q^{-1}), \tag{67}$$

where $I(\nu)$ is the total number of attacking pairs in ν.

Proof. Use Corollary 4.1.4 on the left hand side, ordering the negative letters as $\overline{1} > \overline{2} > \cdots$. Given a negative tableau T on ν, let T^R be the tableau on ν^R obtained by reflecting the tableau along with ν and changing negative letters $\overline{v}$ to positive letters v. Then T^R is an ordinary semistandard tableau, and $T \mapsto T^R$ is a weight preserving bijection from negative tableaux on

ν to $\mathrm{SSYT}(\nu^R)$. An attacking pair in ν is an inversion in T if and only if the corresponding attacking pair in ν^R is a non-inversion in T^R, hence $\mathrm{inv}(T^R) = I(\nu) - \mathrm{inv}(T)$. This implies (67). □

Example 4.1.7. Consider a tuple $\nu = (\nu^{(1)}, \dots, \nu^{(k)})$ in which each skew shape is a column so that ν^R is a tuple of rows. The super tableau of shape ν^R with all entries a positive letter 1 has no inversions, whereas the super tableau T of shape ν with all entries a negative letter $\overline{1}$ has

$$\mathrm{inv}(T) = I(\nu), \tag{68}$$

where $I(\nu)$ is the total number of attacking pairs in ν.

Lemma 4.1.8. *The LLT polynomial $\mathcal{G}_\nu(X; q)$ is a linear combination of Schur functions $s_\lambda(X)$ such that $\ell(\lambda)$ is bounded by the total number of rows in the diagram ν.*

Proof. Let r be the total number of rows in ν. It is equivalent to show that $\omega\, \mathcal{G}_\nu(X; q)$ is a linear combination of monomial symmetric functions $m_\lambda(X)$ such that $\lambda_1 \leq r$. By Proposition 4.1.6, $\omega\, \mathcal{G}_\nu(X; q)$ has a monomial term $q^{I(\nu)-\mathrm{inv}(T)} x^T$ for each semistandard tableau $T \in \mathrm{SSYT}(\nu^R)$ on the tuple of reflected shapes ν^R. Since a letter can appear at most once in each column of T, the exponents of x^T are bounded by r. □

4.2. Reminder on Hecke algebras

We recall, in the case of GL_l, the Hecke algebra action on the group algebra of the weight lattice, as in Lusztig [25] or Macdonald [27] and due originally to Bernstein and Zelevinsky.

For GL_l, we identify the group algebra $\mathbf{k}X$ of the weight lattice $X = \mathbb{Z}^l$ with the Laurent polynomial algebra $\mathbf{k}[x_1^{\pm 1}, \dots, x_l^{\pm 1}]$. Here $\mathbf{k}$ is any ground field containing $\mathbb{Q}(q)$.

The Demazure-Lusztig operators

$$T_i = q s_i + (1 - q) \frac{1}{1 - x_{i+1}/x_i}(s_i - 1) \tag{69}$$

for $i = 1, \dots, l-1$ generate an action of the Hecke algebra of S_l on $\mathbf{k}[x_1^{\pm 1}, \dots, x_l^{\pm 1}]$. We have normalized the generators so that the quadratic relations are $(T_i - q)(T_i + 1) = 0$. The elements T_w, defined by $T_w = T_{i_1} T_{i_2} \cdots T_{i_m}$ for any reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_m}$, form a $\mathbf{k}$-basis of the Hecke algebra, as w ranges over S_l.

Let R_+ be the set of positive roots and $Q_+ = \mathbb{N} R_+$ the cone they generate in the root lattice Q. For dominant weights we define $\lambda \leq \mu$ if $\mu - \lambda \in Q_+$. For polynomial weights of GL_l, this coincides with the standard partial ordering (7) on partitions.

For any weight λ, let λ_+ denote the dominant weight in the orbit $S_l\, \lambda$.

Let $\mathrm{conv}(S_l\, \lambda)$ be the convex hull of $S_l\, \lambda$ in the coset $\lambda + Q$ of the root lattice, i.e., the set of weights that occur with non-zero multiplicity in the irreducible character χ_{λ_+}. Note that $\mathrm{conv}(S_l\, \lambda) \subseteq \mathrm{conv}(S_l\, \mu)$ if and only if $\lambda_+ \leq \mu_+$.

Each orbit $S_l\, \lambda_+$ has a partial ordering induced by the Bruhat ordering on S_l. More explicitly, this ordering is the transitive closure of the relation $s_i \lambda > \lambda$ if $\langle \alpha_i^\vee, \lambda \rangle > 0$. We extend this to a partial ordering on all of $X = \mathbb{Z}^l$ by defining $\lambda \leq \mu$ if $\lambda_+ < \mu_+$, or if $\lambda_+ = \mu_+$ and $\lambda \leq \mu$ in the Bruhat order on $S_l\, \lambda_+$.

Suppose $\langle \alpha_i^\vee, \lambda \rangle \geq 0$. If $\langle \alpha_i^\vee, \lambda \rangle = 0$, that is, if $\lambda = s_i \lambda$, then

$$T_i\, x^\lambda = q\, x^\lambda. \tag{70}$$

$E_{000} = 1$	$F_{000} = 1$
$E_{100} = x_1$	$F_{100} = y_1$
$E_{010} = (1-q^{-1})x_1 + x_2$	$F_{010} = y_2$
$E_{001} = (1-q^{-1})x_1 + (1-q^{-1})x_2 + x_3$	$F_{001} = y_3$
$E_{110} = x_1x_2$	$F_{110} = y_1y_2$
$E_{101} = (1-q^{-1})x_1x_2 + x_1x_3$	$F_{101} = y_1y_3$
$E_{011} = (1-q^{-1})x_1x_2 + (1-q^{-1})x_1x_3 + x_2x_3$	$F_{011} = y_2y_3$
$E_{200} = x_1^2$	$F_{200} = y_1^2 + (1-q)y_1y_2 + (1-q)y_1y_3$
$E_{020} = (1-q^{-1})x_1^2 + (1-q^{-1})x_1x_2 + x_2^2$	$F_{020} = y_2^2 + (1-q)y_2y_3$
$E_{002} = (1-q^{-1})x_1^2 + (1-q^{-1})^2x_1x_2 + (1-q^{-1})x_1x_3 + (1-q^{-1})x_2^2 + (1-q^{-1})x_2x_3 + x_3^2$	$F_{002} = y_3^2$

Table 1. *Non-symmetric Hall-Littlewood polynomials* $E^\sigma_{\mathbf{a}}(x_1, x_2, x_3; q)$ *and* $F^\sigma_{\mathbf{a}}(y_1, y_2, y_3; q)$ *for* $l = 3$, $\sigma = 1$, *and* $|\mathbf{a}| \leq 2$.

Otherwise, if $\langle \alpha_i^\vee, \lambda\rangle > 0$, then

$$\begin{aligned} T_i\, x^\lambda &\equiv q\, x^{s_i\lambda} + (q-1)x^\lambda, \\ T_i\, x^{s_i\lambda} &\equiv x^\lambda \end{aligned} \tag{71}$$

modulo the space spanned by monomials x^μ for μ strictly between λ and $s_i\lambda$ on the root string $\lambda + \mathbb{Z}\alpha_i$. Note that $\mu < \lambda$ for these weights μ, since they lie on orbits strictly inside $\operatorname{conv}(S_l\,\lambda)$. Furthermore, the set of all weights $\mu \leq s_i\lambda$ is s_i-invariant and has convex intersection with every root string $\nu + \mathbb{Z}\alpha_i$, hence the space $\mathbf{k}\,\{x^\mu \mid \mu \leq s_i\lambda\}$ is closed under T_i. It follows that if $\langle \alpha_i^\vee, \lambda\rangle \geq 0$, then T_i applied to any Laurent polynomial of the form $x^\lambda + \sum_{\mu<\lambda} c_\mu x^\mu$ yields a result of the form

$$T_i\,\big(x^\lambda + \sum_{\mu<\lambda} c_\mu x^\mu\big) = q\,x^{s_i\lambda} + \sum_{\mu < s_i\lambda} d_\mu x^\mu\,. \tag{72}$$

4.3. Non-symmetric Hall-Littlewood polynomials

For each GL_l weight $\lambda \in \mathbb{Z}^l$, we define the *non-symmetric Hall-Littlewood polynomial*

$$E_\lambda(x;q) = E_\lambda(x_1,\ldots,x_l;q) = q^{-\ell(w)} T_w x^{\lambda_+}, \tag{73}$$

where $w \in S_l$ is such that $\lambda = w(\lambda_+)$. If λ has non-trivial stabilizer then w is not unique, but it follows from (70)–(72) that $E_\lambda(x;q)$ is independent of the choice of w and has the monic and triangular form

$$E_\lambda(x;q) = x^\lambda + \sum_{\mu<\lambda} c_\mu x^\mu. \tag{74}$$

See Table 1 for examples.

For context, we remark that several distinct notions of 'non-symmetric Hall-Littlewood polynomial' can be found in the literature. Our E_λ (and F_λ, below) coincide with specializations of non-symmetric Macdonald polynomials considered by Ion in [22, Theorem 4.8]. The twisted variants E^σ_λ below are specializations of the 'permuted basement' non-symmetric Macdonald polynomials studied (for GL_l) by Alexandersson [1] and Alexandersson and Sawhney [2]. We also note that $E_\lambda(x;q^{-1})$ and $F_\lambda(x;q)$ have coefficients in $\mathbb{Z}[q]$ and specialize at $q=0$ to Demazure characters and Demazure atoms respectively.

For any $\mu \in \mathbb{R}^l$ we define $\mathrm{Inv}(\mu) = \{(i<j) \mid \mu_i > \mu_j\}$. In the case of a permutation, $\mathrm{Inv}(\sigma)$ is then the usual inversion set of $\sigma = (\sigma(1), \ldots, \sigma(l)) \in S_l$.

Taking ρ as in §2.3 and $\epsilon > 0$ small, the notation $\mathrm{Inv}(\lambda + \epsilon\rho)$ denotes the set of pairs $i<j$ such that $\lambda_i \geq \lambda_j$.

Given $\sigma \in S_l$, we define *twisted* non-symmetric Hall-Littlewood polynomials

$$E^\sigma_\lambda(x;q) = q^{\left|\mathrm{Inv}(\sigma^{-1}) \cap \mathrm{Inv}(\lambda+\epsilon\rho)\right|} T^{-1}_{\sigma^{-1}} E_{\sigma^{-1}(\lambda)}(x;q) \tag{75}$$

$$F^\sigma_\lambda(x;q) = \overline{E^{\sigma w_0}_{-\lambda}(x;q)} = E^{\sigma w_0}_{-\lambda}(x_1^{-1}, \ldots, x_l^{-1}; q^{-1}), \tag{76}$$

where $w_0 \in S_l$ is the longest element, given by $w_0(i) = l+1-i$. The normalization in (75) implies the recurrence

$$E^\sigma_\lambda = \begin{cases} q^{-I(\lambda_i \leq \lambda_{i+1})}\, T_i\, E^{s_i\sigma}_{s_i\lambda}, & s_i\sigma > \sigma \\ q^{I(\lambda_i \geq \lambda_{i+1})}\, T_i^{-1}\, E^{s_i\sigma}_{s_i\lambda}, & s_i\sigma < \sigma, \end{cases} \tag{77}$$

where $I(P) = 1$ if P is true, $I(P) = 0$ if P is false. Together with the initial condition $E^\sigma_\lambda = x^\lambda$ for all σ if $\lambda = \lambda_+$, this determines E^σ_λ for all σ and λ.

Corollary 4.3.1. *E^σ_λ has the monic and triangular form in* (74) *for all σ.*

Proof. This follows from (72) and (77). □

Proposition 4.3.2. *For every $\sigma \in S_l$, the elements $E^\sigma_\lambda(x;q)$ and $\overline{F^\sigma_\lambda(x;q)}$ are dual bases of $\mathbf{k}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]$ with respect to the inner product defined by*

$$\langle f, g\rangle_q = \langle x^0 \rangle\, f\, g \prod_{i<j} \frac{1 - x_i/x_j}{1 - q^{-1} x_i/x_j}. \tag{78}$$

In other words, $\langle E^\sigma_\lambda, \overline{F^\sigma_\mu}\rangle_q = \delta_{\lambda\mu}$ for all $\lambda, \mu \in \mathbb{Z}^l$ and all $\sigma \in S_l$.

To prove Proposition 4.3.2 we need the following lemma.

Lemma 4.3.3. *The Demazure-Lusztig operators T_i in* (69) *are self-adjoint with respect to $\langle -,-\rangle_q$.*

Proof. It's the same to show that $T_i - q$ is self-adjoint. A bit of algebra gives

$$T_i - q = q\, \frac{1 - q^{-1} x_i/x_{i+1}}{1 - x_i/x_{i+1}} (s_i - 1), \tag{79}$$

and therefore

$$\langle (T_i - q) f, g\rangle_q = q\, \langle x^0 \rangle\, (s_i(f)\, g - f\, g) \prod \frac{1 - x_j/x_k}{1 - q^{-1} x_j/x_k}, \tag{80}$$

where the product is over $j<k$ with $(j,k) \neq (i, i+1)$. We want to show that this is symmetric in f and g, i.e., that the right hand side is unchanged if we replace $s_i(f)\, g$ with $f\, s_i(g)$. Let Δ denote the product factor in (80), and note that Δ is symmetric in x_i and x_{i+1}. The constant term $\langle x^0 \rangle\, \varphi(x)$ of any $\varphi(x_1, \ldots, x_l)$ is equal to $\langle x^0 \rangle\, s_i(\varphi(x))$. In particular, $\langle x^0 \rangle s_i(f) g\, \Delta = \langle x^0 \rangle f s_i(g)\, \Delta$, which implies the desired result. □

Proof of Proposition 4.3.2. The desired identity is just a tidy notation for $\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = \delta_{\lambda\mu}$.

By (77), for every i, we have either $\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = q^e\, \langle T_i E^{s_i\sigma}_{s_i\lambda}, T_i^{-1} E^{s_i\sigma w_0}_{-s_i\mu}\rangle_q$ or $\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = q^e\, \langle T_i^{-1} E^{s_i\sigma}_{s_i\lambda}, T_i E^{s_i\sigma w_0}_{-s_i\mu}\rangle_q$, depending on whether $s_i\sigma > \sigma$ or $s_i\sigma < \sigma$, for some exponent e. Moreover, if $\lambda = \mu$, then $q^e = 1$.

Since T_i is self-adjoint, we get $\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = q^e \langle E^{s_i\sigma}_{s_i\lambda}, E^{s_i\sigma w_0}_{-s_i\mu}\rangle_q$ in either case. Repeating this gives an identity

$$\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = q^e \langle E^{v\sigma}_{v\lambda}, E^{v\sigma w_0}_{-v\mu}\rangle_q \tag{81}$$

for all $\lambda, \mu \in \mathbb{Z}^l$ and all $\sigma, v \in S_l$, again with $q^e = 1$ if $\lambda = \mu$.

Choose $v \in S_l$ such that $\mu_- = v(\mu)$ is antidominant. Then (81) gives

$$\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = q^e \langle E^{v\sigma}_{v\lambda}, x^{-(\mu_-)}\rangle_q = q^e\, \langle x^{\mu_-}\rangle\, \Delta\, E^{v\sigma}_{v\lambda}, \tag{82}$$

where now Δ is the product factor in (78). Let $\operatorname{supp}(f)$ denote the set of weights ν for which x^{ν} occurs with non-zero coefficient in f. Since $\operatorname{supp}(\Delta) = Q_+$, and $\operatorname{supp}(E^{v\sigma}_{v\lambda}) \subseteq \operatorname{conv}(S_l\, \lambda)$, it follows from (82) that if $\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q \neq 0$, then $(\mu_- - Q_+) \cap \operatorname{conv}(S_l\, \lambda) \neq \emptyset$ and therefore $\mu_- - \lambda_- \in Q_+$. Since $w_0(Q_+) = -Q_+$, this is equivalent to $\lambda_+ \geq \mu_+$.

By symmetry, exchanging λ with $-\mu$ and σ with σw_0, if $\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q \neq 0$ then we also have $(-\lambda)_- - (-\mu)_- \in Q_+$, hence $\lambda_+ - \mu_+ \in -Q_+$, that is, $\lambda_+ \leq \mu_+$. Hence, $\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q \neq 0$ implies $\lambda_+ = \mu_+$, so λ and μ belong to the same S_l orbit. This reduces the problem to the case that $S_l\, \lambda = S_l\, \mu$.

In this case, $(\mu_- - Q_+) \cap \operatorname{conv}(S_l\, \lambda) = \{\mu_-\}$. Furthermore, if $\lambda \neq \mu$, then $v\lambda \neq \mu_-$, and Corollary 4.3.1 implies that $(\mu_- - Q_+) \cap \operatorname{supp}(E^{v\sigma}_{v\lambda}) = \emptyset$, hence $\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = 0$.

If $\lambda = \mu$, then the right hand side of (82) reduces to $\langle x^{\mu_-}\rangle\, \Delta\, E^{v\sigma}_{\mu_-}$. Since $\operatorname{supp}(\Delta) = Q_+$ and $\operatorname{supp}(E^{v\sigma}_{\mu_-}) \subset \mu_- + Q_+$, only the constant term of Δ and the x^{μ_-} term of $E^{v\sigma}_{\mu_-}$ contribute to the coefficient of x^{μ_-} in $\Delta\, E^{v\sigma}_{\mu_-}$, and we have $\langle x^{\mu_-}\rangle E^{v\sigma}_{\mu_-} = 1$ by Corollary 4.3.1. Hence, $\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\lambda}\rangle_q = 1$. □

Lemma 4.3.4. *Given $\lambda \in \mathbb{Z}^l$, suppose there is an index k such that $\lambda_i \geq \lambda_j$ for all $i \leq k$ and $j > k$. Given $\sigma \in S_l$, let $\sigma_1 \in S_k$ and $\sigma_2 \in S_{l-k}$ be the permutations such that $\sigma_1(1), \dots, \sigma_1(k)$ are in the same relative order as $\sigma(1), \dots, \sigma(k)$, and $\sigma_2(1), \dots, \sigma_2(l-k)$ are in the same relative order as $\sigma(k+1), \dots, \sigma(l)$. Then*

$$E^{\sigma^{-1}}_{\lambda}(x_1, \dots, x_l; q) = E^{\sigma_1^{-1}}_{(\lambda_1,\dots,\lambda_k)}(x_1, \dots, x_k; q)\, E^{\sigma_2^{-1}}_{(\lambda_{k+1},\dots,\lambda_l)}(x_{k+1}, \dots, x_l; q). \tag{83}$$

Proof. If λ is dominant, the result is trivial. Otherwise, the recurrence (77) determines $E^{\sigma^{-1}}_{\lambda}$ by induction on $|\operatorname{Inv}(-\lambda)|$. The condition on λ implies that we only need to use (77) for $i \neq k$, that is, for s_i in the Young subgroup $S_k \times S_{l-k} \subset S_l$. For $i \neq k$, the right hand side of (83) satisfies the same recurrence. □

4.4. LLT series

Definition 4.4.1. Given GL_l weights $\alpha, \beta \in \mathbb{Z}^l$ and a permutation $\sigma \in S_l$, the *LLT series* $\mathcal{L}^{\sigma}_{\beta/\alpha}(x_1, \dots, x_l; q)$ is the infinite formal sum of irreducible GL_l characters in which the coefficient of χ_λ is defined by

$$\langle \chi_\lambda\rangle\, \mathcal{L}^{\sigma^{-1}}_{\beta/\alpha}(x; q^{-1}) = \langle E^{\sigma}_{\beta}\rangle\, \chi_\lambda\, E^{\sigma}_{\alpha}, \tag{84}$$

where $E^{\sigma}_{\lambda}(x; q)$ are the twisted non-symmetric Hall-Littlewood polynomials from §4.3.

We remark that the coefficients of $E^\sigma_\lambda(x;q)$ are polynomials in q^{-1}, so the convention of inverting q in (84) makes the coefficients of $\mathcal{L}^\sigma_{\beta/\alpha}(x;q)$ polynomials in q. Inverting σ as well leads to a more natural statement and proof in Corollary 4.5.7, below.

Proposition 4.4.2. *We have the formula*

$$\mathcal{L}^\sigma_{\beta/\alpha}(x;q) = \mathbf{H}_q(w_0(F^{\sigma^{-1}}_\beta(x;q)\overline{E^{\sigma^{-1}}_\alpha(x;q)})), \tag{85}$$

where $\mathbf{H}_q$ is the Hall-Littlewood symmetrization operator in (23) *and $w_0(i) = l+1-i$ is the longest element in S_l.*

Proof. By Proposition 4.3.2, the coefficient $\langle E^{\sigma^{-1}}_\beta(x;q^{-1})\rangle\, \chi_\lambda\, E^{\sigma^{-1}}_\alpha(x;q^{-1})$ of χ_λ in $\mathcal{L}^\sigma_{\beta/\alpha}$ is given by the constant term

$$\langle x^0\rangle\, \chi_\lambda\, F^{\sigma^{-1}}_\beta(x^{-1};q)\, E^{\sigma^{-1}}_\alpha(x;q^{-1}) \prod_{i<j} \frac{1-x_i/x_j}{1-q\,x_i/x_j}. \tag{86}$$

Substituting $x_i \mapsto x_i^{-1}$ and applying w_0, this is equal to

$$\langle x^0\rangle\, \overline{\chi_\lambda}\, w_0(F^{\sigma^{-1}}_\beta(x;q)\overline{E^{\sigma^{-1}}_\alpha(x;q)}) \prod_{i<j} \frac{1-x_i/x_j}{1-q\,x_i/x_j}. \tag{87}$$

Considering this expression as a formal Laurent series in q and applying (20) coefficient-wise yields

$$\langle \chi_\lambda\rangle \boldsymbol{\sigma} \left(w_0(F^{\sigma^{-1}}_\beta(x;q)\overline{E^{\sigma^{-1}}_\alpha(x;q)}) \prod_{i<j} \frac{1}{1-q\,x_i/x_j} \right), \tag{88}$$

which is $\langle\chi_\lambda\rangle \mathbf{H}_q(w_0(F^{\sigma^{-1}}_\beta(x;q)\overline{E^{\sigma^{-1}}_\alpha(x;q)}))$, as desired. ☐

Remark 4.4.3. All definitions and results in §§2.3–2.4 and 4.2–4.4 extend naturally from the weight lattice and root system of GL_l to those of any reductive algebraic group G, as in [14]. The reader may observe that apart from changes in notation, the arguments given here also apply in the general case.

4.5. Tableaux for LLT series

We now work out a tableau formalism which relates the polynomial part $\mathcal{L}^\sigma_{\beta/\alpha}(x;q)_{\mathrm{pol}}$ to a combinatorial LLT polynomial $\mathcal{G}_\nu(x;q)$.

Lemma 4.5.1. *For all $\sigma \in S_l$, $\lambda \in \mathbb{Z}^l$ and k, the product of the elementary symmetric function $e_k(x)$ and the non-symmetric Hall-Littlewood polynomial $E^{\sigma^{-1}}_\lambda(x;q)$ is given by*

$$e_k(x)\, E^{\sigma^{-1}}_\lambda(x;q) = \sum_{|I|=k} q^{-h_I} E^{\sigma^{-1}}_{\lambda+\varepsilon_I}(x;q), \tag{89}$$

where $I \subseteq \{1,\ldots,l\}$ has k elements, $\varepsilon_I = \sum_{i\in I} \varepsilon_i$ is the indicator vector of I, and

$$h_I = |\,\mathrm{Inv}(\lambda+\varepsilon_I+\epsilon\sigma) \setminus \mathrm{Inv}(\lambda+\epsilon\sigma)\,|. \tag{90}$$

Equivalently, h_I is the number of pairs $i<j$ such that $i \in I$, $j \notin I$, and we have $\lambda_j = \lambda_i$ if $\sigma(i) < \sigma(j)$, or $\lambda_j = \lambda_i + 1$ if $\sigma(i) > \sigma(j)$.

Proof. First consider the case $\sigma = 1$. Being symmetric, $e_k(x)$ commutes with T_w, giving

$$e_k\, E_\lambda = q^{-\ell(w)} T_w\, e_k\, x^{\lambda_+} = q^{-\ell(w)} \sum_{|J|=k} T_w\, x^{\lambda_+ + \varepsilon_J}, \tag{91}$$

where $\lambda = w(\lambda_+)$, as in (73). To fix the choice, we take w maximal in its coset $w\,\mathrm{Stab}(\lambda_+)$.

In each term of the sum in (91), the weight $\mu = \lambda_+ + \varepsilon_J$ can fail to be dominant at worst by having some entries $\mu_j = \mu_i + 1$ for indices $i < j$ such that $(\lambda_+)_i = (\lambda_+)_j$, $i \notin J$ and $j \in J$. Let v be the minimal permutation such that $\mu_+ = v(\mu)$, that is, the permutation that moves indices $j \in J$ to the left within each block of constant entries in λ_+. The formula $T_i x_i^a x_{i+1}^{a+1} = x_i^{a+1} x_{i+1}^a$ is immediate from the definition of T_i, and implies that $T_v x^\mu = x^{\mu_+}$. By the maximality of w, since $v \in \mathrm{Stab}(\lambda_+)$, there is a reduced factorization $w = uv$, hence $T_w = T_u T_v$. Then

$$T_w\, x^{\lambda_+ + \varepsilon_J} = T_u x^{\mu_+} = q^{\ell(u)} E_{\lambda + w(\varepsilon_J)}, \tag{92}$$

since $\lambda + w(\varepsilon_J) = w(\mu) = u(\mu_+)$.

Now, $\ell(v)$ is equal to the number of pairs $i < j$ such that $\mu_i < \mu_j$, that is, such that $(\lambda_+)_i = (\lambda_+)_j$, $i \notin J$ and $j \in J$. By maximality, the permutation w carries these to the pairs $j' = w(i)$, $i' = w(j)$ such that $i' < j'$, $\lambda_{i'} = \lambda_{j'}$, $i' \in I$ and $j' \notin I$, where $I = w(J)$. For $\sigma = 1$, the definition of h_I is the number of such pairs i', j', so we have $\ell(u) - \ell(w) = -\ell(v) = -h_I$. Hence, the term for J in (91) is $q^{-\ell(w)} T_w x^{\lambda_+ + \varepsilon_J} = q^{-h_I} E_{\lambda + \varepsilon_I}$. As J ranges over subsets of size k, so does $I = w(J)$, giving (89) in this case.

Substituting $\sigma(\lambda)$ for λ and $\sigma(I)$ for I in the $\sigma = 1$ case, and acting on both sides with T_σ^{-1}, yields

$$q^{-|\mathrm{Inv}(\sigma)\cap \mathrm{Inv}(\lambda+\epsilon\rho)|} e_k\, E_\lambda^{\sigma^{-1}} = \sum_{|I|=k} q^{-|\,\mathrm{Inv}(\sigma(\lambda+\varepsilon_I))\setminus \mathrm{Inv}(\sigma(\lambda))|} q^{-|\mathrm{Inv}(\sigma)\cap \mathrm{Inv}(\lambda+\varepsilon_I+\epsilon\rho)|} E_{\lambda+\varepsilon_I}^{\sigma^{-1}}. \tag{93}$$

Combining powers of q gives the desired identity (89) if we verify that

$$\begin{aligned} &|\,\mathrm{Inv}(\sigma)\cap \mathrm{Inv}(\lambda+\varepsilon_I+\epsilon\rho)\,| - |\,\mathrm{Inv}(\sigma)\cap \mathrm{Inv}(\lambda+\epsilon\rho)\,| \\ &\qquad = |\,\mathrm{Inv}(\lambda+\varepsilon_I+\epsilon\sigma)\setminus \mathrm{Inv}(\lambda+\epsilon\sigma)\,| - |\,\mathrm{Inv}(\sigma(\lambda+\varepsilon_I))\setminus \mathrm{Inv}(\sigma(\lambda))\,|. \end{aligned} \tag{94}$$

On the left hand side, cancelling the contribution from the intersection of the two sets leaves

$$|\mathrm{Inv}(\sigma)\cap (\mathrm{Inv}(\lambda+\varepsilon_I+\epsilon\rho)\setminus \mathrm{Inv}(\lambda+\epsilon\rho))| - |\mathrm{Inv}(\sigma)\cap (\mathrm{Inv}(\lambda+\epsilon\rho)\setminus \mathrm{Inv}(\lambda+\varepsilon_I+\epsilon\rho))|. \tag{95}$$

The first term in (95) counts pairs $i < j$ such that $i \in I$, $j \notin I$, $\sigma(i) > \sigma(j)$, and $\lambda_j = \lambda_i + 1$. The second term counts pairs $i > j$ such that $i \in I$, $j \notin I$, $\sigma(i) < \sigma(j)$, and $\lambda_j = \lambda_i$. The first term on the right hand side of (94) counts pairs $i < j$ such that $i \in I$, $j \notin I$, and $\lambda_j = \lambda_i$ if $\sigma(i) < \sigma(j)$, or $\lambda_j = \lambda_i + 1$ if $\sigma(i) > \sigma(j)$. The second term on the right hand side of (94) counts the set of pairs whose images under σ^{-1} are pairs i, j (in either order) such that $i \in I$, $j \notin I$, $\sigma(i) < \sigma(j)$ and $\lambda_i = \lambda_j$. The cases in the first term with $\sigma(i) < \sigma(j)$ cancel those in the second term with $i < j$. The remaining cases in the first term, with $\sigma(i) > \sigma(j)$, match the first term in (95), while the remaining cases in the second term, with $i > j$, match the second term in (95). This proves (94) and completes the proof of the lemma. □

Given $\alpha, \beta \in \mathbb{Z}^l$ such that $\alpha_j \leq \beta_j$ for all j, we let β/α denote the tuple of one-row skew shapes $(\beta_j)/(\alpha_j)$ such that the x coordinates of the right edges of boxes in the row that forms the j-th component are the integers $\alpha_j + 1, \ldots, \beta_j$. The boxes just outside this j-th component, adjacent to the left and right ends of the row, then have right edges with x coordinate α_j and

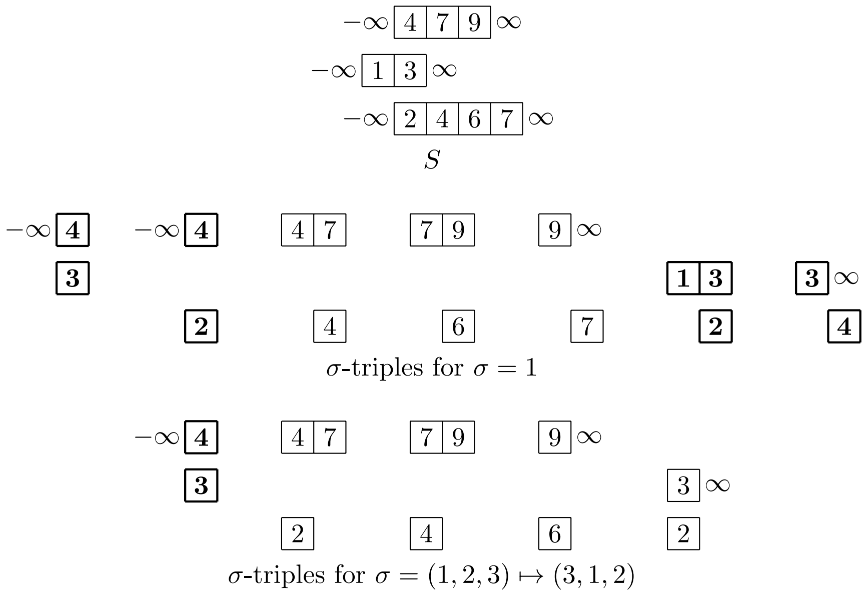


Figure 1. *A negative tableau S on $\beta/\alpha = (6,3,5)/(2,1,2)$ and the σ-triples in β/α, for two choices of σ, shown with their entries from S. Triples in boldface are increasing in S.*

$\beta_j + 1$. In the case of an empty row with $\alpha_j = \beta_j$, we still consider these two boxes to be adjacent to the ends of the row.

For each box a, let $i(a)$ denote the x coordinate of its right edge and $j(a)$ the index of the row containing it.

For $\sigma \in S_l$, we define $\sigma(\beta/\alpha) = \sigma(\beta)/\sigma(\alpha)$. If a is a box in row $j(a)$ of β/α, then $\sigma(a)$ denotes the corresponding box with x coordinate $i(\sigma(a)) = i(a)$ in row $\sigma(j(a))$ of $\sigma(\beta/\alpha)$. The adjusted content of $\sigma(a)$, as defined in §4.1, is then $\tilde{c}(\sigma(a)) = i(a) + \sigma(j(a))\,\epsilon$. Hence, the reading order on $\sigma(\beta/\alpha)$ corresponds via σ to the ordering of boxes in β/α by increasing values of $i(a) + \sigma(j(a))\,\epsilon$.

We define a *σ-triple* in β/α to consist of any three boxes (a, b, c) arranged as follows: boxes a and c are in or adjacent to the same row $j(a) = j(c)$, and are consecutive, that is, $i(c) = i(a)+1$, while box b is in a row $j(b) < j(a)$, and we have $a < b < c$ in the ordering corresponding to the reading order on $\sigma(\beta/\alpha)$. More explicitly, this means that if $\sigma(j(b)) < \sigma(j(a))$, then $i(b) = i(c)$, while if $\sigma(j(b)) > \sigma(j(a))$, then $i(b) = i(a)$. The box b is required to be a box of β/α, but box a is allowed to be outside and adjacent to the left end of a row, while c is similarly allowed to be outside and adjacent to the right end of a row.

An example of a tuple β/α, with all its σ-triples for two different choices of σ, is shown in Figure 1.

A negative tableau on β/α is a map $S\colon \beta/\alpha \to \mathbb{Z}_{>0}$ strictly increasing on each row. In the terminology of §4.1, S is a super tableau on β/α with entries in $\mathbb{Z}_{>0}$, considered as a negative alphabet ordered by $\overline{1} < \overline{2} < \cdots$. We say that a σ-triple (a, b, c) in β/α is *increasing* in S if $S(a) < S(b) < S(c)$, with the convention that $S(a) = -\infty$ if a is just outside the left end of a row, and $S(c) = \infty$ if c is just outside right end of a row. Along with the σ-triples in β/α, Figure 1 also displays which triples are increasing in an example tableau S.

Proposition 4.5.2. *Given $\alpha, \beta \in \mathbb{Z}^l$ such that $\alpha_j \leq \beta_j$ for all j, and $\sigma \in S_l$, let*

$$N^{\sigma}_{\beta/\alpha}(X;q) = \sum_{S \in \mathrm{SSYT}_-(\beta/\alpha)} q^{h_\sigma(S)} x^S \tag{96}$$

be the generating function for negative tableaux S on the tuple of one-row skew diagrams $(\beta_j)/(\alpha_j)$, weighted by $q^{h_\sigma(S)}$, where $h_\sigma(S)$ is the number of increasing σ-triples in S. Then $N^{\sigma}_{\beta/\alpha}(X;q)$ is a symmetric function, and $\omega N^{\sigma}_{\beta/\alpha}(X;q)$ evaluates in l variables to

$$(\omega N^{\sigma}_{\beta/\alpha})(x_1,\ldots,x_l;q) = \mathcal{L}^{\sigma}_{\beta/\alpha}(x_1,\ldots,x_l;q)_{\mathrm{pol}}. \tag{97}$$

If we do not have $\alpha_j \leq \beta_j$ for all j, then $\mathcal{L}^{\sigma}_{\beta/\alpha}(x;q)_{\mathrm{pol}} = 0$.

Proof. Let $L^{\sigma}_{\beta/\alpha}(X;q)$ be the unique symmetric function such that (i) $L^{\sigma}_{\beta/\alpha}(X;q)$ is a linear combination of Schur functions s_λ with $\ell(\lambda) \leq l$, and (ii) in l variables, it evaluates to

$$L^{\sigma}_{\beta/\alpha}(x_1,\ldots,x_l;q) = \mathcal{L}^{\sigma}_{\beta/\alpha}(x_1,\ldots,x_l;q)_{\mathrm{pol}}. \tag{98}$$

What we need to prove is that $\omega\, L^{\sigma}_{\beta/\alpha}(X;q) = N^{\sigma}_{\beta/\alpha}(X;q)$.

The definition of $\mathcal{L}^{\sigma}_{\beta/\alpha}(x;q)$ implies that $L^{\sigma}_{\beta/\alpha}(X;q)$ satisfies

$$\langle s_\lambda(X), L^{\sigma}_{\beta/\alpha}(X;q)\rangle = \langle E^{\sigma^{-1}}_{\beta}(x;q^{-1})\rangle\, s_\lambda(x_1,\ldots,x_l)\, E^{\sigma^{-1}}_{\alpha}(x;q^{-1}) \tag{99}$$

for every partition λ, including when $\ell(\lambda) > l$, since then both sides are zero. By linearity, we can replace s_λ by any symmetric function f, giving

$$\langle f(X),\, L^{\sigma}_{\beta/\alpha}(X;q)\rangle = \langle E^{\sigma^{-1}}_{\beta}(x;q^{-1})\rangle\, f(x)\, E^{\sigma^{-1}}_{\alpha}(x;q^{-1})\,. \tag{100}$$

The coefficient of $m_\mu(X)$ in $\omega\, L^{\sigma}_{\beta/\alpha}(X;q)$ is given by taking $f = e_\mu$.

To show that $\omega\, L^{\sigma}_{\beta/\alpha}(X;q)$ is given by the tableau generating function in (96), we use Lemma 4.5.1 to express

$$\langle E^{\sigma^{-1}}_{\beta}(x;q^{-1})\rangle\, e_\mu(x)\, E^{\sigma^{-1}}_{\alpha}(x;q^{-1}) \tag{101}$$

as a sum of powers of q indexed by negative tableaux. In particular, this coefficient will vanish unless we have $\alpha_j \leq \beta_j$ for all j, giving the last conclusion in the proposition.

Multiplying by e_{μ_1} through e_{μ_n} successively and keeping track of one chosen term in each product gives a sequence of terms $E^{\sigma^{-1}}_{\alpha^{(0)}}, E^{\sigma^{-1}}_{\alpha^{(1)}}, \ldots, E^{\sigma^{-1}}_{\alpha^{(n)}}$, in which $\alpha^{(0)} = \alpha$ and $\alpha^{(m+1)} = \alpha^{(m)} + \varepsilon_I$ for a set of indices I of size μ_m, for each m. Each sequence with $\alpha^{(n)} = \beta$ contributes to (101).

If we record these data in the form of a tableau $S\colon \beta/\alpha \to \mathbb{Z}_{>0}$ with $S(a) = m$ for $a \in (\alpha^{(m)}/\alpha^{(m-1)})$, S satisfies the condition that it is a negative tableau of weight $x^S = x^\mu$. The contribution to (101) from the corresponding sequence of terms is the product of the q^{h_I} with h_I as in (90) for $k = \mu_m$, $\lambda = \alpha^{(m-1)}$, and I the set of indices j such that $S(a) = m$ for some box a in row j.

We now express the h_I corresponding to $(\alpha^{(m)}/\alpha^{(m-1)}) = S^{-1}(\{m\})$ as an attribute of S. For h_I to count a pair $i < j$, we must have $i \in I$, which means that $S(b) = m$ for a box b in row i, and $j \notin I$, and one of the following two situations.

If $\sigma(i) < \sigma(j)$, we must have $\alpha^{(m-1)}_j = \alpha^{(m-1)}_i$. Since there is no m in row j of S, this means that the boxes a and c in row j with coordinates $i(a) = i(b) - 1$, $i(c) = i(b)$ have $S(a) < S(b) < S(c)$, with the same convention as above that $S(a) = -\infty$ if a is to the left of a row of β/α, and $S(c) = \infty$ if c is to the right of a row.

If $\sigma(i) > \sigma(j)$, we must have $\alpha_j^{(m-1)} = \alpha_i^{(m-1)} + 1$. This means that the boxes a and c in row j with coordinates $i(a) = i(b)$, $i(c) = i(b) + 1$ have $S(a) < S(b) < S(c)$, with the same convention as before.

These two cases establish that h_I is equal to the number of increasing σ-triples in S for which $S(b) = m$. Summing them up gives the total number of increasing σ-triples, implying that the coefficient in (101) is the sum of $q^{h_\sigma(S)}$ over negative tableaux S of weight $x^S = x^\mu$ on β/α, where $h_\sigma(S)$ is the number of increasing σ-triples in S. □

Similar to the notation $h_\sigma(S)$ in (96) for the number of increasing triples in a tableau S, we let $h_\sigma(\beta/\alpha)$ denote the total number of triples in a tuple of rows.

Lemma 4.5.3. *Given $\sigma \in S_l$ and $\alpha, \beta \in \mathbb{Z}^l$ with $\alpha_j \le \beta_j$ for all j, we have*

$$h_\sigma(\beta/\alpha) - \operatorname{inv}(T) = h_\sigma(S) \tag{102}$$

for $T \in \mathrm{SSYT}_-(\sigma(\beta/\alpha))$, where $S = \sigma^{-1}(T) = T \circ \sigma$.

Proof. Recall from §4.1 that an attacking inversion in a negative tableau is defined by $T(a) \ge T(b)$, where a, b is an attacking pair with a preceding b in the reading order.

One can verify from the definition of σ-triple that the attacking pairs in $\sigma(\beta/\alpha)$, ordered by the reading order, are precisely the pairs $(\sigma(a), \sigma(b))$ or $(\sigma(b), \sigma(c))$ for (a, b, c) a σ-triple such that the relevant boxes are in β/α. Moreover, every attacking pair occurs in this manner exactly once.

If all three boxes of a σ-triple (a, b, c) are in β/α, and T is a negative tableau on $\sigma(\beta/\alpha)$, then since $T(\sigma(a)) < T(\sigma(c))$, at most one of the attacking pairs $(\sigma(a), \sigma(b))$, $(\sigma(b), \sigma(c))$ can be an attacking inversion in T. The condition that neither pair is an attacking inversion is that $S(a) < S(b) < S(c)$ in the negative tableau $S = \sigma^{-1}(T) = T \circ \sigma$ on β/α. This also holds for triples not contained in β/α with our convention that $S(a) = -\infty$ or $S(c) = \infty$ for a or c outside the tuple β/α. Hence, the result follows. □

Example 4.5.4. Let S be as in Figure 1 and

$$\sigma = (1,2,3) \mapsto (3,1,2), \qquad T = S \circ \sigma^{-1} = \begin{array}{ccccc} & & \boxed{2}\,\boxed{4}\,\boxed{6}\,\boxed{7} \\ & & \boxed{4}\,\boxed{7}\,\boxed{9} & \\ & \boxed{1}\,\boxed{3} & & \end{array},$$

so T is a negative tableau on $\sigma(\beta/\alpha) = (3,5,6)/(1,2,2)$. Reading T by reading order on $\sigma(\beta/\alpha)$ gives 1 3 4 2 7 4 9 6 7. The pairs $(T(a), T(b))$ for attacking inversions (a, b) in T are $(3,2)$, $(4,2)$, $(7,4)$, and $(9,6)$, so $\operatorname{inv}(T) = 4$. From the last diagram in Figure 1, we have $h_\sigma(\beta/\alpha) = 5$, $h_\sigma(S) = 1$, so $h_\sigma(\beta/\alpha) - \operatorname{inv}(T) = h_\sigma(S)$ is indeed satisfied.

Now let $\sigma = 1$ and $T = S \circ \sigma^{-1} = S$. Reading T by reading order on β/α gives 1 2 3 4 4 7 6 9 7. The pairs $(T(a), T(b))$ for the attacking inversions are $(4,4)$, $(7,6)$ and $(9,7)$, so $\operatorname{inv}(T) = 3$; note that equal attacking entries $(4,4)$ count as an inversion in a negative tableau. From the second diagram in Figure 1, we see $h_\sigma(\beta/\alpha) = 7$, $h_\sigma(S) = 4$, so again $h_\sigma(\beta/\alpha) - \operatorname{inv}(T) = h_\sigma(S)$ holds.

Remark 4.5.5. If we define an increasing σ-triple for $T \in \mathrm{SSYT}(\sigma(\beta/\alpha))$ to be any σ-triple (a, b, c) satisfying $T(a) \le T(b) \le T(c)$, then a similar argument gives the relation in (102) for ordinary semistandard tableaux as well, where again $S = \sigma^{-1}(T) = T \circ \sigma$.

Remark 4.5.6. Given a tuple of rows β/α, if we shift the j-th row to the right by $\sigma(j)\,\epsilon$ for a small $\epsilon > 0$, then $h_\sigma(\beta/\alpha)$ is equal to the number of alignments between a box boundary in row j and the interior of a box in row i for $i < j$, where a boundary is the edge common to two adjacent boxes which are either in or adjacent to the row. An empty row has one boundary.

The relation between $\mathcal{G}$ and $\mathcal{L}$ can now be made precise by applying Proposition 4.5.2 and Lemma 4.5.3 to the expression for $q^{h_\sigma(\beta/\alpha)}\mathcal{G}_{\sigma(\beta/\alpha)}(X;q^{-1})$ given in Corollary 4.1.4.

Corollary 4.5.7. *Given* $\alpha,\beta\in\mathbb{Z}^l$ *and* $\sigma\in S_l$,

$$\mathcal{L}^\sigma_{\beta/\alpha}(x;q)_{\mathrm{pol}}=\begin{cases} q^{h_\sigma(\beta/\alpha)}\mathcal{G}_{\sigma(\beta/\alpha)}(x;q^{-1}) & \text{if } \alpha_j\le\beta_j \text{ for all } j\\ 0 & \text{otherwise,}\end{cases} \tag{103}$$

where $h_\sigma(\beta/\alpha)$ *is the number of* σ*-triples in* β/α*, as in Lemma 4.5.3, and the right hand side is evaluated in* l *variables* $x_1,\dots,x_l$.

5. The generalized Shuffle Theorem

5.1. Cauchy identity

In this section we derive our main results, Theorems 5.3.1 and 5.5.1. The key point is the following delightful 'Cauchy identity' for non-symmetric Hall-Littlewood polynomials.

Theorem 5.1.1. *For any permutation* $\sigma\in S_l$*, the twisted non-symmetric Hall-Littlewood polynomials* $E^\sigma_\lambda(x;q)$ *and* $F^\sigma_\lambda(x;q)$ *in (75, 76) satisfy the identity*

$$\frac{\prod_{i<j}(1-q\,t\,x_i\,y_j)}{\prod_{i\le j}(1-t\,x_i\,y_j)}=\sum_{\mathbf{a}\ge 0}t^{|\mathbf{a}|}\,E^\sigma_{\mathbf{a}}(x_1,\dots,x_l;q^{-1})\,F^\sigma_{\mathbf{a}}(y_1,\dots,y_l;q), \tag{104}$$

where the sum is over $\mathbf{a}=(a_1,\dots,a_l)$ *with all* $a_i\ge 0$ *and* $|\mathbf{a}|\overset{\mathrm{def}}{=}a_1+\cdots+a_l$.

Proof. Let $Z(x,y,q,t)$ denote the product on the left hand side.

From the definitions we see that $E^\sigma_{\mathbf{a}}(x;q)$ and $F^\sigma_{\mathbf{a}}(x;q)$ for $\mathbf{a}\ge 0$ belong to the polynomial ring $\mathbb{Q}(q)[x]=\mathbb{Q}(q)[x_1,\dots,x_l]$. The $E^\sigma_{\mathbf{a}}(x;q)$ form a graded basis of $\mathbb{Q}(q)[x]$, since they are homogeneous and $E^\sigma_{\mathbf{a}}(x;q)$ has leading term $x^{\mathbf{a}}$. The $F^\sigma_{\mathbf{a}}(x;q)$ likewise form a graded basis of $\mathbb{Q}(q)[x]$. We are to prove that the expansion of $Z(x,y,q,t)$ as a power series in t, with coefficients expressed in terms of the basis $\{E^\sigma_{\mathbf{a}}(x;q^{-1})F^\sigma_{\mathbf{b}}(y;q)\}$ of $\mathbb{Q}(q)[x,y]$, is given by the formula on the right hand side. Put another way, we are to show that the coefficient of $F^\sigma_{\mathbf{a}}(y;q)$ in $Z(x,y,q,t)$ is equal to $t^{|\mathbf{a}|}E^\sigma_{\mathbf{a}}(x;q^{-1})$, or equivalently that the coefficient of $F^\sigma_{\mathbf{a}}(y^{-1};q^{-1})$ in $Z(x,y^{-1},q^{-1},t)$ is equal to $t^{|\mathbf{a}|}E^\sigma_{\mathbf{a}}(x;q)$.

Using Proposition 4.3.2, this will follow by taking $f(y)=E^\sigma_{\mathbf{a}}(y;q)$ in the identity

$$f(tx)=\langle y^0\rangle\, f(y)\frac{\prod_{i<j}(1-q^{-1}t\,x_i/y_j)}{\prod_{i\le j}(1-t\,x_i/y_j)}\prod_{i<j}\frac{1-y_i/y_j}{1-q^{-1}y_i/y_j}, \tag{105}$$

provided we prove that this identity is valid for all polynomials $f(y)=f(y_1,\dots,y_l)$. Here we mean that $f(y)\in\mathbb{Q}(q)[y]$ is a true polynomial and not a Laurent polynomial. Note that the denominator factors in (105) should be understood as geometric series.

The only factor in (105) that involves negative powers of y_1 is $1/(1-t\,x_1/y_1)$. All the rest is a power series as a function of y_1. For any power series $g(y_1)$, we have $\langle y_1^0\rangle\, g(y_1)/(1-t\,x_1/y_1)=g(tx_1)$. The factors other than $1/(1-t\,x_1/y_1)$ with index $i=1$ in (105) cancel upon setting $y_1=tx_1$. It follows that when we take the constant term in the variable y_1, (105) reduces to the same identity in variables $y_2,\dots,y_l$. We can assume that the latter holds by induction. □

Example 5.1.2. The products $t^{|\mathbf{a}|} E^\sigma_{\mathbf{a}}(x; q^{-1}) F^\sigma_{\mathbf{a}}(y; q)$ for the pairs in Table 1 sum to the t^0 through t^2 terms in the expansion of

$$\frac{(1 - q\, t\, x_1\, y_2)(1 - q\, t\, x_1\, y_3)(1 - q\, t\, x_2\, y_3)}{(1 - t\, x_1\, y_1)(1 - t\, x_1\, y_2)(1 - t\, x_1\, y_3)(1 - t\, x_2\, y_2)(1 - t\, x_2\, y_3)(1 - t\, x_3\, y_3)}. \tag{106}$$

Remark 5.1.3. Using the fact that our non-symmetric Hall-Littlewood polynomials agree with those of Ion in [22], the $\sigma = 1$ case of (104) can be derived from the Cauchy identity for non-symmetric Macdonald polynomials of Mimachi and Noumi [29]. We also note that (104) for $\sigma = 1$ specializes at $q = 0$ to the GL_l case of the non-symmetric Cauchy identities of Fu and Lascoux [9].

5.2. Winding permutations

We will apply Theorem 5.1.1 in cases for which the twisting permutation has a special form, allowing the Hall-Littlewood polynomial $F^\sigma_{\mathbf{a}}(y; q)$ to be written another way.

Definition 5.2.1. Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of a real number x. Let $c_1, \ldots, c_l$ be the sequence of fractional parts $c_i = \{a\, i + b\}$ of an arithmetic progression, where a is assumed irrational, so the c_i are distinct. Let $\sigma \in S_l$ be the permutation such that $\sigma(c_1, \ldots, c_l)$ is increasing, i.e., $\sigma(1), \ldots, \sigma(l)$ are in the same relative order as $c_1, \ldots, c_l$.

A permutation σ of this form is a *winding permutation*. The *descent indicator* of σ is the vector $(\eta_1, \ldots, \eta_{l-1})$ defined by

$$\eta_i = \begin{cases} 1 & \text{if } \sigma(i) > \sigma(i+1), \\ 0 & \text{if } \sigma(i) < \sigma(i+1). \end{cases} \tag{107}$$

The *head* and *tail* of the winding permutation σ are the respective permutations τ, $\theta \in S_{l-1}$ such that $\tau(1), \ldots, \tau(l-1)$ are in the same relative order as $\sigma(1), \ldots, \sigma(l-1)$, and $\theta(1), \ldots, \theta(l-1)$ are in the same relative order as $\sigma(2), \ldots, \sigma(l)$.

Adding an integer to a in the above definition doesn't change the c_i, so we can assume that $0 < a < 1$. In that case the descent indicator of σ is characterized by

$$\begin{aligned} \eta_i = 1 \quad &\Leftrightarrow \quad c_i > c_{i+1} \quad \Leftrightarrow \quad c_{i+1} = c_i + a - 1, \\ \eta_i = 0 \quad &\Leftrightarrow \quad c_i < c_{i+1} \quad \Leftrightarrow \quad c_{i+1} = c_i + a. \end{aligned} \tag{108}$$

Proposition 5.2.2. *Let $\sigma \in S_l$ be a winding permutation, with descent indicator η, and head and tail τ, $\theta \in S_{l-1}$. For every $\lambda \in \mathbb{Z}^{l-1}$ we have identities*

$$E^{\theta^{-1}}_\lambda(x; q) = x^\eta\, E^{\tau^{-1}}_{\lambda-\eta}(x; q), \tag{109}$$

$$F^{\theta^{-1}}_\lambda(x; q) = x^\eta\, F^{\tau^{-1}}_{\lambda-\eta}(x; q) \tag{110}$$

of Laurent polynomials in $x_1, \ldots, x_{l-1}$.

The proof uses the following lemma.

Lemma 5.2.3. *With τ, θ and η as in Proposition 5.2.2, and for every $w \in S_{l-1}$, there is an identity of operators on $\mathbf{k}[x_1^{\pm 1}, \ldots, x_{l-1}^{\pm 1}]$*

$$T^{-1}_{\tau w}\, T_\tau\, x^{-\eta}\, T^{-1}_\theta\, T_{\theta w} = q^e\, x^{-w^{-1}(\eta)}, \tag{111}$$

for some exponent e depending on w.

Proof. We prove (111) by induction on the length of w. The base case $w = 1$ is trivial. Suppose now that $w = vs_i$ is a reduced factorization. We can write the left hand side of (111) as

$$T_i^{\varepsilon_1} T_{\tau v}^{-1} T_\tau x^{-\eta} T_\theta^{-1} T_{\theta v} T_i^{\varepsilon_2}, \tag{112}$$

where

$$\varepsilon_1 = \begin{cases} +1 & \text{if } \tau v s_i < \tau v \\ -1 & \text{if } \tau v s_i > \tau v \end{cases} \qquad \varepsilon_2 = \begin{cases} +1 & \text{if } \theta v s_i > \theta v \\ -1 & \text{if } \theta v s_i < \theta v \end{cases}. \tag{113}$$

Assuming by induction that (111) holds for v, and substituting this into (112), we are left to show that

$$T_i^{\varepsilon_1} x^{-v^{-1}(\eta)} T_i^{\varepsilon_2} = q^e x^{-s_i v^{-1}(\eta)} \tag{114}$$

for some exponent e. We now consider the possible values for $\langle \alpha_i^\vee, -v^{-1}(\eta)\rangle$, which is equal to $\eta_k - \eta_j$, where $v(i) = j$, $v(i+1) = k$.

Case 1: If $\eta_j = \eta_k$, we see from (108) that $c_{j+1} - c_j = c_{k+1} - c_k$, hence $c_{j+1} < c_{k+1} \Leftrightarrow c_j < c_k$. This implies that $\sigma(j+1) < \sigma(k+1) \Leftrightarrow \sigma(j) < \sigma(k)$, and therefore that $\tau v(i) < \tau v(i+1) \Leftrightarrow \theta v(i) < \theta v(i+1)$. Hence, in this case we have $\varepsilon_1 = -\varepsilon_2$.

Case 2: If $\eta_j = 1$ and $\eta_k = 0$, then from (108) we get $c_{j+1} = c_j + a - 1$ and $c_{k+1} = c_k + a$. Then $c_{k+1} - c_{j+1} = c_k - c_j + 1$. Since $c_{k+1} - c_{j+1}$ and $c_k - c_j$ both have absolute value less than 1, this implies $c_k < c_j$ and $c_{k+1} > c_{j+1}$. It follows in the same way as in Case 1 that $\tau v(i) > \tau v(i+1)$ and $\theta v(i) < \theta v(i+1)$. Hence, in this case we have $\varepsilon_1 = \varepsilon_2 = 1$.

Case 3: If $\eta_j = 0$ and $\eta_k = 1$ we reason as in Case 2, but with j and k exchanged, to conclude that in this case we have $\varepsilon_1 = \varepsilon_2 = -1$.

In each case, (114) now follows from the well-known affine Hecke algebra identities

$$\begin{aligned} T_i^{-1} x^\mu T_i = T_i x^\mu T_i^{-1} = x^\mu &= x^{s_i\mu} && \text{if } \langle \alpha_i^\vee, \mu\rangle = 0, \\ T_i x^\mu T_i &= q\, x^{s_i\mu} && \text{if } \langle \alpha_i^\vee, \mu\rangle = -1, \\ T_i^{-1} x^\mu T_i^{-1} &= q^{-1} x^{s_i\mu} && \text{if } \langle \alpha_i^\vee, \mu\rangle = 1, \end{aligned} \tag{115}$$

which can be verified directly from the definition of T_i. □

Proof of Proposition 5.2.2. Let $w_0 \in S_l$ and $w_0' \in S_{l-1}$ be the longest permutations. Then $w_0'\tau$, $w_0'\theta$ are the head and tail of the winding permutation $w_0\sigma$, and the descent indicator of $w_0\sigma$ is $\eta_i' = 1 - \eta_i$. Using these facts and the definition (76) of the F's in terms of the E's, one can check that (109) implies (110).

To prove (109), we begin by observing that for any given λ there exists $w \in S_l$ such that both $w^{-1}(\lambda)$ and $w^{-1}(\lambda - \eta)$ are dominant. To see this, first choose any v such that $v(\lambda) = \lambda_+$ is dominant. Since η is $\{0,1\}$-valued, the weight $\mu = v(\lambda - \eta) = \lambda_+ - v(\eta)$ has the property that for all $i < j$, if $\mu_i < \mu_j$ then $(\lambda_+)_i = (\lambda_+)_j$. Hence, there is a permutation u that fixes λ_+ and sorts μ into weakly decreasing order, so $u(\mu) = uv(\lambda - \eta)$ is dominant. Since $uv(\lambda) = \lambda_+$ is also dominant, $w^{-1} = uv$ works.

Now, Lemma 5.2.3 implies

$$T_{\tau w}^{-1} T_\tau x^{-\eta} T_\theta^{-1} T_{\theta w}(x^{w^{-1}(\lambda)}) \sim x^{-w^{-1}(\eta)} x^{w^{-1}(\lambda)}, \tag{116}$$

where $\sim$ signifies that the expressions are equal up to a q power factor. Equivalently,

$$T_\theta^{-1} T_{\theta w} x^{w^{-1}(\lambda)} \sim x^\eta T_\tau^{-1} T_{\tau w} x^{w^{-1}(\lambda-\eta)}. \tag{117}$$

Writing out the definitions of $E_\lambda^{\theta^{-1}}$ and $E_{\lambda-\eta}^{\tau^{-1}}$, while ignoring q power factors, and using the fact that $\lambda_+ = w^{-1}(\lambda)$ and $(\lambda - \eta)_+ = w^{-1}(\lambda - \eta)$ for this w, (117) implies that (109) holds up to a scalar factor q^e. But we know that the x^λ term on each side has coefficient 1, so (109) holds exactly. $\square$

5.3. Stable Shuffle Theorem

We now prove an identity of formal power series with coefficients in GL_l characters, that is, symmetric Laurent polynomials in variables $x_1, \ldots, x_l$. When truncated to the polynomial part, this identity will reduce to our Shuffle Theorem for paths under a line (Theorem 5.5.1).

Theorem 5.3.1. *Let p, s be real numbers with p positive and irrational. For $i = 1, \ldots, l$, let*

$$b_i = \lfloor s - p(i-1) \rfloor - \lfloor s - pi \rfloor.$$

Let $c_i = \{s - p(i-1)\}$, and let $\sigma \in S_l$ be the permutation such that $\sigma(1), \ldots, \sigma(l)$ are in the same relative order as $c_l, \ldots, c_1$, i.e., $\sigma(c_l, \ldots, c_1)$ is increasing. For any non-negative integers u, v we have the identity of formal power series in t

$$\mathcal{H}_{b_1+u, b_2, \ldots, b_{l-1}, b_l - v} = \sum_{a_1, \ldots, a_{l-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}^{\sigma}_{((b_l, \ldots, b_1)+(-v, a_{l-1}, \ldots, a_1))/(a_{l-1}, \ldots, a_1, -u)}(x; q), \tag{118}$$

where $\mathcal{H}_{\mathbf{b}}$ is given by Definition 3.7.1.

Remark 5.3.2. If δ is the highest south-east lattice path weakly below the line $y + px = s$, starting at $(0, \lfloor s \rfloor)$ and extending forever (not stopping at the x axis), then b_i is the number of south steps in δ along the line $x = i - 1$, and c_i is the gap along $x = i - 1$ between the given line and the highest point of δ beneath it.

Proof of Theorem 5.3.1. We will prove that for $\mathbf{b}$, σ as in the hypothesis of the theorem, we have the stronger 'unstraightened' identity

$$x_1^u x_l^{-v} x^{\mathbf{b}} \frac{\prod_{i+1<j}(1 - q\, t\, x_i/x_j)}{\prod_{i<j}(1 - t\, x_i/x_j)} \\ = \sum_{a_1, \ldots, a_{l-1} \geq 0} t^{|\mathbf{a}|} w_0 \big(F^{\sigma^{-1}}_{(b_l, \ldots, b_1)+(-v, a_{l-1}, \ldots, a_1)}(x; q) \overline{E^{\sigma^{-1}}_{(a_{l-1}, \ldots, a_1, -u)}(x; q)} \big). \tag{119}$$

By Proposition 4.4.2, applying the Hall-Littlewood raising operator $\mathbf{H}_q$ to both sides of (119) yields (118).

By construction, the b_i take only values $\lfloor p \rfloor$ or $\lceil p \rceil$, and since $b_i + c_i - c_{i+1} = p$, we have

$$b_i = \lfloor p \rfloor \quad \Leftrightarrow \quad c_i > c_{i+1} \quad \Leftrightarrow \quad \sigma(l-i) < \sigma(l-i+1). \tag{120}$$

In particular, $b_l \leq b_{l-1} + 1$, hence $b_l - v \leq b_{l-1} + a_{l-1} + 1$, and if equality holds, then $b_{l-1} = \lfloor p \rfloor$, so $\sigma(1) < \sigma(2)$. Using Lemma 4.3.4, and recalling that the definition (76) of $F^{\sigma}_{\lambda}(x; q)$ is $\overline{E^{\sigma w_0}_{-\lambda}(x; q)}$, we have

$$E^{\sigma^{-1}}_{(a_{l-1}, \ldots, a_1, -u)}(x; q) = x_l^{-u} E^{\tau^{-1}}_{(a_{l-1}, \ldots, a_1)}(x_1, \ldots, x_{l-1}; q) \tag{121}$$

$$F^{\sigma^{-1}}_{(b_l, \ldots, b_1)+(-v, a_{l-1}, \ldots, a_1)}(x; q) = x_1^{b_l - v} F^{\theta^{-1}}_{(b_{l-1}, \ldots, b_1)+(a_{l-1}, \ldots, a_1)}(x_2, \ldots, x_l; q), \tag{122}$$

where τ, θ are the head and tail of σ, as in Proposition 5.2.2. Note that σ is a winding permutation. From (120) we also see that $(b_{l-1}, \ldots, b_1) = \eta + \lfloor p \rfloor \cdot (1, \ldots, 1)$, where η is the descent indicator of σ. Adding a constant vector $k \cdot (1, \ldots, 1)$ to the index λ multiplies any

$F^\pi_\lambda(x;q)$ by $(\prod_i x_i)^k$. Using this and Proposition 5.2.2, we can replace (122) with

$$F^{\sigma^{-1}}_{(b_l,\dots,b_1)+(-v,a_{l-1},\dots,a_1)}(x;q) = x_1^{b_l-v} x_2^{b_{l-1}} \cdots x_l^{b_1} F^{\tau^{-1}}_{(a_{l-1},\dots,a_1)}(x_2,\dots,x_l;q) \tag{123}$$

Using the Cauchy identity (104) from Theorem 5.1.1 in $l-1$ variables, with twisting permutation τ^{-1}, and substituting x_i^{-1} for x_i, we obtain

$$\frac{\prod_{i<j}(1-q\,t\,y_j/x_i)}{\prod_{i\leq j}(1-t\,y_j/x_i)} = \sum_{\mathbf{a}\geq 0} t^{|\mathbf{a}|}\, F^{\tau^{-1}}_{\mathbf{a}}(y_1,\dots,y_{l-1};q)\overline{E^{\tau^{-1}}_{\mathbf{a}}(x_1,\dots,x_{l-1};q)}. \tag{124}$$

Setting $y_i = x_{i+1}$ in (124) and multiplying both sides by $w_0(x_1^u x_l^{-v} x^{\mathbf{b}})$, then using (121) and (123), and finally applying w_0 to both sides, yields (119). □

5.4. LLT data and the dinv statistic

Our next goal is to deduce the combinatorial version of our Shuffle Theorem—that is, the identity (1) previewed in the introduction and restated as (133), below—from Theorem 5.3.1. To do this we first need to define the data that will serve to attach LLT polynomials to lattice paths, and relate these to the combinatorial statistic $\operatorname{dinv}_p(\lambda)$.

We will be concerned with lattice paths λ lying weakly below the line segment

$$y + p\,x = s \qquad (p = s/r) \tag{125}$$

between arbitrary points $(0,s)$ and $(r,0)$ on the positive y and x axes.

We always assume that the slope $-p$ of the line is irrational. Clearly it is possible to perturb any line slightly so as make its slope irrational, without changing the set of lattice points, and therefore also the set of lattice paths, that lie below the line. All dependence on p in the combinatorial constructions to follow comes from comparisons between p and various rational numbers. By taking p to be irrational, we avoid the need to resolve ambiguities that would result from equality occurring in the comparisons.

Definition 5.4.1. Let λ be a south-east lattice path in the first quadrant with endpoints on the axes. Let Y be the Young diagram enclosed by the positive axes and λ. The *arm* and *leg* of a box $y \in Y$ are, as usual, the number of boxes in Y strictly east of y and strictly north of y, respectively. Given a positive irrational number p, we define $\operatorname{dinv}_p(\lambda)$ to be the number of boxes in Y whose arm a and leg ℓ satisfy

$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a}, \tag{126}$$

where we interpret $(\ell+1)/a$ as $+\infty$ if $a=0$.

Geometrically, condition (126) means that some line of slope $-p$ crosses both the east step in λ at the top of the leg and the south step at the end of the arm, as shown in Figure 2. Since p is irrational, such a line can always be assumed to pass through the interiors of the two steps.

To each lattice path weakly below the line (125) we now attach a tuple of one-row skew shapes β/α and a permutation σ. The index $\nu(\lambda)$ of the LLT polynomial in (1) and (133) will be defined in terms of these data.

Definition 5.4.2. Let λ be a south-east lattice path from $(0,\lfloor s\rfloor)$ to $(\lfloor r\rfloor,0)$ which is weakly below the line $y+p\,x=s$ in (125), where $p=s/r$ is irrational, and let $l=\lfloor r\rfloor+1$. For $i=1,\dots,l$, let

$$d_i = \lfloor s - p(i-1)\rfloor \tag{127}$$

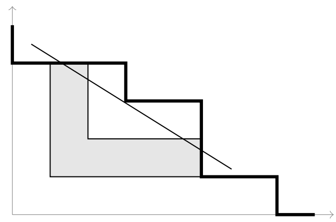


Figure 2. *A p-balanced hook in a Young diagram, where the diagonal line has slope $-p$.*

be the y coordinate of the highest lattice point weakly below the given line at $x = i - 1$. Let

$$\alpha = (\alpha_l, \dots, \alpha_1), \quad \beta = (\beta_l, \dots, \beta_1) \tag{128}$$

be the vectors of integers $0 \le \alpha_i \le \beta_i$, written in reverse order, such that the south steps in λ on the line $x = i - 1$ go from $y = d_i - \alpha_i$ to $y = d_i - \beta_i$. Let

$$c_i = s - p(i-1) - d_i = \{s - p(i-1)\} \tag{129}$$

be the gap between the given line and the highest lattice point weakly below it along the line $x = i - 1$. Let $\sigma \in S_l$ be the permutation with $\sigma(1), \dots, \sigma(l)$ in the same relative order as $c_l, \dots, c_1$, i.e., such that $\sigma(c_l, \dots, c_1)$ is increasing. The vectors α and β and the permutation σ are the *LLT data* associated with λ and the given line.

Example 5.4.3. The first diagram in Figure 3 shows a line $y + px = s$ with $p \approx 1.36$, $s \approx 9.27$, and a path λ below it from $(0, \lfloor s \rfloor) = (0, 9)$ to $(l - 1, 0) = (6, 0)$ with $l = 7$.

In this example, the y coordinates of the highest lattice points below the line at $x = 0, \dots, 6$ are $(d_1, \dots, d_7) = (9, 7, 6, 5, 3, 2, 1)$. The runs of south steps in λ go from y coordinates $(9, 6, 6, 3, 1, 1, 0)$ to $(6, 6, 3, 1, 1, 0, 0)$. Subtracting these from the d_i and listing them in reverse order gives

$$\alpha = (1, 1, 2, 2, 0, 1, 0), \quad \beta = (1, 2, 2, 4, 3, 1, 3). \tag{130}$$

The gaps, in reverse order, are $(c_7, \dots, c_1) \approx (.11, .47, .83, .19, .55, .91, .27)$, giving

$$\sigma = (1, 4, 6, 2, 5, 7, 3). \tag{131}$$

Proposition 5.4.4. *Given the line* (125) *and a lattice path λ weakly below it satisfying the conditions in Definition 5.4.2, let α, β, σ be the associated LLT data. Then*

$$\operatorname{dinv}_p(\lambda) = h_\sigma(\beta/\alpha), \tag{132}$$

where $\operatorname{dinv}_p(\lambda)$ *is given by Definition 5.4.1 and* $h_\sigma(\beta/\alpha)$ *is as in Corollary 4.5.7.*

Proof. Let λ' be the image of λ under the transformation in the plane that sends (x, y) to $(x, y + px)$. Then λ' is a path composed of unit south steps and sloped steps $(1, p)$ (transforms of east steps), which starts at $(0, \lfloor s \rfloor)$ and stays weakly below the horizontal line $y = s$ (transform of the line $y + p\,x = s$).

The south steps in λ' on the line $x = i - 1$ run from $y = s - (c_i + \alpha_i)$ to $y = s - (c_i + \beta_i)$. This means that if we offset the i-th row $(\beta_{l+1-i})/(\alpha_{l+1-i})$ in the tuple of one-row skew diagrams β/α by c_{l+1-i}, then the x coordinate on each box of β/α covers the same unit interval as does the distance below the line $y = s$ on the south step in λ' corresponding to that box. See Figure 3 for an example.

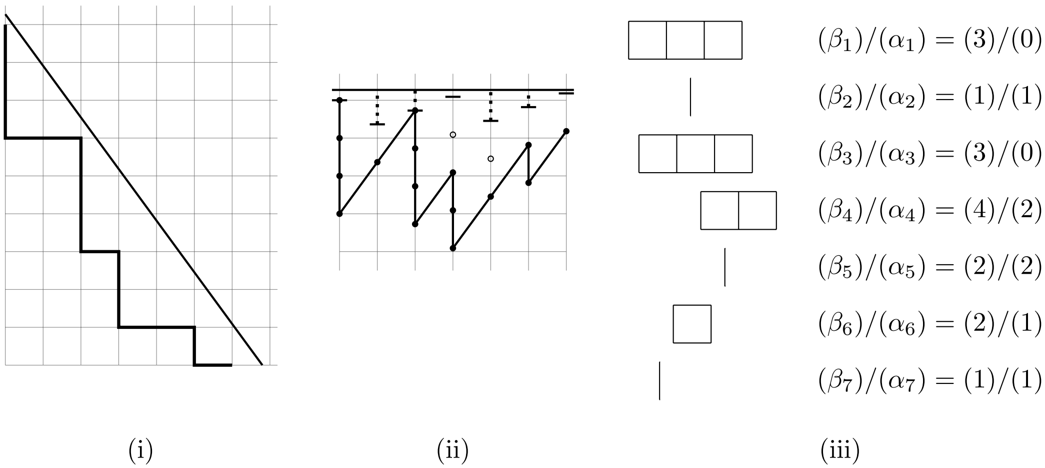


Figure 3. *(i) A path λ under $y+px = s$ with $p \approx 1.36$, $s \approx 9.27$, $l = 7$. (ii) Transformed path λ' under $y = s$, with gaps c_i marked. (iii) Bottom to top: tuple of rows $(\beta_7, \dots, \beta_1)/(\alpha_7, \dots, \alpha_1)$ offset by $(c_7, \dots, c_1)$.*

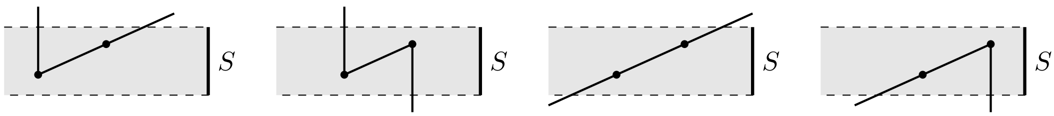


Figure 4. *Types of connected components C in the proof of Proposition 5.4.4.*

Since $0 < c_i < 1$ and the numbers $c_l, \dots, c_1$ are in the same relative order as $\sigma(1), \dots, \sigma(l)$, the description of $h_\sigma(\beta/\alpha)$ in Remark 4.5.6 still applies if we offset row i by c_{l+1-i} instead of $\sigma(i)\epsilon$. Mapping this onto λ', we see that $h_\sigma(\beta/\alpha)$ is the number of horizontal alignments between any endpoint of a step in λ' and the interior of a south step occurring later in λ'. To put this another way, for each south step S in λ', let B_S denote the interior of the horizontal band of height 1 to the left of S in the plane. Then $h_\sigma(\beta/\alpha)$ is the number of pairs consisting of a south step S and a point $P \in B_S$ which is an endpoint of a step in λ'.

For comparison, $\mathrm{dinv}_p(\lambda)$ is the number of pairs consisting of a south step S in λ' and a sloped step which meets B_S. To complete the proof it suffices to show that each band B_S contains the same number of step endpoints P as the number of sloped steps that meet B_S. In fact, we make the following stronger claim: *within each band B_S, step endpoints alternate from left to right with fragments of sloped steps, starting with a step endpoint and ending with a sloped step fragment.*

To see this, consider a connected component C of $\lambda' \cap B_S$. Each component C either enters B_S from above along a south step or from below along a sloped step, and exits B_S either at the top along a sloped step or at the bottom along a south step, except in two degenerate situations. One of these occurs if C contains the starting point $(0, \lfloor s \rfloor)$ of λ'. In this case we regard C as entering B_S from above. The other is if C contains a sloped step that adjoins S. Then we regard C as exiting B_S at the top.

Each component C thus belongs to one of four cases shown in Figure 4. Note that since B_S has height 1, it cannot contain a full south step of λ'. In Figure 4 we have chosen $p < 1$ in order to illustrate the possibility that B_S might contain full sloped steps of λ'. If $p > 1$, then B_S can only meet sloped steps in proper fragments.

On each component C, step endpoints clearly alternate with sloped step fragments, starting with a step endpoint if C enters from above, or with a sloped step fragment if C enters from below, and ending with a step endpoint if C exits at the bottom, or with a sloped step fragment if C exits at the top. Since the distance from the line $y = s$ to the starting point of λ' is less than 1, the leftmost component C of $\lambda' \cap B_S$ always enters B_S from the top. Each subsequent component from left to right must enter B_S from the same side (top or bottom) that the previous component exited. This implies the claim stated above. □

5.5. Shuffle Theorem for paths under a line

We now prove the identity previewed as (1) in the introduction.

Theorem 5.5.1. *Let r, s be positive real numbers with $p = s/r$ irrational. We have the identity*

$$D_{b_1,\dots,b_l} \cdot 1 = \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega(\mathcal{G}_{\nu(\lambda)}(X; q^{-1})), \tag{133}$$

where the sum is over lattice paths λ from $(0, \lfloor s \rfloor)$ to $(\lfloor r \rfloor, 0)$ lying weakly below the line (125) through $(0, s)$ and $(r, 0)$, and the other pieces of (133) are defined as follows.

The integer $a(\lambda)$ is the number of lattice squares enclosed between λ and δ, where δ is the highest path from $(0, \lfloor s \rfloor)$ to $(\lfloor r \rfloor, 0)$ weakly below the given line. The index b_i is the number of south steps in δ along the line $x = i - 1$, for $i = 1, \dots, l$, where $l = \lfloor r \rfloor + 1$. The integer $\mathrm{dinv}_p(\lambda)$ is given by Definition 5.4.1.

The LLT polynomial $\mathcal{G}_{\nu(\lambda)}(X; q)$ is indexed by the tuple of one-row skew shapes $\nu(\lambda) = \sigma(\beta/\alpha)$, where α, β and σ are the LLT data associated to λ in Definition 5.4.2. More explicitly, $\sigma(\beta/\alpha) = (\beta_{w_0\sigma^{-1}(1)}, \dots, \beta_{w_0\sigma^{-1}(l)})/(\alpha_{w_0\sigma^{-1}(1)}, \dots, \alpha_{w_0\sigma^{-1}(l)})$, where $\alpha = (\alpha_l, \dots, \alpha_1)$ and $\beta = (\beta_l, \dots, \beta_1)$.

The operator $D_{b_1,\dots,b_l}$ on the left hand side is a Negut element in $\mathcal{E}$, as defined in §3.6, so that $D_{b_1,\dots,b_l} \cdot 1$ satisfies (58).

Proof. We prove the equivalent identity

$$\omega(D_{b_1,\dots,b_l} \cdot 1) = \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1}). \tag{134}$$

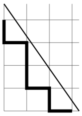
By Corollary 3.7.2 and Lemma 4.1.8, both sides of (134) involve only Schur functions $s_\lambda(X)$ indexed by partitions such that $\ell(\lambda) \le l$. It therefore suffices to prove that (134) holds when evaluated in l variables $x_1, \dots, x_l$. After doing this and using the formula (58) from Corollary 3.7.2, the desired identity becomes

$$(\mathcal{H}_{\mathbf{b}})_{\mathrm{pol}} = \sum_{\lambda} t^{a(\lambda)}\, q^{\mathrm{dinv}_p(\lambda)}\, \mathcal{G}_{\nu(\lambda)}(x_1, \dots, x_l; q^{-1}). \tag{135}$$

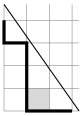
This is the same identity (24) that was mentioned in the introduction to §3. We now prove it using Theorem 5.3.1.

Let $b'_i = \lfloor s - p(i-1) \rfloor - \lfloor s - pi \rfloor$. As in Remark 5.3.2, this is the number of south steps along $x = i - 1$ in the highest south-east path δ' under our given line, where δ' starts at $(0, \lfloor s \rfloor)$ and extends forever. For $i < l$ we have $b'_i = b_i$. On the line $x = l - 1 = \lfloor r \rfloor$, however, the path δ stops at $(l - 1, 0)$, while δ' may extend below the x-axis, giving $b_l \le b'_l$.

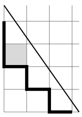
We now apply Theorem 5.3.1 with b'_i in place of b_i, $u = 0$, and $v = b'_l - b_l$, and then take the polynomial part on both sides of (118). This gives the same left hand side as in (135). On the right hand side, by Corollary 4.5.7, only those terms survive for which the index $\mathbf{a}$ satisfies



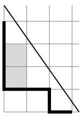
$a(\lambda) = 0$, $\operatorname{dinv}_p(\lambda) = 4$
$\mathcal{G}_{2011/0000}(X;q) =$
$s_4 + (q^2+q)s_{31} + q^2 s_{22} + q^3 s_{211}$



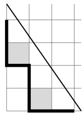
$a(\lambda) = 1$, $\operatorname{dinv}_p(\lambda) = 2$
$\mathcal{G}_{3011/0001}(X;q) = s_4 + q\,s_{31}$



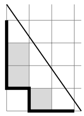
$a(\lambda) = 1$, $\operatorname{dinv}_p(\lambda) = 3$
$\mathcal{G}_{2021/1000}(X;q) =$
$s_4 + (q^2+q)s_{31} + q^2 s_{22} + q^3 s_{211}$



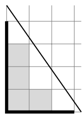
$a(\lambda) = 2$, $\operatorname{dinv}_p(\lambda) = 1$
$\mathcal{G}_{2031/2000}(X;q) = s_4 + q\,s_{31}$



$a(\lambda) = 2$, $\operatorname{dinv}_p(\lambda) = 2$
$\mathcal{G}_{3021/1001}(X;q) =$
$s_4 + q\,s_{31} + q^2 s_{22}$



$a(\lambda) = 3$, $\operatorname{dinv}_p(\lambda) = 1$
$\mathcal{G}_{3031/2001}(X;q) = s_4 + q\,s_{31}$



$a(\lambda) = 4$, $\operatorname{dinv}_p(\lambda) = 0$
$\mathcal{G}_{3041/3001}(X;q) = s_4$

Figure 5. *An illustration of Theorem 5.5.1 as described in Example 5.5.3.*

$(a_{l-1}, \dots, a_1, 0) \leq (b_l, \dots, b_1) + (0, a_{l-1}, \dots, a_1)$ in each coordinate, that is, for which

$$a_{l-1} \leq b_l \quad \text{and} \quad a_i \leq a_{i+1} + b_{i+1} \text{ for } i = 1, \dots, l-2. \tag{136}$$

Now, (136) is precisely the condition for there to exist a (unique) lattice path λ from $(0, \lfloor s \rfloor)$ to $(\lfloor r \rfloor, 0)$ such that a_i is the number of lattice squares in the i-th column (defined by $x \in [i-1, i]$) of the region between λ and the highest path δ. Moreover, when (136) holds, the LLT data for λ, as in Definition 5.4.2, are given by

$$\begin{aligned} \beta &= (b_l, \dots, b_1) + (0, a_{l-1}, \dots, a_1), \\ \alpha &= (a_{l-1}, \dots, a_1, 0), \end{aligned} \tag{137}$$

and $\sigma \in S_l$ such that $\sigma(1), \dots, \sigma(l)$ are in the same relative order as $c_l, \dots, c_1$, where $c_i = \{s - p(i-1)\}$, as in Theorem 5.3.1. Hence, by Corollary 4.5.7 and Proposition 5.4.4, we have

$$\mathcal{L}^{\sigma}_{((b_l,\dots,b_1)+(0,a_{l-1},\dots,a_1))/(a_{l-1},\dots,a_1,0)}(x;q)_{\mathrm{pol}} = q^{\operatorname{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x;q^{-1}). \tag{138}$$

When (136) holds we clearly also have $a(\lambda) = |\mathbf{a}|$. This shows that the polynomial part of the right hand side in (118) is the same as the right hand side of (135). ☐

Remark 5.5.2. The preceding argument also goes through with $u > 0$ in Theorem 5.3.1 to give a slightly more general version of Theorem 5.5.1 in which the sum is over lattice paths λ that start at a higher point $(0, n)$ on the y axis, with $n = \lfloor s \rfloor + u$, go directly south to $(0, \lfloor s \rfloor)$, and then continue below the given line to $(l-1, 0)$ as before.

The corresponding modifications to (133) are *(i)* the index b_1 on the left hand side is the number of south steps in λ on the y axis including the extension to $(0, n)$, and *(ii)* the row in $\nu(\lambda)$ corresponding to south steps in λ on the y axis is also extended accordingly.

Example 5.5.3. Figure 5 illustrates Theorem 5.5.1 for $s \approx 4.7$, $r \approx 3.31$. We have $(c_4, c_3, c_2, c_1) \approx (.44, .86, .28, .70)$ and $\sigma = (1,2,3,4) \mapsto (2,4,1,3)$. The paths λ are shown along with the corresponding statistics and LLT polynomials $\mathcal{G}_{\nu(\lambda)}(X;q) = \mathcal{G}_{\sigma(\beta)/\sigma(\alpha)}(X;q)$.

The highest path δ is the one at the top left in the figure and $(b_1, b_2, b_3, b_4) = (1, 2, 1, 0)$. The left side of (134), evaluated in $l = 4$ variables, is then

$$\omega(D_{1,2,1,0} \cdot 1)(x) = \mathcal{H}_{(1,2,1,0)}(x)_{\mathrm{pol}} = \boldsymbol{\sigma}\Big(\frac{x_1 x_2^2 x_3 (1 - q\,t\,x_1/x_3)(1 - q\,t\,x_2/x_4)(1 - q\,t\,x_1/x_4)}{\prod_{1 \le i < j \le 4}(1 - q\,x_i/x_j)(1 - t\,x_i/x_j)}\Big)_{\mathrm{pol}}. \tag{139}$$

To see that (134) holds at $t = 0$, observe that the right hand side of (139) becomes the Hall-Littlewood polynomial $\mathbf{H}_q(x_1 x_2^2 x_3)_{\mathrm{pol}} = q H_{2110}(x; q)$, which agrees with the area 0 contribution $q^4 \mathcal{G}_{2011/0000}(x; q^{-1})$.

Definition 5.5.4. For $\mathbf{b} \in \mathbb{Z}^l$, the *generalized q, t-Catalan number $C_{\mathbf{b}}(q, t)$* is the coefficient of the single row Schur function $s_{(|\mathbf{b}|)}(X)$ in $\omega(D_{\mathbf{b}} \cdot 1)$.

When $\mathbf{b} = 1^l$, $C_{\mathbf{b}}(q, t)$ is the q, t-Catalan number introduced by Garsia and the second author [10]. The generalized q, t-Catalan numbers have been studied in [13]—see §7.2.

Corollary 5.5.5. *With $\mathbf{b} = (b_1, \ldots, b_l)$, r, s, and $p = s/r$ as in Theorem 5.5.1,*

$$C_{\mathbf{b}}(q, t) = \sum_{\lambda} t^{a(\lambda)} q^{\mathrm{dinv}_p(\lambda)}, \tag{140}$$

where the sum is over lattice paths λ from $(0, \lfloor s \rfloor)$ to $(\lfloor r \rfloor, 0)$ lying weakly below the line through $(0, s)$ and $(r, 0)$.

6. Relation to previous Shuffle Theorems

Theorem 5.5.1 is formulated a little differently than the classical and (km, kn) Shuffle Theorems in [4, 16], although these also have an algebraic side and a combinatorial side resembling ours. We now explain how to recover them from our version by transforming each side of (133) into its counterpart in the (km, kn) and classical Shuffle Conjectures.

For the (km, kn) Shuffle Conjecture, we take the line $y + px = s$ in (125) to be a perturbation of the line from $(0, kn)$ to $(km, 0)$, with $s = kn$ and r slightly larger than km. Our perturbed line has the same lattice points and paths under it as the line from $(0, kn)$ to $(km, 0)$, but it now has slope $-p$, where $p = n/m - \epsilon$ for a small $\epsilon > 0$. The classical Shuffle Conjectures in [16] are the special cases of the (km, kn) conjecture with $n = 1$. For these we perturb the line from $(0, k)$ to $(km, 0)$ in the same way. Note that for our chosen line we have $l = km + 1$, and every lattice path λ under it has $b_l = 0$ south steps at $x = km$.

The classical Shuffle Conjecture was formulated in [16, Conjecture 6.2.2] as the identity

$$\nabla^m e_k = \sum_{\lambda} \sum_{P \in \mathrm{SSYT}\,((\lambda + (1^k))/\lambda)} t^{a(\lambda)} q^{\mathrm{dinv}_m(P)} x^P, \tag{141}$$

where the sum is over lattice paths λ below the bounding line, and $\mathrm{dinv}_m(P)$ is a statistic defined in [16], attached to each labelling P of the south steps in λ by non-negative integers strictly increasing from south to north along each vertical run. Rather than recall the original definition of $\mathrm{dinv}_m(P)$, we will use results from [16] to obtain a formula for it in (143), below. The left hand side of (141) is $e_k[-MX^{m,1}] \cdot 1$ by Corollary 3.7.4. This agrees with the left hand side $D_{b_1, \ldots, b_l} \cdot 1$ of (133) by Corollary 3.7.3.

It was noted and used in [16] that the combinatorial side of (141) can be phrased in terms of LLT polynomials, but to explicitly match with our formulation requires that we transform the right hand side of (133) as follows. For the given $\nu(\lambda) = \sigma(\beta/\alpha)$, apply Proposition 4.1.6

to replace $\omega\mathcal{G}_{\nu(\lambda)}(X;q^{-1})$ with $q^{-I(\nu(\lambda)^R)}\mathcal{G}_{\nu(\lambda)^R}(X;q)$. Then writing out $\mathcal{G}_{\nu(\lambda)^R}(X;q)$ term by term with tableaux on the tuple $\nu(\lambda)^R$ of one-column diagrams using Definition 4.1.2 gives

$$D_{b_1,\ldots,b_l}\cdot 1=\sum_{\lambda}\sum_{T\in \mathrm{SSYT}(\nu(\lambda)^R)} t^{a(\lambda)}q^{\mathrm{dinv}_p(\lambda)-I(\nu(\lambda)^R)+\mathrm{inv}(T)}x^T\,, \tag{142}$$

where $\mathrm{inv}(T)$ is the number of attacking inversions, as in Definition 4.1.2, and $I(\nu(\lambda)^R)$ is the number of attacking pairs in the tuple $\nu(\lambda)^R$ given by Definition 4.1.5.

By the construction, boxes in each column of $\nu(\lambda)^R$, from top to bottom, correspond to south steps u in a vertical run in λ, from north to south. Semistandard tableaux $T\in \mathrm{SSYT}(\nu(\lambda)^R)$ therefore biject with labellings $P_T\colon \{\text{south steps in }\lambda\}\to\mathbb{Z}_{>0}$ such that P_T is strictly increasing from south to north on each vertical run in λ; namely, $P_T\in \mathrm{SSYT}((\lambda{+}(1^k))/\lambda)$. Changing (142) to instead sum over labellings, we can match the right hand sides of (142) and (141) by showing that for $p=1/m-\epsilon$,

$$\mathrm{dinv}_m(P_T)=\mathrm{dinv}_p(\lambda)-I(\nu(\lambda)^R)+\mathrm{inv}(T). \tag{143}$$

For any super tableau T, [16, Corollary 6.4.2] implies that $\mathrm{dinv}_m(P_T)=u_\lambda+\mathrm{inv}(T)$ for an offset u_λ not depending on T. For the tableau T_0 with all entries $\bar{1}$, [16, Lemma 6.3.3] gives that $\mathrm{dinv}_m(P_{T_0})=b_m(\lambda)$, where we note that $b_m(\lambda)$ defined in [16, (100)] is simply $\mathrm{dinv}_p(\lambda)$ with $p=1/m-\epsilon$. Therefore, $u_\lambda=\mathrm{dinv}_m(P_{T_0})-\mathrm{inv}(T_0)=\mathrm{dinv}_p(\lambda)-I(\nu(\lambda)^R)$ by (68).

In fact, there is a direct correspondence between the combinatorics of $\mathrm{dinv}_m(P)$ for paths, as defined in [16], and that of triples in negative tableaux on a tuple of one-row shapes, as considered in §4.5.

Proposition 6.1.1. *Let λ be a lattice path from $(0,k)$ to $(km,0)$, lying weakly below the bounding line $y+p\,x=k$ with $p=1/m-\epsilon$. Let α, β, σ be the LLT data associated to λ for this p. There is a weight-preserving bijection from labellings $P\in\mathrm{SSYT}((\lambda+(1^k))/\lambda)$ to negative tableaux $S\in \mathrm{SSYT}_-(\beta/\alpha)$ such that*

$$\mathrm{dinv}_m(P)=h_\sigma(S). \tag{144}$$

Proof. The labelling $P=P_T\in \mathrm{SSYT}((\lambda+(1^k))/\lambda)$ corresponds naturally to a semistandard tableau $T\in\mathrm{SSYT}(\nu(\lambda)^R)$. Their statistics are related by (143), into which we can substitute $\mathrm{dinv}_p(\lambda)=h_\sigma(\beta/\alpha)$ by Proposition 5.4.4. The bijection $T\mapsto T^R$ in the proof of Proposition 4.1.6 satisfies $\mathrm{inv}(T)-I(\nu(\lambda)^R)=-\mathrm{inv}(T^R)$. Hence, $\mathrm{dinv}_m(P_T)=h_\sigma(\beta/\alpha)-\mathrm{inv}(T^R)$. To complete the bijection, take $S=T^R\circ\sigma$. Then $h_\sigma(\beta/\alpha)-\mathrm{inv}(T^R)=h_\sigma(S)$ by Lemma 4.5.3, proving (144). □

See Figure 6 for an example with $m=1$ and $p=1-\epsilon$. Note that these values give $\sigma=w_0$ in the LLT data.

Next we turn to the non-compositional (km,kn) Shuffle Conjecture from [4]. Its symmetric function side is precisely the Schiffmann algebra operator expression that we denote here by $e_k[-MX^{m,n}]\cdot 1$. By Corollary 3.7.3, this agrees with the left hand side $D_{b_1,\ldots,b_l}\cdot 1$ of (133).

The combinatorial side of the (km,kn) Shuffle Conjecture can be written as in [4, §7], using notation defined there, as

$$\sum_{u}\sum_{\pi\in\mathrm{Park}(u)} t^{\mathrm{area}(u)}\,q^{\mathrm{dinv}(u)+\mathrm{tdinv}(\pi)-\mathrm{maxtdinv}(u)}F_{\mathrm{ides}(\pi)}(x). \tag{145}$$

Here u encodes a north-east lattice path lying above the line from $(0,0)$ to (km,kn), $\mathrm{Park}(u)$ encodes the set of standard Young tableaux on a tuple of columns corresponding to vertical runs in the path encoded by u, and $F_\gamma(x)$ is a Gessel fundamental quasi-symmetric function.

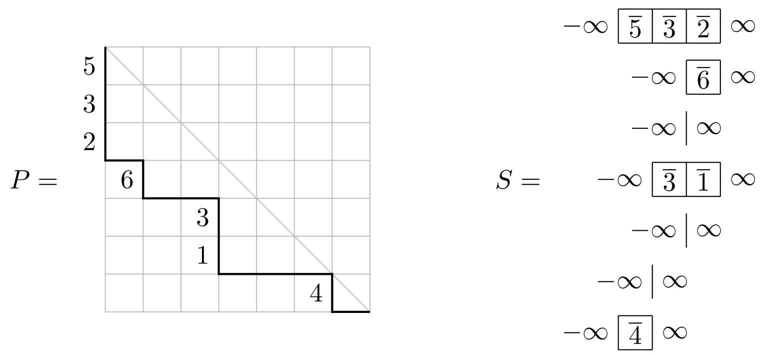


Figure 6. *Example of the bijection $P = P_T \leftrightarrow T \leftrightarrow T^R \leftrightarrow S = T^R \circ \sigma$ in Proposition 6.1.1, with $m = 1$, $p = 1-\epsilon$, $\sigma = w_0$. Letters in S are ordered $\overline{1} > \overline{2} > \cdots$. We see $\operatorname{dinv}_1(P) = h_{w_0}(S) = 6$.*

To make u correspond to a lattice path λ under the line from $(0, kn)$ to $(km, 0)$, as in (142), we need to flip the picture over, replacing each entry $\pi(j)$ with $kn + 1 - \pi(j)$ so the resulting standard tableau T on $\nu(\lambda)^R$ has columns increasing upwards, as it should, instead of decreasing. Using [4, Definition 7.1] and taking account the modification of π to give T, we can translate notation in (145) as follows: $\operatorname{area}(u) = a(\lambda)$, $\operatorname{tdinv}(\pi) = \operatorname{inv}(T)$, $\operatorname{maxtdinv}(u) = I(\nu(\lambda)^R)$, and $\operatorname{dinv}(u) = \operatorname{dinv}_p(\lambda)$, where $p = n/m - \epsilon$.

Finally, the definition of $\operatorname{ides}(\pi)$ becomes the descent set of T relative to the reading order on $\nu(\lambda)^R$. This implies that expanding $F_{\operatorname{ides}(\pi)}(x)$ into monomials gives a sum with semistandard tableaux T in place of standard tableaux and x^T in place of $F_{\operatorname{ides}(\pi)}(x)$. After these substitutions, (145) coincides with the right hand side of (142).

7. A positivity conjecture

7.1. The conjecture

Theorem 5.5.1, Corollary 3.7.2 and [16, Proposition 5.3.1] imply that the symmetric function

$$\omega(D_{\mathbf{b}} \cdot 1) = \mathbf{H}_{q,t} \left(\frac{x^{\mathbf{b}}}{\prod_i (1 - q\, t\, x_i/x_{i+1})} \right)_{\mathrm{pol}} \tag{146}$$

is q, t Schur positive when b_i is the number of south steps along $x = i-1$ on the highest lattice path below a line with endpoints on the positive x and y axes. Computational evidence leads us to conjecture that (146) is q, t Schur positive under a more general geometric condition on $\mathbf{b}$.

Let C be a convex curve (meaning that the region above C is convex) in the first quadrant with endpoints $(r, 0)$ and $(0, s)$ on the positive x and y axes. Let δ be the highest lattice path from $(0, \lfloor s \rfloor)$ to $(\lfloor r \rfloor, 0)$ weakly below C. Let b_i be the number of south steps in δ along $x = i - 1$ for $i = 1, \ldots, l$, where $l = \lfloor r \rfloor + 1$. Algebraically, this means that there are real numbers $s_0 \geq s_1 \geq \cdots \geq s_l = 0$ with weakly decreasing differences $s_{i-1} - s_i \geq s_i - s_{i+1}$, such that $b_i = \lfloor s_{i-1} \rfloor - \lfloor s_i \rfloor$.

Note that if δ is the highest path strictly below a convex curve C', then it is also the highest lattice path weakly below a slightly lower curve C, and vice versa, so it doesn't matter whether we use 'weakly below' or 'strictly below' to formulate the condition on δ.

Conjecture 7.1.1. *When b_i is the number of south steps along $x = i-1$ in the highest lattice path below a convex curve, as above, the symmetric function in* (146) *is a linear combination of Schur functions with coefficients in $\mathbb{N}[q, t]$.*

At $q = 1$, the q-Kostka coefficients reduce to $K_{\lambda,\mu}(1) = K_{\lambda,\mu} = \langle s_\lambda, h_\mu \rangle$. Hence, the Hall-Littlewood symmetrization operator reduces to $\mathbf{H}_q(x^\mu)_{\mathrm{pol}}|_{q=1} = h_\mu(x)$ if $\mu_i \geq 0$ for all i, and otherwise $\mathbf{H}_q(x^\mu)_{\mathrm{pol}} = 0$. At $q = 1$, the factors containing t in (46) cancel, so $\mathbf{H}_{q,t}$ reduces to the same thing as $\mathbf{H}_q$.

It follows that (146) specializes at $q = 1$ to

$$\omega(D_{\mathbf{b}} \cdot 1)|_{q=1} = \sum_{a_1,\dots,a_{l-1} \geq 0} t^{|\mathbf{a}|} h_{\mathbf{b}+(a_1, a_2-a_1, \dots, a_{l-1}-a_{l-2}, -a_{l-1})}, \tag{147}$$

with the convention that $h_\mu = 0$ if $\mu_i < 0$ for any i. As in Theorem 5.5.1, the index $\mathbf{b}+(a_1, a_2 - a_1, \dots, a_{l-1} - a_{l-2}, -a_{l-1})$ is non-negative precisely when it is the sequence $b(\lambda)$ of lengths of south runs in a lattice path λ lying below the path δ whose south runs are given by $\mathbf{b}$. Here a_i is the number of lattice squares in column i between λ and δ, so $|\mathbf{a}|$ is the area $a(\lambda)$ enclosed between the two paths. This gives a combinatorial expression

$$\omega(D_{\mathbf{b}} \cdot 1)|_{q=1} = \sum_\lambda t^{a(\lambda)} h_{b(\lambda)}, \tag{148}$$

for (146) at $q = 1$, which is positive in terms of complete homogeneous symmetric functions h_λ, hence t Schur positive. We may conjecture that when the hypothesis of Conjecture 7.1.1 holds, $\omega(D_{\mathbf{b}} \cdot 1)$ is given by some Schur positive combinatorial q-analog of (148), but it remains an open problem to formulate such a conjecture precisely.

Of course, (148) cannot be considered evidence for Conjecture 7.1.1, since (148) holds for any $\mathbf{b} \geq 0$, whether the convexity hypothesis is satisfied or not.

7.2. Relation to previous conjectures

The generalized q,t-Catalan numbers $C_{\mathbf{b}}(q,t) = \langle s_{(|\mathbf{b}|)}(X), \omega(D_{\mathbf{b}} \cdot 1) \rangle$ from Definition 5.5.4 coincide with the functions denoted $F(b_2, \dots, b_l)$ in [13], where several equivalent expressions for them were obtained. To see that $C_{\mathbf{b}}(q,t) = F(b_2, \dots, b_l)$, one can compare the formula in Proposition 7.2.1, below, with the equation just before (2.6) in [13]. It was also shown in [13] that this quantity does not depend on b_1, hence the notation $F(b_2, \dots, b_l)$.

Conjecture 7.1.1 implies a conjecture of Negut, announced in [13], which asserts that $C_{\mathbf{b}}(q,t) \in \mathbb{N}[q,t]$ when $b_2 \geq \cdots \geq b_l$. Conjecture 7.1.1 is stronger than Negut's conjecture in two ways: the weight $\mathbf{b}$ is generalized from a partition to the highest path below a convex curve, and the coefficient of $s_{(|\mathbf{b}|)}(X)$ in $\omega(D_{\mathbf{b}} \cdot 1)$ is generalized to the coefficient of any Schur function.

Proposition 7.2.1. *For any $\mathbf{b} \in \mathbb{Z}^l$, the generalized q,t-Catalan number $C_{\mathbf{b}}(q,t)$ has the following description as a series coefficient:*

$$C_{\mathbf{b}}(q,t) = \langle z^{-\mathbf{b}} \rangle \prod_{i=1}^{l} \frac{1}{1 - z_i^{-1}} \prod_{i=1}^{l-1} \frac{1}{1 - q\,t\,z_i/z_{i+1}} \prod_{i<j} \frac{(1 - z_i/z_j)(1 - q\,t\,z_i/z_j)}{(1 - q\,z_i/z_j)(1 - t\,z_i/z_j)}. \tag{149}$$

Proof. From (50) we have

$$\omega(D_{\mathbf{b}} \cdot 1) = \langle z^0 \rangle \frac{z^{\mathbf{b}}}{\prod_{i=1}^{l-1}(1 - q\,t\,z_i/z_{i+1})} \prod_{i<j} \frac{1 - q\,t\,z_i/z_j}{(1 - q\,z_i/z_j)(1 - t\,z_i/z_j)} \Omega[\overline{Z}X] \prod_{i<j}(1 - z_i/z_j).$$

Specializing $X = 1$ gives the result. ☐

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