A COMBINATORIAL FORMULA FOR NON-SYMMETRIC MACDONALD POLYNOMIALS

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Abstract. We give a combinatorial formula for the non-symmetric Macdonald polynomials $E_\mu(x; q, t)$. The formula generalizes our previous combinatorial interpretation of the integral form symmetric Macdonald polynomials $J_\mu(x; q, t)$. We prove the new formula by verifying that it satisfies a recurrence, due to Knop and Sahi, that characterizes the non-symmetric Macdonald polynomials.

1. Introduction

In a previous paper [6], we gave a combinatorial formula for Macdonald's symmetric functions (for root systems of type A). In fact, we gave two formulas, one for the transformed Macdonald polynomials $\tilde{H}_\mu(x; q, t)$, which appear in the positivity theorem of [8] and are connected to the geometry of Hilbert schemes, and another [6, Proposition 8.1] for the *integral forms* $J_\mu(x; q, t)$ [15], which are scalar multiples of the classical monic forms $P_\mu(x; q, t)$, and thus closer to the original viewpoint on Macdonald polynomials as orthogonal polynomials associated with a root system [17].

The symmetric Macdonald polynomials $P_\mu(x; q, t)$ can be reconstructed from a broader theory of *non-symmetric Macdonald polynomials* $E_\mu(x; q, t)$, defined and developed by Opdam and Heckman [21], Macdonald [16], and Cherednik [3]. Many important aspects of the symmetric theory are best understood with the help of extra structure available in the non-symmetric theory. These include the norm and evaluation formulas [1], [2], as well as the earlier combinatorial formula of Knop and Sahi [14] for the *Jack polynomials* $P_\mu(x; \alpha) = \lim_{t\to 1} P_\mu(x; t^\alpha, t)$. In fact, Knop and Sahi's formula was derived from its non-symmetric counterpart. It is similarly possible to derive our formula for $P_\mu(x; q, t)$ in [6] from the formula for $E_\mu(x; q, t)$ obtained here—see §5.2. Apart from its value as a path to results in the symmetric theory, the non-symmetric theory has lately taken on a life of its own, with works such as [10], [11], [20] suggesting that the $E_\mu(x; q, t)$ for any root system should possess intrinsic Lie-theoretic significance. It also seems to us that the best hope for progress on the problem of extending the combinatorial theory to other root systems lies in the non-symmetric setting.

Date: February 9, 2007.
2000 *Mathematics Subject Classification.* Primary: 05E10; Secondary: 05A30, 33D52.
Work supported by NSA grant MSPF-02G-193 (J.H.).
Work supported by NSF grant DMS-0301072 (M.H.).
Work supported by NSF postdoctoral research fellowship DMS-0303178 (N.L.).

The proof of our formula uses a recurrence discovered independently by Knop [12] and Sahi [23] and used by them to prove Macdonald's integrality conjecture. In retrospect, The Knop–Sahi recurrence is the special case in type A of Cherednik's intertwiner formula [4].

In the next section we fix notation, review the definition of non-symmetric Macdonald polynomials, and explain the Knop–Sahi recurrence. In §3 we state our main theorem, the combinatorial formula. The proof is in §4. In §5 we compare the new formula to our earlier one for the symmetric case. Finally, we give a table of type A non-symmetric Macdonald polynomials in an appendix, for the convenience of the reader who might wish to compare examples calculated from our formula against known values.

2. Notation and definitions

2.1. **Non-symmetric Macdonald polynomials.** Our notational conventions mostly conform to those in Cherednik's paper [3] and the expositions by Cherednik [5, Chapter 3], Macdonald [18], and the second author [9], particularized to GL_n as indicated below. In particular, t is the Hecke algebra parameter, and q is the formal exponential of the null root.

The integer n is fixed throughout. The weight lattice of GL_n is $X = \mathbb{Z}^n$; the simple roots are $\alpha_i = e_i - e_{i+1}$, where e_i is the i-th unit vector. Identifying the co-weight lattice $X^\vee$ with X using the standard inner product on $\mathbb{Z}^n$, the simple coroots $\alpha_i^\vee$ coincide with the simple roots α_i. Hence the dominant weights, *i.e.*, $\lambda \in X$ such that $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all i, are the weakly decreasing sequences $(\lambda_1 \geq \cdots \geq \lambda_n)$. We write $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ for the formal exponential of a weight λ.

The affine weight lattice is $\widehat{X} = X \oplus \mathbb{Z}\delta$, where δ is the smallest positive imaginary root, or *null root.* The simple affine roots are $\alpha_1, \ldots, \alpha_n$ as above, and $\alpha_0 = \delta - \theta$, where $\theta = e_1 - e_n$ is the highest root of GL_n. The set of positive affine (real) roots is $\widehat{R}_+ = \{e_i - e_j + k\delta : i \neq j,\ k \in \mathbb{N}, \text{ and } k > 0 \text{ if } i > j\}$. The formal exponential x^δ is denoted by q, so $x^{\alpha_i} = x_i/x_{i+1}$ for $i \neq 0$, and $x^{\alpha_0} = qx_n/x_1$. The group algebra $\mathbb{Q}(t)\widehat{X}$ is thereby identified with the Laurent polynomial ring $\mathbb{Q}(t)[q^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Extending scalars to $\mathbb{Q}(q,t)$, we further identify $\mathbb{Q}(t)\widehat{X}$ with a subring of $\mathbb{Q}(q,t)X = \mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Cherednik's inner product on $\mathbb{Q}(q,t)X$ is defined by

$$\langle f, g \rangle_{q,t} = [x^0](f\overline{g}\Delta_1), \tag{1}$$

where $\overline{\cdot}$ is the involution $\overline{q} = q^{-1}$, $\overline{t} = t^{-1}$, $\overline{x_i} = x_i^{-1}$, and $\Delta_1 = \Delta/([x^0]\Delta)$, with

$$\Delta = \prod_{\alpha \in \widehat{R}_+} \frac{1 - x^\alpha}{1 - tx^\alpha} = \prod_{i<j} \prod_{k=0}^{\infty} \frac{(1 - q^k x_i/x_j)(1 - q^{k+1} x_j/x_i)}{(1 - tq^k x_i/x_j)(1 - tq^{k+1} x_j/x_i)}. \tag{2}$$

Here $\Delta \in \mathbb{Q}[[q,t]]$ is a formal power series in q, t. For such a series f, $[x^\lambda]f \in \mathbb{Q}[[q,t]]$ denotes its coefficient of x^λ, taken term by term. It is known that $[x^\lambda]\Delta_1$ is a rational function of q, t for all λ, and that $\overline{\Delta_1} = \Delta_1$. Hence $\langle f, g \rangle_{q,t} \in \mathbb{Q}(q,t)$, and $\langle g, f \rangle_{q,t} = \overline{\langle f, g \rangle_{q,t}}$.

The *Bruhat order* is the partial ordering $<$ on X induced by identifying X with the set of minimal coset representatives in $\widehat{W}/W_0$, where $W_0 = S_n$ is the Weyl group of GL_n, and

$\widehat{W} = W_0 \ltimes X$ is the extended affine Weyl group, equipped with its usual Bruhat order. To make it explicit for GL_n, if $i < j$, $\lambda_i < \lambda_j$ and σ_{ij} is the transposition $(i\ j)$, then $\lambda > \sigma_{ij}(\lambda)$, and if in addition $\lambda_j - \lambda_i > 1$, then $\sigma_{ij}(\lambda) > \lambda + e_i - e_j$. The Bruhat order is the transitive closure of these relations. Note that distinct cosets in X of the root lattice $Q = \mathbb{Z}\{\alpha_1, \ldots, \alpha_n\}$ are incomparable in the Bruhat order.

Definition 2.1.1. The *non-symmetric Macdonald polynomials* $E_\mu(x; q, t) \in \mathbb{Q}(q, t)X$ ($\mu \in X$) are uniquely characterized by the conditions

(i) triangularity: $E_\mu \in x^\mu + \mathbb{Q}(q, t)\{x^\lambda : \lambda < \mu\}$,
(ii) orthogonality: $\langle E_\lambda, E_\mu \rangle_{q,t} = 0$ for $\lambda \neq \mu$.

We remark that Opdam [21] and Macdonald [16], [18] use a stronger ordering that is superficially easier to describe but not as natural, since, as Sahi [24] observed, the coefficient of x^λ in $E_\mu(x; q, t)$ in non-zero if and only if $\lambda \leq \mu$ in Bruhat order.

The proof that the polynomials E_μ exist uses an affine Hecke algebra representation which is also needed in order to state the Knop–Sahi recurrence, and which we now recall. The *affine Hecke algebra* is the $\mathbb{Q}(t)$-algebra $\mathcal{H}$ with generators T_i ($i = 0, \ldots, n-1$) satisfying the braid relations

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\ T_i T_j &= T_j T_i \qquad (i - j \neq \pm 1), \end{aligned} \tag{3}$$

where all indices are modulo n, and the quadratic relations

$$(T_i - t)(T_i + 1) = 0. \tag{4}$$

Here we depart slightly from [5], [9], [18], which take $(T_i - t^{1/2})(T_i + t^{-1/2}) = 0$ instead. We will not use the parameter $t^{1/2}$.

The (unextended) affine Weyl group W_a is a Coxeter group with generators s_i ($i = 0, \ldots, n-1$) satisfying the above braid relations. They act on $\widehat{X}$, and by extension on $\mathbb{Q}(t)\widehat{X}$ and $\mathbb{Q}(q, t)X$, as affine reflections

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i. \tag{5}$$

For $i \neq 0$, s_i is just the transposition $\sigma_{i,i+1}$, while s_0 is given explicitly by

$$s_0 f(x_1, \ldots, x_n) = f(q x_n, x_2, \ldots, x_{n-1}, x_1/q). \tag{6}$$

Cherednik's representation of $\mathcal{H}$ on $\mathbb{Q}(q, t)X$ is defined by the formula

$$T_i\, x^\lambda = t\, x^{s_i(\lambda)} + (t - 1)\frac{x^\lambda - x^{s_i(\lambda)}}{1 - x^{\alpha_i}}, \quad \text{for all } i = 0, \ldots, n-1, \tag{7}$$

where for $i = 0$, we recall that $x^{\alpha_0} = q x_n / x_1$.

Denote the standard basis elements of $\mathcal{H}$ by T_w ($w \in W_a$). For $\tau(\beta) \in W_a$ corresponding to translation by a dominant weight $\beta \in Q^\vee = Q$, set $Y^\beta = t^{-\langle \beta, \rho^\vee \rangle} T_{\tau(\beta)}$, acting on $\mathbb{Q}(q, t)X$ via Cherednik's representation. Here $\rho^\vee = (\sum_{\alpha \in R^+} \alpha^\vee)/2$, as usual. The operators Y^β commute, are unitary with respect to Cherednik's inner product, and are lower triangular with respect to Bruhat order on the basis $\{x^\lambda : \lambda \in X\}$. The polynomials $E_\mu(x;q,t)$ are their simultaneous eigenfunctions. Bernstein's relations [18, (4.2.4)] in $\mathcal{H}$ and a computation of the eigenvalues of Y^β imply the relations

$$E_{s_i(\mu)}(x;q,t) = \left(T_i + \frac{1-t}{1-q^{\langle \mu,\alpha_i^\vee\rangle} t^{\langle w_\mu(\rho),\alpha_i^\vee\rangle}}\right) E_\mu(x;q,t), \quad (i \neq 0) \tag{8}$$

for $\mu_i > \mu_{i+1}$ (that is, $s_i(\mu) > \mu$), where $w_\mu \in W_0$ is the *maximum-length* permutation such that $w_\mu^{-1}(\mu)$ is dominant, and $\rho = (n-1, n-2, \ldots, 0)$. A more detailed argument can be found in [12, Theorem 4.2b].

Cherednik's intertwiner theory yields a version of (8) for $i = 0$, but for GL_n it is simpler to make use of the symmetry of the affine root system. Define automorphisms π of X and Ψ of $\mathbb{Q}(q,t)X$ by

$$\begin{aligned} \pi(\lambda_1,\ldots,\lambda_n) &= (\lambda_n+1, \lambda_1, \ldots, \lambda_{n-1}) \\ \Psi f(x_1,\ldots,x_n) &= x_1 f(x_2,\ldots,x_n,q^{-1}x_1). \end{aligned} \tag{9}$$

If we define a linear automorphism ϕ of $\widehat{X}$ by $\phi(\lambda_1,\ldots,\lambda_n) = (\lambda_n,\lambda_1,\ldots,\lambda_{n-1}) - \lambda_n\delta$, $\phi\delta = \delta$, then $\Psi f = x_1\Psi' f$, where $\Psi'(x^\lambda) = x^{\phi(\lambda)}$. Now, ϕ stabilizes $\widehat{R}_+$, hence Ψ' fixes Δ and is an isometry of Cherednik's inner product $\langle\cdot,\cdot\rangle_{q,t}$, which in turn implies that Ψ is an isometry of $\langle\cdot,\cdot\rangle_{q,t}$. Moreover, $\Psi(x^\lambda) = q^{-\lambda_n}x^{\pi(\lambda)}$, and π preserves the Bruhat order. It therefore follows from Definition 2.1.1 that

$$E_{\pi(\mu)}(x;q,t) = q^{\mu_n}\Psi E_\mu(x;q,t). \tag{10}$$

Equations (8) and (10) are the Knop–Sahi recurrence.

It is immediate from the definition or from (10) that if $\mu' = \mu + (r,r,\ldots,r)$, then $E_{\mu'} = (x_1\cdots x_n)^r E_\mu$. Therefore, without loss of generality, we may restrict attention to weights $\mu \in \mathbb{N}^n$, and we will do so from now on. Then μ is a *composition*, and dominant weights are *partitions*.

Lemma 2.1.2. *The Macdonald polynomials E_μ for $\mu \in \mathbb{N}^n$ are uniquely determined by the the initial value $E_{\mathbf{0}} = 1$, together with*

(a) *equation* (10)*, and*
(b) *the special case of* (8) *in which $\mu_{i+1} = 0$.*

Proof. If $\mu_1 > 0$, then $\mu = \pi(\nu)$, where again $\nu \in \mathbb{N}^n$. By induction on the sum of the parts, we can assume E_ν already determined, and apply (a). If $\mu_1 = 0$ and $\mu_j > 0$ for some j, we can reduce to the case $\mu_1 > 0$ by repeated applications of (b). □

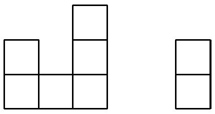
We should point out that notational conventions in earlier literature sometimes differ from ours. Ion [10], [11] and Marshall [19] change q, t to q^{-1}, t^{-1}, and our t is q^k in Macdonald [16]. Both Knop [12] and Marshall [19] reverse the indexing of the variables $x_1,\ldots,x_n$, so for them, monomials x^λ with $\lambda_1 \geq \cdots \geq \lambda_n$ correspond to *anti-dominant* weights. The equation $\Phi\xi_1 = q\xi_n\Phi$ in [12, §4] contains a typographical error, and should instead read $\Phi\xi_1 = q^{-1}\xi_n\Phi$.

2.2. **Diagrams.** We visualize a composition $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$ as a diagram consisting of n columns, with μ_i boxes in column i. Formally, the *column diagram* of μ is the set

$$\mathrm{dg}'(\mu) = \{(i,j) \in \mathbb{N}^2 : 1 \le i \le n,\ 1 \le j \le \mu_i\} \tag{11}$$

in Cartesian coordinates, the abscissa i indexing the columns, and the ordinate j indexing the rows. The prime serves as a reminder that the parts of μ are the columns rather than the rows, transpose to the usual conventions for partition diagrams.

For example, $\mu = (2,1,3,0,0,2)$ is represented by the diagram

(12) $\mathrm{dg}'(\mu) =$  .

We will also need the *augmented diagram* of μ, defined by

$$\widehat{\mathrm{dg}}(\mu) = \mathrm{dg}'(\mu) \cup \{(i,0) : 1 \le i \le n\}, \tag{13}$$

i.e., we adjoin n extra boxes in row 0, thus adding a box to the bottom of each column.

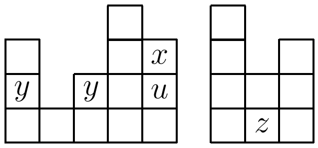
2.3. **Arms and legs.** Given $\mu \in \mathbb{N}^n$ and $u = (i,j) \in \mathrm{dg}'(\mu)$, define

$$\begin{aligned} \mathrm{leg}(u) &= \{(i,j') \in \mathrm{dg}'(\mu) : j' > j\} \\ \mathrm{arm}^{\mathrm{left}}(u) &= \{(i',j) \in \mathrm{dg}'(\mu) : i' < i,\ \mu_{i'} \le \mu_i\} \\ \mathrm{arm}^{\mathrm{right}}(u) &= \{(i',j-1) \in \widehat{\mathrm{dg}}(\mu) : i' > i,\ \mu_{i'} < \mu_i\} \\ \mathrm{arm}(u) &= \mathrm{arm}^{\mathrm{left}}(u) \cup \mathrm{arm}^{\mathrm{right}}(u), \end{aligned} \tag{14}$$

and set

$$\begin{aligned} l(u) &= |\,\mathrm{leg}(u)| = \mu_i - j \\ a(u) &= |\,\mathrm{arm}(u)|. \end{aligned} \tag{15}$$

For example, for $\mu = (3,1,2,4,3,0,4,2,3)$ and $u = (5,2)$, the cells belonging to $\mathrm{leg}(u)$, $\mathrm{arm}^{\mathrm{left}}(u)$, and $\mathrm{arm}^{\mathrm{right}}(u)$ are marked by x, y, z in the following figure:

(16) $\mathrm{dg}'(\mu) =$ 

,

giving $l(u) = 1$, $a(u) = 3$. Note that $\mathrm{arm}(u)$ is a subset of the augmented diagram, and may contain boxes in row 0, if u is in row 1. When μ is anti-dominant, $\mathrm{dg}'(\mu)$ is a reversed partition diagram, all right arms are empty, and the leg and left arm reduce to the usual notions for partition diagrams. We remark that the preceding definitions agree with those in [12], [14], after a suitable change of coordinates, and exchanging arms with legs.

With these definitions, if $\mu_i > \mu_{i+1}$, and $u = (i,\ \mu_{i+1}+1)$, then (8) takes the form

$$E_{s_i(\mu)}(x;q,t) = \left(T_i + \frac{1-t}{1-q^{l(u)+1}t^{a(u)}}\right) E_\mu(x;q,t). \tag{17}$$

2.4. **Integral forms.** For $\mu \in \mathbb{N}^n$, the *integral form* non-symmetric Macdonald polynomials are defined by

$$\mathcal{E}_\mu(x;q,t) = \prod_{u \in \mathrm{dg}'(\mu)} \left(1 - q^{l(u)+1} t^{a(u)+1}\right) E_\mu(x;q,t). \tag{18}$$

Knop [12, Corollary 5.2] proved that $\mathcal{E}_\mu$ has coefficients in $\mathbb{Z}[q,t]$, as will also be seen from our formula.

3. The combinatorial formula

In this section we present our main result, Theorem 3.5.1, a combinatorial formula for the non-symmetric Macdonald polynomials E_μ. The formula is a sum whose terms are indexed by combinatorial objects σ called *fillings of* μ. Each term in the sum depends on a pair of combinatorial statistics $\mathrm{maj}(\widehat{\sigma})$ and $\mathrm{coinv}(\widehat{\sigma})$, which are variants of their counterparts defined in [6], [7] for the symmetric case. We begin by defining these combinatorial data.

3.1. **Fillings.** A *filling of* μ is a function

$$\sigma\colon\ \mathrm{dg}'(\mu) \to [n], \tag{19}$$

where $[n] = \{1, \dots, n\}$. The associated *augmented filling* is the filling $\widehat{\sigma}\colon \widehat{\mathrm{dg}}(\mu) \to [n]$ of the augmented diagram such that $\widehat{\sigma}$ agrees with σ on $\mathrm{dg}'(\mu)$, and $\widehat{\sigma}((j,0)) = j$ for $j = 1, \dots, n$. Distinct lattice squares $u, v \in \mathbb{N}^2$ are said to *attack* each other if either

(a) they are in the same row, *i.e.*, they have the form $(i,j), (i',j)$, or
(b) they are in consecutive rows, and the box in the lower row is to the right of the one in the upper row, *i.e.*, they have the form $(i,j), (i', j-1)$ with $i < i'$.

A filling $\widehat{\sigma}\colon \widehat{\mathrm{dg}}(\mu) \to [n]$ is *non-attacking* if $\widehat{\sigma}(u) \neq \widehat{\sigma}(v)$ for every pair of attacking boxes $u, v \in \widehat{\mathrm{dg}}(\mu)$. We say that a filling $\sigma\colon\ \mathrm{dg}'(\mu) \to [n]$ of μ is *non-attacking* if its associated augmented filling $\widehat{\sigma}$ is non-attacking (non-attacking fillings are called *admissible* in [14]).

3.2. **Descents and major index.** We write $d(u) = (i, j-1)$ for the box directly below a box $u = (i,j)$. For any lattice-square diagram $S \subseteq \mathbb{N}^2$, a *descent* of a filling $\widehat{\sigma}\colon S \to [n]$ is a box $u \in S$ such that $d(u) \in S$ and $\widehat{\sigma}(u) > \widehat{\sigma}(d(u))$. Note that the descents of a filling $\widehat{\sigma}$ of $\widehat{\mathrm{dg}}(\mu)$ are contained in $\mathrm{dg}'(\mu)$. For such a filling, we define

$$\mathrm{Des}(\widehat{\sigma}) = \{\text{descents of } \widehat{\sigma}\}, \tag{20}$$

$$\mathrm{maj}(\widehat{\sigma}) = \sum_{u \in \mathrm{Des}(\widehat{\sigma})} (l(u) + 1). \tag{21}$$

3.3. **Inversions.** The *reading order* on a lattice-square diagram S is the total ordering of the boxes in S row by row, from top to bottom, and from *right to left* within each row. In symbols, $(i,j) < (i',j')$ if $j > j'$, or if $j = j'$ and $i > i'$. An *inversion* of a filling $\widehat{\sigma}\colon S \to [n]$ is a pair of boxes $u, v \in S$ such that u and v attack each other, $u < v$ in the reading order, and $\widehat{\sigma}(u) > \widehat{\sigma}(v)$. When σ is a filling of μ and $\widehat{\sigma}$ is its associated augmented filling, we define

$$\operatorname{Inv}(\widehat{\sigma}) = \{\text{inversions of } \widehat{\sigma}\}, \tag{22}$$

$$\operatorname{inv}(\widehat{\sigma}) = |\operatorname{Inv}(\widehat{\sigma})| - |\{i < j : \mu_i \leq \mu_j\}| - \sum_{u \in \operatorname{Des}(\widehat{\sigma})} a(u), \tag{23}$$

$$\operatorname{coinv}(\widehat{\sigma}) = \Big(\sum_{u \in \operatorname{dg}'(\mu)} a(u)\Big) - \operatorname{inv}(\widehat{\sigma}). \tag{24}$$

Note that every pair of boxes in row 0 counts as an inversion of $\widehat{\sigma}$, although the second term in the expression for $\operatorname{inv}(\widehat{\sigma})$ has the effect of cancelling part of this contribution.

3.4. **Example.** The figure below shows the augmentation $\widehat{\sigma}$ of a non-attacking filling σ of $\mu = (2,1,3,0,0,2)$, and the arm-lengths $a(u)$ for each box $u \in \operatorname{dg}'(\mu)$.

$$\widehat{\sigma} = \begin{array}{|c|c|c|c|c|c|} \hline & & 2 & & & \\ \hline \mathbf{6} & & \mathbf{4} & & & 5 \\ \hline 1 & 2 & 3 & & & 5 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array}\,, \qquad a(u) = \begin{array}{|c|c|c|c|c|c|} \hline & & 1 & & & \\ \hline 1 & & 2 & & & 1 \\ \hline 3 & 2 & 5 & & & 2 \\ \hline & & & & & \\ \hline \end{array}\,. \tag{25}$$

The bottom row is row 0. The descent set $\operatorname{Des}(\widehat{\sigma})$ consists of the two boxes marked in boldface, giving $\operatorname{maj}(\widehat{\sigma}) = 3$. There are $\binom{6}{2} = 15$ inversions in row 0, $\binom{4}{2} = 6$ in row 1, and 1 in row 2, plus 3 inversions between rows 2 and 1, for a total of 25 inversions, giving $\operatorname{inv}(\widehat{\sigma}) = 25 - 7 - 3 = 15$, $\operatorname{coinv}(\widehat{\sigma}) = 17 - \operatorname{inv}(\widehat{\sigma}) = 2$.

3.5. **The formula.** We can now state our main result.

Theorem 3.5.1. *The non-symmetric Macdonald polynomials E_μ are given by the formula*

$$E_\mu(x;q,t) = \sum_{\substack{\sigma\colon \mu \to [n] \\ \textit{non-attacking}}} x^\sigma q^{\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u \in \operatorname{dg}'(\mu) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} \frac{1-t}{1 - q^{l(u)+1} t^{a(u)+1}}, \tag{26}$$

where $x^\sigma = \prod_{u \in \operatorname{dg}'(\mu)} x_{\sigma(u)}$.

Corollary 3.5.2. *The integral form non-symmetric Macdonald polynomials $\mathcal{E}_\mu$ in* (18) *are given by*

$$\mathcal{E}_\mu(x;q,t) = \sum_{\substack{\sigma\colon \mu \to [n] \\ \textit{non-attacking}}} x^\sigma q^{\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u \in \operatorname{dg}'(\mu) \\ \widehat{\sigma}(u) = \widehat{\sigma}(d(u))}} (1 - q^{l(u)+1} t^{a(u)+1}) \prod_{\substack{u \in \operatorname{dg}'(\mu) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} (1-t). \tag{27}$$

Let us make some remarks about the combinatorial nature of (26) and (27). It follows trivially from (8) and (10) that there exist expressions for E_μ as a sum of terms of the form

$$\pm x^\lambda q^c t^d \prod_{(a,b)\in S} \frac{1-t}{1-q^a t^b}. \tag{28}$$

In fact, using Cherednik's intertwiners, the same conclusion can be drawn for non-symmetric Macdonald polynomials associated with any root system. The reader might think at first that in (26) we have merely organized the terms in such an expression, but this is not so.

For one thing, (26) and (27) are *positive* formulas in at least two senses. First, if the parameters are specialized to real numbers $0 < q, t < 1$ then (26) becomes a polynomial in x with positive coefficients. Second, if we set $q = t^k$ for any integer $k \geq 0$, then (27) shows that

$$\mathcal{E}_\mu(x; t^k, t)/(1-t)^{|\mu|} \tag{29}$$

is a polynomial in x and t with positive coefficients.

A more subtle point is that (26) is a *canonical* formula. In order to apply (8) and (10) directly, one must first choose a reduced factorization in the affine Weyl group for the minimal element w that represents $\mu \in X \cong \widehat{W}/W_0$, and the resulting expression for E_μ will depend on this choice. Formula (26) involves no such auxiliary choices. This is not to say, however, that (26) is the unique, or even the simplest, positive combinatorial formula for E_μ. For example, if $\mu = (r, r, \dots, r) - (\nu_n, \dots, \nu_1)$, it is easy to deduce from the definition the following identity:

$$E_\mu(x_1, \dots, x_n; q, t) = (x_1 \cdots x_n)^r E_\nu(x_n^{-1}, \dots, x_1^{-1}; q, t). \tag{30}$$

It may happen that ν has fewer non-attacking fillings than μ, in which case (30) combined with formula (26) for E_ν yields a simpler formula for E_μ than (26). It would be interesting to know whether there is some other combinatorial formula that would manifest the symmetry in (30) directly.

3.6. **Inversion triples.** As in [6], we can reformulate $\mathrm{inv}(\widehat{\sigma})$ as the number of suitably defined *inversion triples*. The definition (23) and the reformulation below will both be used in the proof of Theorem 3.5.1 in §4. We have placed the reformulation in this section because it clarifies the nature of the statistics $\mathrm{inv}(\widehat{\sigma})$ and $\mathrm{coinv}(\widehat{\sigma})$, showing in particular that they are non-negative.

A *triple* consists of three boxes (u, v, w) in $\widehat{\mathrm{dg}}(\mu)$ such that

$$w = d(u) \text{ and } v \in \mathrm{arm}(u). \tag{31}$$

Given a filling $\widehat{\sigma}$ and boxes x, y in $\widehat{\mathrm{dg}}(\mu)$, with $x < y$ in the reading order, set

$$\chi_{xy}(\widehat{\sigma}) = \begin{cases} 1 & \text{if } \widehat{\sigma}(x) > \widehat{\sigma}(y) \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

If (u,v,w) is a triple, then v attacks u and w, so we have $\chi_{uv}(\widehat{\sigma}) = 1 \Leftrightarrow (u,v) \in \operatorname{Inv}(\widehat{\sigma})$, $\chi_{vw}(\widehat{\sigma}) = 1 \Leftrightarrow (v,w) \in \operatorname{Inv}(\widehat{\sigma})$, and $\chi_{uw}(\widehat{\sigma}) = 1 \Leftrightarrow u \in \operatorname{Des}(\widehat{\sigma})$. It follows from the transitive law for inequalities that we always have

$$\chi_{uv}(\widehat{\sigma}) + \chi_{vw}(\widehat{\sigma}) - \chi_{uw}(\widehat{\sigma}) \in \{0,1\}. \tag{33}$$

We say that the triple (u,v,w) is an *inversion triple* of σ if $\chi_{uv}(\widehat{\sigma}) + \chi_{vw}(\widehat{\sigma}) - \chi_{uw}(\widehat{\sigma}) = 1$. Otherwise (u,v,w) is a *co-inversion triple.*

Lemma 3.6.1. *Every pair of attacking boxes in $\widehat{\operatorname{dg}}(\mu)$ occurs as either $\{u,v\}$ or $\{v,w\}$ in a unique triple (u,v,w), with the exception that an attacking pair $\{(i,0),(i',0)\}$ in row 0, such that $i < i'$ and $\mu_i \le \mu_{i'}$, does not belong to any triple.*

Proof. Consider an attacking pair $\{x = (i,j), y = (i',j)\}$ in the same row, where $i < i'$. If $\mu_i \le \mu_{i'}$, then $x = (i,j)$ is in the arm of $y = (i',j)$, so $(y, x, w = d(y))$ is a triple, provided $j \neq 0$. If $\mu_i > \mu_{i'}$, then $u = (i, j+1)$ is a box of $\operatorname{dg}'(\mu)$, and y is in the arm of u, so (u,y,x) is a triple. Moreover, in the first case, y is not in the arm of u (possibly u is not even in the diagram), so (u,y,x) is not a triple, while in the second case, x is not in the arm of y, so $(y,x,w = d(y))$ is not a triple. These are the only two sets of three boxes that might potentially form a triple containing $\{x,y\}$, so the triple with this property is unique.

We leave the similar argument for an attacking pair in consecutive rows to the reader. □

Proposition 3.6.2. *Let σ be a filling of μ (not necessarily non-attacking). The number of inversion triples (resp. co-inversion triples) of $\widehat{\sigma}$ is equal to $\operatorname{inv}(\widehat{\sigma})$ (resp. $\operatorname{coinv}(\widehat{\sigma})$).*

Proof. The number of inversion triples is given by the sum over all triples

$$\sum_{(u,v,w)} \big(\chi_{uv}(\widehat{\sigma}) + \chi_{vw}(\widehat{\sigma}) - \chi_{uw}(\widehat{\sigma})\big). \tag{34}$$

Lemma 3.6.1 implies that the contribution to the sum from the first two terms is equal to $|\operatorname{Inv}(\widehat{\sigma})| - |\{i < j : \mu_i \le \mu_j\}|$. The contribution from the last term is $-\sum_{u \in \operatorname{Des}(u)} a(u)$. This proves that $\operatorname{inv}(\widehat{\sigma})$ is equal to the number of inversion triples. There is exactly one triple (u,v,w) for every $u \in \operatorname{dg}'(\mu)$ and $v \in \operatorname{arm}(u)$. Hence the total number of triples is equal to $\sum_{u \in \operatorname{dg}'(\mu)} a(u)$, and it follows that $\operatorname{coinv}(\widehat{\sigma})$ is equal to the number of co-inversion triples. □

A more pictorial characterization of inversion versus co-inversion triples is sometimes useful. Observe that the pattern of boxes in any triple (u,v,w) is one of the following two types:

$$\underset{\text{Type I}}{\begin{array}{cc} \boxed{u} & \\ \boxed{w} & \boxed{v} \end{array}}, \qquad \underset{\text{Type II}}{\begin{array}{cc} \boxed{v} & \boxed{u} \\ & \boxed{w} \end{array}}, \tag{35}$$

with the further proviso that the column containing u, w is strictly taller than the column containing v in Type I, and weakly taller in Type II. In either type, one easily checks the following criterion.

Lemma 3.6.3. *If σ is a non-attacking filling, then a triple (u, v, w) is a co-inversion triple if and only if $\widehat{\sigma}(u) < \widehat{\sigma}(v) < \widehat{\sigma}(w)$ or $\widehat{\sigma}(v) < \widehat{\sigma}(w) < \widehat{\sigma}(u)$ or $\widehat{\sigma}(w) < \widehat{\sigma}(u) < \widehat{\sigma}(v)$.*

Informally speaking, a triple in a non-attacking filling is a co-inversion triple if its entries increase *clockwise* in Type I, or *counterclockwise* in Type II. For example, the two co-inversion triples in (25) are a Type I formed by the 3 and the 5 in row 1 with the 4 in row 2, and a Type II formed by the 6 and the 4 in row 2 with the 3 in row 1.

We may recast (26) as a formula for the "opposite" Macdonald polynomials $E_\mu(x; q^{-1}, t^{-1})$, and interpret the result conveniently in terms of inversion triples. Define

$$\begin{aligned} \operatorname{maj}'(\widehat{\sigma}) &= \Big(\sum_{\substack{u \in \operatorname{dg}'(\mu) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} (l(u)+1) \Big) - \operatorname{maj}(\widehat{\sigma}) = \sum_{\substack{u \in \operatorname{dg}'(\mu) \\ \widehat{\sigma}(u) < \widehat{\sigma}(d(u))}} (l(u)+1), \\ \operatorname{coinv}'(\widehat{\sigma}) &= \Big(\sum_{\substack{u \in \operatorname{dg}'(\mu) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} a(u) \Big) - \operatorname{coinv}(\widehat{\sigma}). \end{aligned} \tag{36}$$

Then $\operatorname{coinv}'(\widehat{\sigma})$ is the number of inversion triples with distinct entries—or, if one prefers, $\operatorname{coinv}'(\widehat{\sigma}) = \operatorname{coinv}(\widehat{\sigma}')$ and $\operatorname{maj}'(\widehat{\sigma}) = \operatorname{maj}(\widehat{\sigma}')$, where $\widehat{\sigma}'(u) = n + 1 - \widehat{\sigma}(u)$. In particular, $\operatorname{coinv}'(\widehat{\sigma})$ and $\operatorname{maj}'(\widehat{\sigma})$ are non-negative. We have the following corollary to Theorem 3.5.1.

Corollary 3.6.4.

$$E_\mu(x; q^{-1}, t^{-1}) = \sum_{\substack{\sigma \colon \mu \to [n] \\ \textit{non-attacking}}} x^\sigma q^{\operatorname{maj}'(\widehat{\sigma})} t^{\operatorname{coinv}'(\widehat{\sigma})} \prod_{\substack{u \in \operatorname{dg}'(\mu) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} \frac{1-t}{1 - q^{l(u)+1} t^{a(u)+1}}. \tag{37}$$

4. Proof of the formula

This section is devoted to the proof of Theorem 3.5.1. To avoid confusion between the two quantities which we want to prove are equal, we will henceforth denote the right-hand side of (26) by $C_\mu(x; q, t)$. We will prove the theorem by verifying that the special cases of (8) and (10) isolated in Lemma 2.1.2 hold with C_μ in place of E_μ.

4.1. **The cyclic shift symmetry π.** Our first task is to verify that C_μ satisfies (10). It will be convenient to extend the symmetry π in (9) first to *diagrams*, defining

$$\pi \colon \widehat{\operatorname{dg}}(\mu) \to \widehat{\operatorname{dg}}(\pi(\mu)) \tag{38}$$

by $\pi((i, j)) = (i + 1, j)$ for $i < n$, and $\pi((n, j)) = (1, j + 1)$, also to *values* of fillings, defining

$$\pi \colon [n] \to [n] \tag{39}$$

by $\pi(i) = i + 1$ for $i < n$, and $\pi(n) = 1$, and finally to *fillings*, defining ${}^{\pi}\sigma$ to be the unique filling of $\operatorname{dg}'(\pi(\mu))$ whose augmented filling satisfies

$$\widehat{{}^{\pi}\sigma}(\pi(u)) = \pi(\widehat{\sigma}(u)) \tag{40}$$

for all $u \in \widehat{\mathrm{dg}}(\mu)$. Note that this definition is compatible with the requirement that $\widehat{{}^{\pi}\sigma}((j,0)) = j$, so $\widehat{{}^{\pi}\sigma}$ is a well-defined augmented filling. Also note that $\pi\colon \widehat{\mathrm{dg}}(\mu) \to \widehat{\mathrm{dg}}(\pi(\mu))$ maps attacking pairs to attacking pairs; hence if σ is non-attacking, then so is ${}^{\pi}\sigma$.

Lemma 4.1.1. *Given $u \in \mathrm{dg}'(\mu)$, $v \in \widehat{\mathrm{dg}}(\mu)$, we have $v \in arm(u)$ if and only if $\pi(v) \in arm(\pi(u))$.*

Proof. Immediate from the definitions. □

Proposition 4.1.2. *Let $\sigma\colon \mathrm{dg}'(\mu) \to [n]$ be a non-attacking filling. Then $\mathrm{coinv}(\widehat{{}^{\pi}\sigma}) = \mathrm{coinv}(\widehat{\sigma})$, and $\mathrm{maj}(\widehat{{}^{\pi}\sigma}) = \mathrm{maj}(\widehat{\sigma}) + \mu_n - r$, where $r = |\sigma^{-1}(\{n\})|$.*

Proof. By Lemma 4.1.1, if (u, v, w) is a triple in $\widehat{\mathrm{dg}}(\mu)$, then $(u', v', w') = (\pi(u), \pi(v), \pi(w))$ is a triple in $\widehat{\mathrm{dg}}(\pi(\mu))$. From Lemma 3.6.3, we see that (u', v', w') is a co-inversion triple of $\widehat{{}^{\pi}\sigma}$ if and only if (u, v, w) is a co-inversion triple of $\widehat{\sigma}$. Moreover, every triple in $\widehat{\mathrm{dg}}(\pi(\mu))$ is the image under π of a triple in $\widehat{\mathrm{dg}}(\mu)$, except for those triples (u', v', w') in which $w' = (1, 0)$. But these exceptional triples satisfy $\widehat{{}^{\pi}\sigma}(u') = \widehat{{}^{\pi}\sigma}(w') = 1$, so they are not co-inversion triples. This proves that $\mathrm{coinv}(\widehat{{}^{\pi}\sigma}) = \mathrm{coinv}(\widehat{\sigma})$.

For the major index, let $S = \sigma^{-1}(\{n\}) = \{u \in \mathrm{dg}'(\mu) : \widehat{\sigma}(u) = n\}$, and let $S' = \{u \in \mathrm{dg}'(\mu) : \widehat{\sigma}(d(u)) = n\}$. For $u \in \mathrm{dg}'(\mu)$, we have $\pi(u) \in \mathrm{Des}(\widehat{{}^{\pi}\sigma})$ if and only if either $u \in \mathrm{Des}(\widehat{\sigma}) \setminus S$, or $u \in S' \setminus S$. Note that the only box of $\mathrm{dg}'(\pi(\mu))$ not in the image $\pi(\mathrm{dg}'(\mu))$ is $(1, 1)$, which is not a descent of $\widehat{{}^{\pi}\sigma}$, and that $l(\pi(u)) = l(u)$ for all $u \in \mathrm{dg}'(\mu)$. Therefore,

$$\mathrm{maj}(\widehat{{}^{\pi}\sigma}) = \sum_{u \in \mathrm{Des}(\widehat{\sigma}) \setminus S} (l(u) + 1) + \sum_{u \in S' \setminus S} (l(u) + 1). \tag{41}$$

Now, $S \cap \mathrm{Des}(\widehat{\sigma}) = S \setminus S'$, and using this, (41) is equivalent to

$$\mathrm{maj}(\widehat{{}^{\pi}\sigma}) = \sum_{u \in \mathrm{Des}(\widehat{\sigma})} (l(u) + 1) + \sum_{u \in S'} (l(u) + 1) - \sum_{u \in S} (l(u) + 1). \tag{42}$$

Observe that if $\mu_n > 0$, then S' contains the box $v = (n, 1)$, and that $l(v) + 1 = \mu_n$. The map $u' \to d(u')$ is a bijection from $S' \setminus \{v\}$ onto the set of boxes $u \in S$ such that $l(u) \neq 0$. Hence the second term in (42) is equal to

$$\mu_n + \sum_{\substack{u \in S \\ l(u) \neq 0}} l(u) = \mu_n + \sum_{u \in S} l(u), \tag{43}$$

so we obtain $\mathrm{maj}(\widehat{{}^{\pi}\sigma}) = \mathrm{maj}(\widehat{\sigma}) + \mu_n - |S|$, as desired. □

Corollary 4.1.3. *The combinatorial expression C_μ satisfies* (10)*, that is,*

$$C_{\pi(\mu)}(x; q, t) = q^{\mu_n} \Psi C_\mu(x; q, t) \underset{def}{=} q^{\mu_n} x_1 C_\mu(x_2, \ldots, x_n, q^{-1} x_1; q, t). \tag{44}$$

Proof. Every non-attacking filling of $\pi(\mu)$ satisfies $\sigma((1, 1)) = 1$, since $(1, 1)$ attacks all the boxes $(j, 0)$ for $j = 2, \ldots, n$. Therefore π is a bijection from non-attacking fillings of μ to non-attacking fillings of $\pi(\mu)$. Clearly, $\Psi(x^\sigma) = q^{-r} x^{({}^\pi\sigma)}$, where $r = |\sigma^{-1}(\{n\})|$. Hence $q^{\mu_n}\Psi(x^\sigma) q^{\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} = x^{({}^\pi\sigma)} q^{\operatorname{maj}(\widehat{{}^\pi\sigma})} t^{\operatorname{coinv}(\widehat{{}^\pi\sigma})}$, by Proposition 4.1.2. Moreover, π induces a bijection from $\{u \in \operatorname{dg}'(\mu) : \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))\}$ to $\{u \in \operatorname{dg}'(\pi(\mu)) : \widehat{{}^\pi\sigma}(u) \neq \widehat{{}^\pi\sigma}(d(u))\}$. It is obvious that $l(\pi(u)) = l(u)$, and Lemma 4.1.1 gives $a(\pi(u)) = a(u)$. Hence the term of $C_{\pi(\mu)}$ corresponding to ${}^\pi\sigma$ contains the same factor

$$\prod_{\substack{u \in \operatorname{dg}'(\mu) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} \frac{1-t}{1 - q^{l(u)+1} t^{a(u)+1}} \tag{45}$$

as the term of C_μ corresponding to σ. This shows that (44) holds term by term. □

4.2. **Partial symmetry of generating functions for fillings.** It remains to verify that C_μ satisfies suitable cases of (8). We cannot do this simply by applying the operator T_i in (7) to the formula for C_μ, as the resulting expressions are intractable. Instead, we will take an indirect approach, based on the combinatorial machinery associated with LLT polynomials, which we used in [6] to prove that the formulas given there for symmetric Macdonald polynomials are indeed symmetric. Clearly we must expect less in the non-symmetric case, but it turns out that we will be able to decompose C_μ into parts that are symmetric, or nearly so, in the two variables x_i and x_{i+1}, which is enough to compute $T_i C_\mu$.

Fix the signed alphabet

$$\mathcal{A}(n) = \{1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}\} \tag{46}$$

with the ordering that we denoted by $<_1$ in [6]. Let $S \subseteq \mathbb{N}^2$ be an arbitrary finite lattice-square diagram. A *signed filling* is a map $\sigma : S \to \mathcal{A}(n)$. Descents and inversions of σ are defined as in §3.2–3.3, with the modification that pairs of equal negative entries count as descents or inversions. Denote

$$\begin{aligned} \operatorname{Inv}(\sigma) &= \{\text{inversions of } \sigma\} \\ \operatorname{Des}(\sigma) &= \{\text{descents of } \sigma\}. \end{aligned} \tag{47}$$

In [6] we proved the following basic results. (The definitions of reading order and attacking pairs in [6] differ from those used here by a change of coordinates, and the results are stated for partition diagrams. The coordinate change is inconsequential, however, and the proofs also go through for arbitrary diagrams.)

Lemma 4.2.1 ([6, Proposition 3.4])**.** *Given a lattice-square diagram S and a subset $D \subseteq S$, the polynomial*

$$F_{S,D}(x;t) \underset{def}{=} \sum_{\substack{\sigma\colon S \to [n] \\ \operatorname{Des}(\sigma) = D}} x^\sigma t^{|\operatorname{Inv}(\sigma)|} \tag{48}$$

is a symmetric function; in fact it is an LLT polynomial.

Lemma 4.2.2 (see proof of [6, eqs. (38), (40)]). *For a signed filling $\sigma\colon S \to \mathcal{A}(n)$, let $p(\sigma)$ and $m(\sigma)$ denote the number of positive and negative entries $\sigma(u)$, respectively. Then with S and D as in Lemma 4.2.1, and with $X = x_1 + \cdots + x_n$, we have, using plethystic notation,*

$$F_{S,D}[X(t-1);t] = \sum_{\substack{\sigma\colon S\to\mathcal{A}(n)\\ \mathrm{Des}(\sigma)=D}} x^{|\sigma|}(-1)^{m(\sigma)}t^{p(\sigma)+|\operatorname{Inv}(\sigma)|}. \tag{49}$$

In particular, the sum on the right-hand side is a symmetric function.

Define a signed filling σ to be *non-attacking* if $|\sigma|$ is non-attacking.

Lemma 4.2.3 ([6, Lemma 5.1 and remark after it]). *The polynomial $F_{S,D}[X(t-1);t]$ in Lemma 4.2.2 is also given by*

$$F_{S,D}[X(t-1);t] = \sum_{\substack{\sigma\colon S\to\mathcal{A}(n)\\ \mathrm{Des}(\sigma)=D\\ \sigma \text{ non-attacking}}} x^{|\sigma|}(-1)^{m(\sigma)}t^{p(\sigma)+|\operatorname{Inv}(\sigma)|}. \tag{50}$$

In particular, the sum on the right-hand side is a symmetric function.

Now assign arbitrary "arm" and "leg" values $a(u)$, $l(u)$ to each $u \in S$ such that $d(u) \in S$, and define for any filling σ of S,

$$\begin{aligned} \operatorname{inv}(\sigma) &= |\operatorname{Inv}(\sigma)| - \sum_{u\in\mathrm{Des}(\sigma)} a(u)\\ \operatorname{maj}(\sigma) &= \sum_{u\in\mathrm{Des}(\sigma)} (l(u)+1). \end{aligned} \tag{51}$$

Lemma 4.2.4. *We have the identity*

$$\begin{aligned} &\sum_{\substack{\sigma\colon S\to\mathcal{A}(n)\\ \text{non-attacking}}} x^{|\sigma|}(-1)^{m(\sigma)}q^{\operatorname{maj}(\sigma)}t^{|S|-(p(\sigma)+\operatorname{inv}(\sigma))}\\ &\qquad = \sum_{\substack{\sigma\colon S\to[n]\\ \text{non-attacking}}} x^{\sigma}q^{\operatorname{maj}(\sigma)}t^{-\operatorname{inv}(\sigma)} \prod_{\substack{u,d(u)\in S\\ \sigma(u)=\sigma(d(u))}} \left(1-q^{l(u)+1}t^{a(u)+1}\right) \prod_{\substack{u\in S\\ \sigma(u)\neq\sigma(d(u))}} (1-t), \end{aligned} \tag{52}$$

where boxes $u \in S$ such that $d(u) \notin S$ contribute to the last factor. In particular, the sum on the right-hand side is a symmetric function.

Proof. On the left-hand side, consider the partial sum over terms for which $|\sigma|$ is a given filling $\tau\colon S \to [n]$. Set $V = \{u \in S : d(u) \in S,\ \tau(u) = \tau(d(u))\}$. Since τ is non-attacking, we have $\operatorname{Inv}(\sigma) = \operatorname{Inv}(\tau)$. We also have $\mathrm{Des}(\sigma) = \mathrm{Des}(\tau) \cup \{u \in V : \sigma(u) \in \overline{[n]}\}$. In particular, when $\sigma = \tau$, we have $\mathrm{Des}(\sigma) = \mathrm{Des}(\tau)$, and the corresponding term on the left-hand side of (52) is $x^{\tau}q^{\operatorname{maj}(\tau)}t^{-\operatorname{inv}(\tau)}$. In general, this term gets multiplied by a factor $q^{l(u)+1}t^{a(u)}\cdot(-t)$ for each $u \in V$ such that $\sigma(u)$ is negative, and by a factor $-t$ for each $u \in S \setminus V$ such that $\sigma(u)$ is negative. Summing over all choices of signs gives the term

$$x^{\tau} q^{\mathrm{maj}(\tau)} t^{-\mathrm{inv}(\tau)} \prod_{u \in V} \bigl(1 - q^{l(u)+1} t^{a(u)+1}\bigr) \prod_{u \in S \setminus V} (1-t), \tag{53}$$

and summing this over all τ gives (52). On the left-hand side of (52), the partial sum for each fixed descent set $\mathrm{Des}(\sigma) = D$ is a constant multiple of (50) (with t^{-1} in place of t). Hence each side of (52) is a symmetric function. □

We need to work in the more general setting of augmented fillings $\widehat{\sigma}$ that take prescribed values on the extra boxes in a larger diagram $\widehat{S} \supseteq S$. In this setting, of course, we lose the full symmetry, but we sometimes keep a partial symmetry.

Proposition 4.2.5. *Suppose given two disjoint lattice-square diagrams S and T, two disjoint subsets $Y, Z \in S$, and a filling $\tau\colon T \to [n]$ such that $\tau(T) \cap \{i, i+1\} = \emptyset$. Let $\widehat{S} = S \cup T$ and fix arm and leg values $a(u)$, $l(u)$ for each box $u \in S$ such that $d(u) \in \widehat{S}$. For any filling $\sigma\colon S \to [n]$, set $\widehat{\sigma} = \sigma \cup \tau$. With $\mathrm{maj}(\widehat{\sigma})$ and $\mathrm{inv}(\widehat{\sigma})$ defined as in (51), the sum*

$$\sum_{\substack{\sigma\colon S \to [n] \\ \widehat{\sigma}\ \text{non-attacking} \\ \sigma(Y) \cap \{i,i+1\} = \emptyset \\ \sigma(Z) \subseteq \{i,i+1\}}} x^{\sigma} q^{\mathrm{maj}(\widehat{\sigma})} t^{-\mathrm{inv}(\widehat{\sigma})} \prod_{\substack{u \in S, d(u) \in \widehat{S} \\ \widehat{\sigma}(u) = \widehat{\sigma}(d(u))}} \bigl(1 - q^{l(u)+1} t^{a(u)+1}\bigr) \prod_{\substack{u \in S \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} (1-t), \tag{54}$$

with any boxes $u \in S$ such that $d(u) \notin \widehat{S}$ included in the last factor, is symmetric in x_i and x_{i+1}.

Proof. Consider the partial sum over fillings for which the set $Z' = \sigma^{-1}(\{i, i+1\})$ and the restriction of σ to $S \setminus Z'$ are fixed. To evaluate this sum, we may as well adjoin Z' to Z and $S \setminus Z'$ to T, thus reducing the problem to the case that $Y = \emptyset$, $Z = S$, and the sum is over fillings $\sigma\colon S \to \{i, i+1\}$.

For each $u \in T$, the entry $\tau(u)$ stands in the same relative order to both i and $i+1$. Hence the only descents, inversions, and pairs $\widehat{\sigma}(u) = \widehat{\sigma}(d(u))$ which depend on σ are the ones created by pairs of boxes that are both in S. The whole sum (54) is therefore the product of a fixed polynomial $c(x; q, t)$ and the corresponding sum for the diagram S without the augmentation. In other words, the problem is further reduced to the case $T = \emptyset$, which is Lemma 4.2.4. □

Corollary 4.2.6. *Keep the notation of Proposition 4.2.5. The sum*

$$\sum_{\substack{\sigma\colon S \to [n] \\ \widehat{\sigma}\ \text{non-attacking} \\ \sigma(Y) \cap \{i,i+1\} = \emptyset \\ \sigma(Z) \subseteq \{i,i+1\}}} x^{\sigma} q^{\mathrm{maj}(\widehat{\sigma})} t^{-\mathrm{inv}(\widehat{\sigma})} \prod_{\substack{u \in S, d(u) \in \widehat{S} \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} \frac{(1-t)}{\bigl(1 - q^{l(u)+1} t^{a(u)+1}\bigr)}, \tag{55}$$

is symmetric in x_i, x_{i+1}.

Proof. Divide (54) by $\prod_{u\in S, d(u)\in \widehat{S}} \left(1-q^{l(u)+1}t^{a(u)+1}\right) \prod_{u\in S, d(u)\notin S}(1-t)$. $\square$

4.3. **Operators T_i and symmetry.** The following lemma will enable us to apply the machinery in §4.2.

Lemma 4.3.1. *For any $G_1, G_2 \in \mathbb{Q}(q,t)X$, and $0<i<n$, the following conditions are equivalent:*

(i) $G_2 = T_i G_1$;
(ii) $G_1 + G_2$ *and* $tx_{i+1}G_1 + x_i G_2$ *are symmetric in* x_i, x_{i+1}.

Proof. Let $A = (\mathbb{Q}(q,t)X)^{s_i}$ be the subring of Laurent polynomials which are symmetric in x_i, x_{i+1}. For every $f \in \mathbb{Q}(q,t)X$, there exist $a, b\in A$ such that

$$f = a + x_i b, \quad a, b \in A. \tag{56}$$

Specifically, we can take $b = (f - s_i(f))/(x_i - x_{i+1})$ and $a = (x_{i+1}f - x_i s_i(f))/(x_{i+1} - x_i)$. Now, T_i is an A-linear operator, satisfying the identity

$$T_i(a + x_i b) = ta + x_{i+1}b, \quad a, b\in A, \tag{57}$$

since $T_i(1) = t$ and $T_i(x_i) = x_{i+1}$.

Suppose (i) holds. Express G_1 in the form

$$G_1 = a + x_i b. \tag{58}$$

Then we have

$$G_2 = ta + x_{i+1}b, \tag{59}$$

$$G_1 + G_2 = (1+t)a + (x_i + x_{i+1})b, \tag{60}$$

$$tx_{i+1}G_1 + x_i G_2 = t(x_i + x_{i+1})a + (1+t)x_i x_{i+1} b. \tag{61}$$

The right-hand sides of (60)–(61) are manifestly symmetric, giving (ii). Conversely, suppose (ii) holds. We can view (60)–(61) as a non-singular system of linear equations for unknown rational functions a and b, with symmetric coefficients. Hence there exist $a, b \in \mathbb{Q}(q,t,x)^{s_i}$ such that (60)–(61) hold, and therefore (58)–(59) hold also. The operator T_i extends to $\mathbb{Q}(q,t,x)$ and is linear over $\mathbb{Q}(q,t,x)^{s_i}$, so it follows that $T_i G_1 = ta + x_{i+1}b = G_2$. $\square$

4.4. **Conclusion of the proof.** Fix μ and i such that $\mu_{i+1} = 0$, as in Lemma 2.1.2(b). We must show that C_μ satisfies (8), or equivalently, (17). Explicitly, let $u = (i,1) \in \mathrm{dg}'(\mu)$. We are to prove

$$C_{s_i(\mu)}(x;q,t) = \left(T_i + \frac{1-t}{1-q^{\mu_i}t^{a(u)}}\right) C_\mu(x;q,t). \tag{62}$$

Let

$$C_\mu = G_0 + G_1, \tag{63}$$

where G_0 is the partial sum in (26) over fillings σ such that $\sigma(u) \neq i$, and G_1 is the sum over fillings such that $\sigma(u) = i$.

Lemma 4.4.1. *G_0 is symmetric in x_i, x_{i+1}.*

Proof. In Corollary 4.2.6, take $S = \mathrm{dg}'(\mu)$, $T = \{(j,0) : j \in [n] \setminus \{i, i+1\}\}$, $\tau((j,0)) = j$, $Y = \{(j,1) \in \mathrm{dg}'(\mu) : j \leq i\}$, and $Z = \emptyset$. The only conditions imposed on a non-attacking filling σ of μ by the augmentation in boxes $(i,0)$ and $(i+1,0)$ are that $\sigma(y) \notin \{i, i+1\}$ for $y \in Y \setminus \{u\}$ and $\sigma(u) \neq i+1$. Hence the non-attacking augmented fillings $\widehat{\sigma}'$ of $\widehat{S} = \widehat{\mathrm{dg}}(\mu) \setminus \{(i,0),(i+1,0)\}$ which satisfy $\widehat{\sigma}'(y) \notin \{i, i+1\}$ for $y \in Y$ are precisely the restrictions to $\widehat{S}$ of the augmentations $\widehat{\sigma}$ of fillings σ of μ such that $\sigma(u) \neq i$.

Now, G_0 is the partial sum in (26) over the above fillings σ, while the expression in (55) is a sum over their restrictions $\widehat{\sigma}'$. Clearly, $\mathrm{maj}(\widehat{\sigma}') = \mathrm{maj}(\widehat{\sigma})$, since u is never a descent of $\widehat{\sigma}$. For every box $(j,1)$ in row 1 of $\mathrm{dg}'(\mu)$, we have $\sigma((j,1)) \leq j$, since $(j,1)$ attacks $(k,0)$ for all $k > j$. Hence $\widehat{\sigma}$ has no inversions between boxes in row 1 and boxes in row 0, and therefore $\mathrm{Inv}(\widehat{\sigma})$ is the union of $\mathrm{Inv}(\widehat{\sigma}')$ with the fixed set of $2n-3$ additional inversions in row 0 involving the boxes $(i,0)$ or $(i+1,0)$. This shows that $\mathrm{coinv}(\widehat{\sigma})$ differs from $-\mathrm{inv}(\widehat{\sigma}')$ by a constant independent of σ. Finally, since $\sigma(u) \neq i$, each term of G_0 contains a factor $(1-t)/(1-q^{\mu_i} t^{a(u)})$ not in the corresponding term of (55), but the remaining factors are the same for $\widehat{\sigma}$ and $\widehat{\sigma}'$. Hence G_0 is a constant multiple of the expression in (55). $\square$

Now define $s_i\colon \mathrm{dg}'(\mu) \to \mathrm{dg}'(s_i(\mu))$ by $s_i((j,k)) = (s_i(j), k)$. Let G_2 be the sum of the terms of $C_{s_i(\mu)}$ corresponding to fillings σ such that $\sigma(s_i(u)) = i+1$.

Lemma 4.4.2. *$G_1 + G_2$ is symmetric in x_i, x_{i+1}.*

Proof. In Corollary 4.2.6, take $S = \mathrm{dg}'(\mu)$, $T = \{(j,0) : j \in [n] \setminus \{i, i+1\}\}$, $\tau((j,0)) = j$, $Y = \{(j,1) \in \mathrm{dg}'(\mu) : j < i\}$, and $Z = \{u\}$. By similar reasoning to that in the proof of the previous lemma, the fillings $\sigma'\colon S \to [n]$ in (55) such that $\sigma'(u) = i+1$ correspond bijectively to non-attacking fillings $\sigma = \sigma' \circ s_i$ of $s_i(\mu)$ such that $\sigma(s_i(u)) = i+1$. Again, $\mathrm{coinv}(\widehat{\sigma})$ differs from $-\mathrm{inv}(\widehat{\sigma}')$ by a constant C independent of σ. Moreover, $\widehat{\sigma}(d(s_i(u))) = i+1 = \widehat{\sigma}(s_i(u))$, and for every $v \in \mathrm{dg}'(\mu) \setminus \{u\}$, we have $l(s_i(v)) = l(v)$, $a(s_i(v)) = a(v)$. Hence the terms in (55) for $\sigma'(u) = i+1$ sum to $t^{-C} G_2$. Similarly, the terms for $\sigma'(u) = i$ sum to $t^{-C} G_1$, so the expression in (55) is equal to $t^{-C}(G_1 + G_2)$. $\square$

Lemma 4.4.3. *$t x_{i+1} G_1 + x_i G_2$ is symmetric in x_i, x_{i+1}.*

Proof. Let $v = (i+1, 0)$. In Corollary 4.2.6, take $S = \mathrm{dg}'(\mu) \cup \{v\}$, $T = \{(j,0) : j \in [n] \setminus \{i, i+1\}\}$, $\tau((j,0)) = j$, $Y = \{(j,1) \in \mathrm{dg}'(\mu) : j < i\}$, and $Z = \{u, v\}$. Every non-attacking filling $\sigma'\colon S \to [n]$ such that $\sigma'(v) = i$ has $\sigma'(u) = i+1$. As in the proof of the previous lemma, these fillings correspond bijectively to fillings σ of $s_i(\mu)$ such that $\sigma(s_i(u)) = i+1$, and there is a constant C such that $\mathrm{coinv}(\widehat{\sigma}) = C - \mathrm{inv}(\widehat{\sigma}')$. Since $\sigma'(v) = i$, we have $x^{\sigma'} = x_i x^{\sigma}$, so the contribution to (55) from these terms is $t^{-C} x_i G_2$. Note that C includes a contribution from the inversion $(u,v) \in \mathrm{Inv}(\widehat{\sigma}') \setminus \mathrm{Inv}(\widehat{\sigma})$.

The remaining terms, with $\sigma'(v) = i+1$ and $\sigma'(u) = i$, correspond to fillings σ of μ such that $\sigma(u) = i$. In this case, $(u,v) \notin \mathrm{Inv}(\widehat{\sigma})$, so we now have $\mathrm{coinv}(\widehat{\sigma}) = C - 1 - \mathrm{inv}(\widehat{\sigma}')$. The contribution from these terms is therefore $t^{1-C} x_{i+1} G_1$, and the expression in (55) is equal to $t^{-C}(t x_{i+1} G_1 + x_i G_2)$. $\square$

To complete the proof of Theorem 3.5.1, observe that $T_i G_0 = tG_0$, by Lemma 4.4.1, and $T_i G_1 = G_2$, by Lemmas 4.3.1, 4.4.2 and 4.4.3. This yields

$$\left(T_i + \frac{1-t}{1-q^{\mu_i}t^{a(u)}}\right) C_\mu = \frac{1-q^{\mu_i}t^{a(u)+1}}{1-q^{\mu_i}t^{a(u)}} G_0 + \frac{1-t}{1-q^{\mu_i}t^{a(u)}} G_1 + G_2. \tag{64}$$

The fillings σ' of $s_i(\mu)$ such that $\sigma'(s_i(u)) \notin \{i, i+1\}$ correspond bijectively to fillings σ of μ such that $\sigma(u) \neq i$, by the rule $\sigma' = \sigma \circ s_i$. Since $a(s_i(u)) = a(u) - 1$, and $a(s_i(v)) = a(v)$ for all $v \neq u$ in $\mathrm{dg}'(\mu)$, the term in $C_{s_i(\mu)}$ corresponding to σ' differs from the term in C_μ corresponding to σ by the factor

$$\frac{1-q^{\mu_i}t^{a(u)+1}}{1-q^{\mu_i}t^{a(s_i(u))+1}} = \frac{1-q^{\mu_i}t^{a(u)+1}}{1-q^{\mu_i}t^{a(u)}}. \tag{65}$$

The first term in (64) is the sum of these terms. Similarly, the second term in (64) is the sum of terms of $C_{s_i(\mu)}$ corresponding to σ' such that $\sigma'(s_i(u)) = i$. The third term in (64) is by definition the sum of the remaining terms in $C_{s_i(\mu)}$, corresponding to σ' such that $\sigma'(s_i(u)) = i+1$. Hence (64) gives (62).

5. Comparison with symmetric Macdonald polynomials

In this section we indicate some of the connections between the combinatorial theory of non-symmetric Macdonald polynomials and the corresponding theory in the symmetric case. We will refer to three variants of symmetric Macdonald polynomials: the monic forms $P_\lambda(x; q, t)$ and integral forms $J_\lambda(x; q, t)$, as defined in [15], and the transformed integral forms $\tilde{H}_\lambda(x; q, t)$ as in [6], [7]. Throughout, λ denotes a partition $(\lambda_1 \geq \cdots \geq \lambda_n)$, possibly with some parts equal to zero.

5.1. More general formulas for $\tilde{H}_\lambda$.

The main theorem in our previous paper [6, Theorem 2.2] was a combinatorial formula for $\tilde{H}_\lambda$, expressed as a sum over fillings of the diagram of λ. By virtue of the identity

$$\tilde{H}_{\lambda'}(x; q, t) = \tilde{H}_\lambda(x; t, q), \tag{66}$$

the same thing can also be expressed as a sum over fillings of the diagram of the transpose partition λ', that is, of $\mathrm{dg}'(\lambda)$. We shall now generalize this to a sum over fillings of any rearrangement of the columns of $\mathrm{dg}'(\lambda)$. Our original result is equivalent to the special case of the following theorem in which μ is weakly increasing, *i.e.*, μ is the anti-dominant weight in the Weyl group orbit of λ.

Theorem 5.1.1. *For any composition $\mu \in \mathbb{N}^n$ and any $m > 0$, define*

$$D_\mu(x_1, \ldots, x_m; q, t) = \sum_{\sigma\colon\ \mathrm{dg}'(\mu) \to [m]} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)}, \tag{67}$$

where the sum is over unaugmented, possibly attacking fillings σ, and $\mathrm{maj}(\sigma)$, $\mathrm{inv}(\sigma)$ are as in (51), using the arm and leg values defined by (14)–(15). (Note that, although the arm of *a box in row* 1 *is defined with reference to the augmented diagram, these arms play no role in formula* (67)*, since a box in row* 1 *is never a descent of the unaugmented filling* σ*.) Then*

$$\tilde{H}_\lambda(x;q,t) = D_\mu(x;q,t), \tag{68}$$

where μ *is any rearrangement of the parts of the partition* λ*.*

Proof. As a matter of fact, the proof of [6, Theorem 2.2] goes through essentially unchanged in this more general setting. We will just briefly point out what properties of fillings of a partition diagram were actually used in [6], in order to see that the arguments there remain valid. The reader should bear in mind that for compatibility with the non-symmetric case, in (67) we use the right-to-left reading word, instead of the left-to-right reading word used in [6].

First, the proof of [6, Theorem 3.1] applies verbatim to show that D_μ is a symmetric function, and the derivation of [6, eqs. (40)–(41)] applies equally to D_μ.

Next, in [6, §5.1], we used two properties of the diagram and the statistics $\operatorname{inv}(\sigma)$ and $\operatorname{maj}(\sigma)$: (i) the fact that $\operatorname{inv}(\sigma)$ is equal to $|\operatorname{Inv}(\sigma)|$ plus a constant that only depends on $\operatorname{Des}(\sigma)$; and (ii) the fact that λ_i is the number of boxes in row i of the diagram. Both properties are preserved by rearranging the columns of the diagram and redefining the arm-lengths $a(u)$.

Finally, in [6, §5.2], we used (i) the fact that $\operatorname{inv}(\sigma)$ enumerates certain "inversion triples," plus inversions of σ between certain fixed pairs of boxes in row 1 of the diagram; and (ii) that the inversion property of a triple with respect to the ordering $<_2$ in [6, eq. (30)] is invariant under a sign change $a \mapsto \overline{a}$ applied to the entry in any one box v, provided that (a) for every box $u < v$ in the reading order, $x = \sigma(u)$ satisfies $|x| > |a|$, and (b) for every box u at most one row below v, $x = \sigma(u)$ satisfies $|x| \geq |a|$. In the present context, we can define inversion triples for unaugmented, signed, possibly attacking fillings by the same rule as in §3.6, with the modification that $\chi_{uv}(\sigma) = 1$ if $\sigma(u) = \sigma(v) = \overline{a}$, where $\overline{a}$ is a negative letter. Then property (ii) holds by essentially the same case checking as in the original proof. Lemma 3.6.1 holds for triples in the unaugmented diagram $\operatorname{dg}'(\mu)$ with "row 0" changed to "row 1." The counterpart of Proposition 3.6.2 (the proof is the same) is then that $\operatorname{inv}(\sigma)$ is the number of inversion triples plus the number of inversions between boxes $(i,1)$, $(j,1)$, where $i < j$ and $\mu_i \leq \mu_j$. Hence property (i) also holds. $\square$

The analog of Theorem 5.1.1 for the integral forms $J_\mu(x;q,t)$, generalizing [6, Proposition 8.1], is as follows.

Corollary 5.1.2. *Let* $n(\lambda) = \sum_i (i-1)\lambda_i$*. The integral form Macdonald polynomials are given by the sum over unaugmented, non-attacking fillings*

$$\begin{aligned} J_\lambda(x;q,t) &= t^{n(\lambda)} D_\mu[X(1-t^{-1});q,t^{-1}] \\ &= \sum_{\substack{\sigma\colon \operatorname{dg}'(\mu)\to[m] \\ \textit{non-attacking}}} x^\sigma q^{\operatorname{maj}(\sigma)} t^{n(\lambda)-\operatorname{inv}(\sigma)} \prod_{\sigma(u)=\sigma(d(u))} \left(1 - q^{l(u)+1} t^{a(u)+1}\right) \prod_{\sigma(u)\neq\sigma(d(u))} (1-t), \end{aligned} \tag{69}$$

where μ is any rearrangement of the parts of the partition λ. Any boxes u in row 1 of $\mathrm{dg}'(\mu)$ are included in the last factor on the second line.

Proof. The first identity is Theorem 5.1.1, combined with the definition of $\widetilde{H}_\lambda$. The proof of the second identity is the same as the proof of [6, Proposition 8.1] (see also Lemmas 4.2.3 and 4.2.4). $\square$

5.2. **Stable limits of non-symmetric Macdonald polynomials.** Let $(\nu;\mu)$ denote the concatenation of two compositions, and let $0^m \in \mathbb{N}^m$ denote the zero composition. The diagram $\mathrm{dg}'((0^m;\mu))$ is just the diagram of μ, shifted m columns to the right. We will identify fillings of these two diagrams in the obvious way. Note that $a(u) = a(u')$ for any $u \in \mathrm{dg}'(\mu)$ and its corresponding box $u' \in \mathrm{dg}'((0^m;\mu))$.

A general fact about non-symmetric Macdonald polynomials, which follows easily from the definition, is that if $s_i\mu = \mu$, then E_μ is s_i-invariant. In particular, $E_{(0^m;\mu)}$ is symmetric in the variables $x_1,\ldots,x_m$. Setting the remaining variables to zero therefore gives a symmetric function, which we will denote $E_{(0^m;\mu)}(x_1,\ldots,x_m;q,t)$, by slight abuse of notation. It is easy to see from Theorem 3.5.1 that as $m \to \infty$, this converges to a well-defined symmetric function in infinitely many variables, the *stable limit* of the non-symmetric Macdonald polynomial E_μ.

Theorem 5.2.1. *The stable limits of the integral form non-symmetric Macdonald polynomials are the integral form symmetric Macdonald polynomials. Precisely,*

$$J_\lambda(x_1,\ldots,x_m;q,t) = \mathcal{E}_{(0^m;\mu)}(x_1,\ldots,x_m;q,t), \tag{70}$$

for any μ which is a rearrangement of the parts of λ.

Proof. Every non-attacking filling σ of the unaugmented diagram $\mathrm{dg}'(\mu)$ with values in $[m]$ extends to a non-attacking augmented filling $\widehat{\sigma}$ of $\widehat{\mathrm{dg}}((0^m;\mu))$, because the boxes $(j,0)$ for $j \leq m$ only attack other boxes in row 0, and the boxes $(j,0)$ for $j > m$ contain entries $\widehat{\sigma}((j,0)) > m$. It is clear that $\mathrm{maj}(\widehat{\sigma}) = \mathrm{maj}(\sigma)$, and that $\widehat{\sigma}$ has no descents in row 1. We also have $\mathrm{coinv}(\widehat{\sigma}) = C - \mathrm{inv}(\sigma)$, where $C = \sum_{u\in \mathrm{dg}'(\mu)} a(u) - |\{i<j : \mu_i > \mu_j\}|$. Comparing (27) and (69), we see that the result follows if $C = n(\lambda)$. If $\mu_1 \leq \cdots \leq \mu_n$, so $\mathrm{dg}'(\mu)$ is the reversed diagram of the partition λ', this is immediate, since $n(\lambda)$ is the sum of the arms of the boxes in the diagram of λ'. If $\mu_i < \mu_{i+1}$, then transposing these two parts of μ increases $|\{i<j : \mu_i > \mu_j\}|$ by 1. On the other hand, for every box $u \in \mathrm{dg}'(\mu)$, we have $a(s_i(u)) = a(u)$, with one exception: for $u = (i+1, \mu_i+1)$, we have $a(s_i(u)) = a(u)+1$. So the transposition increases $\sum_{u\in\mathrm{dg}'(\mu)} a(u)$ by 1, and C remains invariant. It follows that $C = n(\lambda)$ for every rearrangement μ of λ. $\square$

Corollary 5.2.2. *Let λ° be the rearrangement of the parts of λ in weakly increasing order. For any rearrangement μ of λ, we have*

$$P_\lambda(x_1,\ldots,x_m;q,t) = \left(\frac{\prod_{u\in\mathrm{dg}'(\mu)} 1 - q^{l(u)+1}t^{a(u)+1}}{\prod_{u\in\mathrm{dg}'(\lambda^\circ)} 1 - q^{l(u)}t^{a(u)+1}}\right) E_{(0^m;\mu)}(x_1,\ldots,x_m;q,t). \tag{71}$$

Remarks. (1) A related stable limit, but not truncated to the symmetric part, is considered by Knop in a most interesting preprint [13], where he conjectures an extension of the positivity theorem for Macdonald polynomials to the non-symmetric case.

(2) The Jack polynomial version of Theorem 5.2.1 is due to Knop and Sahi [14, Theorem 4.10]. It is possible to prove Theorem 5.2.1 directly from the orthogonality of Macdonald polynomials, as Knop and Sahi did in the Jack case. Then one can reverse the arguments and deduce Corollary 5.1.2 and Theorem 5.1.1 from Theorem 5.2.1 and Corollary 3.5.2. In principle, this gives an alternative proof of [6, Theorem 2.2], although the only part of the original proof that it avoids is [6, Lemma 5.2].

5.3. **Symmetrization of non-symmetric Macdonald polynomials.** For any root system, the Macdonald polynomial P_λ is the unique monic symmetric linear combination of the polynomials E_μ for μ in the Weyl group orbit of λ. The coefficient of E_μ in P_μ can be computed explicitly using intertwiners [3]. In the GL_n case, this is most conveniently expressed in terms of the polynomials $E_\mu(x; q^{-1}, t^{-1})$, using the fact that $P_\mu(x; q, t) = P_\mu(x; q^{-1}, t^{-1})$, as follows.

Proposition 5.3.1 ([16], [19, Lemma 2.5(a)])**.** *With λ° as in Corollary 5.2.2, we have*

$$P_\lambda(x; q, t) = \prod_{u \in \mathrm{dg}'(\lambda^\circ)} (1 - q^{l(u)+1} t^{a(u)}) \cdot \sum_{\mu \sim \lambda} \frac{E_\mu(x; q^{-1}, t^{-1})}{\prod_{u \in \mathrm{dg}'(\mu)} (1 - q^{l(u)+1} t^{a(u)})}, \tag{72}$$

where the sum is over all rearrangements μ of λ.

Combined with Theorem 3.5.1 and Corollary 3.6.4, we can regard (71) and (72) as different kinds of combinatorial formulas for P_λ. In particular, letting $q, t \to 0$ in (72) yields a curious combinatorial expansion for the Schur function $s_\lambda(x)$, namely,

$$\begin{aligned} s_\lambda(x) &= \sum_{\mu \sim \lambda} E_\mu(x; \infty, \infty), \quad \text{where} \\ E_\mu(x; \infty, \infty) &= \sum_{\substack{\sigma\colon \mathrm{dg}'(\mu) \to [n] \\ \text{non-attacking} \\ \mathrm{maj}'(\widehat{\sigma}) = \mathrm{coinv}'(\widehat{\sigma}) = 0}} x^\sigma \end{aligned} \tag{73}$$

It is natural to regard the polynomials $E_\mu(x; \infty, \infty)$, which coincide with what combinatorialists refer to as *key polynomials* [22], as “non-symmetric Schur functions.” They are closely related but not equal to the polynomials $E_\mu(x; 0, 0)$, which are Demazure characters [10], [11]. A further study of $E_\mu(x; \infty, \infty)$ has been made by Sarah Mason [20], who among other things found a direct, bijective proof of (73).

Appendix: table of $E_\mu(x;q,t)$

The case $n=1$ is trivial. A general closed formula for $n=2$ can be found in Macdonald's book [16, (6.2.7–8)]. We give a short table for $n=3$.

$$
\begin{aligned}
E_{(0,0,0)} &= 1\\
E_{(1,0,0)} &= x_1\\
E_{(0,1,0)} &= x_2 + \frac{1-t}{1-q\,t^2}\,x_1\\
E_{(0,0,1)} &= x_3 + \frac{1-t}{1-q\,t}(x_1+x_2)\\
E_{(1,1,0)} &= x_1\,x_2\\
E_{(1,0,1)} &= x_1\,x_3 + \frac{1-t}{1-q\,t^2}\,x_1\,x_2\\
E_{(0,1,1)} &= x_2\,x_3 + \frac{1-t}{1-q\,t}(x_1\,x_2+x_1\,x_3)\\
E_{(2,0,0)} &= {x_1}^2 + \frac{q\,(1-t)}{1-q\,t}(x_1\,x_2+x_1\,x_3)\\
E_{(0,2,0)} &= {x_2}^2 + \frac{1-t}{(1-q^2\,t^2)}\,{x_1}^2 + \frac{q\,(1-t)}{1-q\,t}\,x_2\,x_3\\
&\qquad + \frac{q\,(1-t)^2}{(1-q\,t)\,(1-q^2\,t^2)}\,x_1\,x_3 + \frac{(1-t)\,(1+q-q\,t-q^2\,t^2)}{(1-q\,t)\,(1-q^2\,t^2)}\,x_1\,x_2
\end{aligned}
$$

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