Cherednik algebras, Macdonald polynomials and combinatorics

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Abstract. In the first part of this article we review the general theory of Cherednik algebras and non-symmetric Macdonald polynomials, including a formulation and proof of the fundamental *duality theorem* in its proper general context. In the last section we summarize some of the combinatorial results in this area which we have recently obtained in collaboration with J. Haglund and N. Loehr.

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1. Introduction

The record is very long. The facts are few and may be briefly stated.
—Miller v. San Francisco Methodist Episcopal (1932)

This article consists of an overview of the theory of Cherednik algebras and non-symmetric Macdonald polynomials, followed by the combinatorial formula for non-symmetric Macdonald polynomials of type A_{n-1} recently obtained by Haglund, Loehr and the author.

The main points in the theory are duality (Theorems 4.10, 5.11), and its consequence, the intertwiner recurrence for Macdonald polynomials (Corollary 6.15), which is the key to the combinatorial study of non-symmetric Macdonald polynomials. The intertwiner recurrence can also be used to deduce other important results in the theory, such as the norm and evaluation formulas, but I have omitted those for lack of space.

The theory of course did not spring into being in the tidy form in which I have attempted to package it here. Rather, it has been gradually clarified over almost twenty years through the efforts of many people, in a large literature which I will not attempt to cite in full. Let me only mention the origins of the theory in the works of Macdonald [13, 14, 15], Opdam [17], and Cherednik [1, 2] and remark that further important contributions were made by Ion, Knop, Koornwinder, Sahi, and van Diejen, among others.

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The overview given here necessarily has much in common with Macdonald's monograph [16], which serves a similar purpose, but there are also several differences. I have systematically used the lattice formulation for root systems, because it is most natural from related points of view (algebraic groups, quantum groups), because it puts affine and other root systems on an equal footing, and because important elements of the theory (§2, 5.1–5.5, 5.13–5.15) apply to arbitrary root systems. I give a new and somewhat more general proof of the duality theorem; Macdonald's proof, strictly speaking, applies to the root system of SL_n, for instance, but not GL_n or PGL_n, although it can be adjusted to cover these cases. For the triangularity property of the Macdonald polynomials E_λ (Theorem 6.6), I use the affine Bruhat order on the weight lattice X, rather than the orbit-lexicographic order used by Macdonald. This simplifies some arguments, and is more natural in that the coefficient of x^μ in E_λ is non-zero if and only if $\mu < \lambda$ in Bruhat order. I have also tried to use more transparent notation.

2. Root systems

2.1. We always consider root systems realized in a lattice. So, for us, a *root system* $\bigl(X, (\alpha_i), (\alpha_i^\vee)\bigr)$ consists of a a finite-rank free abelian group X, whose dual lattice $\operatorname{Hom}(X, \mathbb{Z})$ is denoted $X^\vee$, a finite set of vectors $\alpha_1, \ldots, \alpha_n \in X$, called *simple roots*, and a finite set of covectors $\alpha_1^\vee, \ldots, \alpha_n^\vee \in X^\vee$, called *simple coroots.* We denote by $X_\mathbb{Q}$ (resp. $X_\mathbb{R}$) the $\mathbb{Q}$-vector space $X \otimes_\mathbb{Z} \mathbb{Q}$ (resp. $\mathbb{R}$-vector space $X \otimes_\mathbb{Z} \mathbb{R}$) spanned by X.

The $n \times n$ matrix A with entries $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ is assumed to be a *generalized Cartan matrix*, satisfying the axioms

$$\begin{array}{cl} \text{(i)} & \langle \alpha_i, \alpha_i^\vee \rangle = 2, \\ \text{(ii)} & \langle \alpha_j, \alpha_i^\vee \rangle \leq 0 \text{ for all } j \neq i, \\ \text{(iii)} & \langle \alpha_j, \alpha_i^\vee \rangle = 0 \text{ if and only if } \langle \alpha_i, \alpha_j^\vee \rangle = 0. \end{array}$$

The *Dynkin diagram* is the graph with nodes $i = 1, \ldots, n$ and an edge $\{i, j\}$ for each $a_{ij} \neq 0$, usually with some decoration on the edges to indicate the values of a_{ij}, a_{ji}. If the Dynkin diagram is connected, A is *indecomposable.* If there exist non-zero integers d_i such that $\langle \alpha_j, d_i \alpha_i^\vee \rangle = \langle \alpha_i, d_j \alpha_j^\vee \rangle$ for all i, j, then A is *symmetrizable.* The integers d_i can be assumed positive. If A is symmetrizable and indecomposable, the d_i are unique up to an overall common factor. Then d_i is *length* of the root α_i. If there are only two root lengths, we call them *long* and *short.* If there is only one root length, every root is both long and short.

2.2. Let $\alpha \in X$ and $\alpha^\vee \in X^\vee$ satisfy $\langle \alpha, \alpha^\vee \rangle = 2$. The linear automorphism

$$s_{\alpha\alpha^\vee}(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \, \alpha$$

of X is a *reflection.* It fixes the hyperplane $\langle \lambda, \alpha^\vee \rangle = 0$ pointwise, and sends α to $-\alpha$. Thus $(s_{\alpha,\alpha^\vee})^2 = 1$. The reflection on $X^\vee$ dual to $s_{\alpha,\alpha^\vee}$ is equal to $s_{\alpha^\vee,\alpha}$.

If $\alpha^\vee$ is implicitly associated to α, we write s_α for both $s_{\alpha,\alpha^\vee}$ and $s_{\alpha^\vee,\alpha}$. When $\alpha = \alpha_i$ and $\alpha^\vee = \alpha_i^\vee$ are a simple root and corresponding coroot, we write s_i for s_{α_i}. The s_i are called *simple reflections*.

2.3. The root system $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$ is *non-degenerate* if the simple roots α_i are linearly independent. When the Cartan matrix A is non-singular, *e.g.*, for any finite root system, then both X and its dual $\big(X^\vee, (\alpha_i^\vee), (\alpha_i)\big)$ are necessarily non-degenerate. When A is singular, for instance if the root system is affine (Definition 3.1), it is often convenient to take the simple roots to be a basis of $X_{\mathbb{Q}}$, in which case X is non-degenerate but its dual is degenerate.

2.4. Assume in what follows that $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$ is non-degenerate. The *Weyl group* W is the group of automorphisms of X (and of $X^\vee$) generated by the simple reflections s_i. The sets of *roots* and *coroots* are

$$R = \bigcup_i W(\alpha_i), \qquad R^\vee = \bigcup_i W(\alpha_i^\vee).$$

The *root* and *coroot lattices* are

$$Q = \mathbb{Z}\{\alpha_1, \dots, \alpha_n\} \subseteq X, \qquad Q^\vee = \mathbb{Z}\{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subseteq X^\vee$$

The set of *positive roots* is $R_+ = R \cap Q_+$, where

$$Q_+ = \mathbb{N}\{\alpha_1, \dots, \alpha_n\}.$$

The *dominant weights* are the elements of the cone

$$X_+ = \{\lambda \in X : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i\}.$$

The root system $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$ is *finite* if W is a finite group, or equivalently, R is a finite set. The Cartan matrix A of a finite root system is symmetrizable, with positive definite symmetrization DA. Conversely, if A has a positive definite symmetrization, then R is finite. The finite root systems classify reductive algebraic groups G over any algebraically closed field k. Then X is the character group of a maximal torus in G, or *weight lattice*.

Example 2.5. Let $X = \mathbb{Z}^n$, and identify $X^\vee$ with X using the standard inner product on $\mathbb{Z}^n$ such that the unit vectors e_i are orthogonal. Let $\alpha_i = \alpha_i^\vee = e_i - e_{i+1}$ for $i = 1, \dots, n-1$. This gives the root system of the group GL_n.

Replacing X with the root lattice Q and restricting the simple coroots to Q, we obtain the root system of the adjoint group PGL_n (GL_n modulo its center).

The constant vector $\varepsilon = e_1 + \cdots + e_n$ satisfies $\langle \varepsilon, \alpha_i^\vee \rangle = 0$ for all i. Let $X' = X/(\mathbb{Z}\varepsilon)$, with simple roots and coroots induced by those of X. This gives the root system of the simply connected group SL_n. It is dual to the root system of PGL_n. All three root systems have the same Cartan matrix, of type A_{n-1}.

2.6. We recall some standard facts. First, $R = R_+ \cup -R_+$, *i.e.*, every root is positive or negative (note that $R = -R$, since $s_i(\alpha_i) = -\alpha_i$ for all i). The Weyl group W, with its generating set S of simple reflections s_i, is a Coxeter group with defining relations

$$s_i^2 = 1 \tag{1}$$

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots \quad (m_{ij} \text{ factors on each side}), \tag{2}$$

where if $a_{ij}a_{ji} = 0$, 1, 2 or 3, then $m_{ij} = 2$, 3, 4, or 6, respectively, and if $a_{ij}a_{ji} \geq 4$, there is no relation between s_i, s_j.

The *length* $l(w)$ of $w \in W$ is the minimal l such that $w = s_{i_1} \cdots s_{i_l}$. Such an expression is called a reduced factorization. More generally, if $w = u_1 u_2 \cdots u_r$ with $l(w) = l(u_1) + \cdots + l(u_r)$ we call $u_1 \cdot u_2 \cdots u_r$ a reduced factorization.

If $w = s_{j_1} \cdots s_{j_l}$ is a second reduced factorization, then the identity $s_{j_1} \cdots s_{j_l} = s_{i_1} \cdots s_{i_l}$ holds in the monoid with generators s_i and the *braid relations* (2), that is, it does not depend on the relations $s_i^2 = 1$.

The length of w is equal to the number of positive roots carried into negative roots by w, *i.e.*, $l(w) = |R_+ \cap w^{-1}(-R_+)|$. In particular, α_i is the only positive root α such that $s_i(\alpha) \in -R_+$. The following conditions are equivalent: (i) $l(ws_i) < l(w)$; (ii) $w(\alpha_i) \in -R_+$; (iii) some reduced factorization of w ends with s_i. We abbreviate these conditions to $ws_i < w$, and write $s_i w < w$ when $w^{-1}s_i < w^{-1}$.

If $\alpha = w(\alpha_i) = w'(\alpha_j)$, then $w(\alpha_i^\vee) = w'(\alpha_j^\vee)$, so there is a well-defined coroot $\alpha^\vee = w(\alpha_i^\vee)$ associated to α and satisfying $\langle \alpha, \alpha^\vee \rangle = 2$, and accordingly a well-defined reflection $s_\alpha = s_{\alpha,\alpha^\vee} = w s_i w^{-1}$. Warning: the correspondence $\alpha \mapsto \alpha^\vee$ need not be bijective if the dual root system is degenerate.

The map $W \to \{\pm 1\}$, $w \mapsto (-1)^{l(w)}$ is a group homomorphism. In particular, $l(s_\alpha)$ is always odd, and $l(ws_\alpha) \neq l(w)$. We put $ws_\alpha < w$ if $l(ws_\alpha) < l(w)$. The *Bruhat order* is the partial order on W given by the transitive closure of these relations.

2.7. The *braid group* $\mathcal{B}(W)$ is the group with generators T_i and the braid relations (2) with T_i in place of s_i. If $w = s_{i_1} \cdots s_{i_l}$ is a reduced factorization, we set $T_w = T_{i_1} \cdots T_{i_l}$. These elements are well-defined and satisfy

$$T_u T_v = T_{uv} \quad \text{when } uv = u \cdot v \text{ is a reduced factorization.} \tag{3}$$

There is a canonical homomorphism $\mathcal{B}(W) \to W$, $T_i \mapsto s_i$. By the symmetry of the braid relations, there is an automorphism $T_i \leftrightarrow T_i^{-1}$ of $\mathcal{B}(W)$.

2.8. The *affine Weyl group* of $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$ is the semidirect product $W \ltimes X$. In this context, we use multiplicative notation for the group X, denoting $\lambda \in X$ by x^λ. Explicitly, $W \ltimes X$ is generated by its subgroups W and X with the additional relations

$$s_i x^\lambda s_i = x^{s_i(\lambda)}. \tag{4}$$

2.9. The (left) *affine braid group* $\mathcal{B}(W, X)$ of $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$ is the group generated by $\mathcal{B}(W)$ and X, with the additional relations

$$T_i x^\lambda = x^\lambda T_i \qquad \text{if } \langle \lambda, \alpha_i^\vee \rangle = 0 \ \ (\textit{i.e.}, \text{ if } s_i(\lambda) = \lambda); \tag{5}$$

$$T_i x^\lambda T_i = x^{s_i(\lambda)} \qquad \text{if } \langle \lambda, \alpha_i^\vee \rangle = 1. \tag{6}$$

These two relations may be combined into the following analog of (4):

$$T_i^a x^\lambda T_i^b = x^{s_i(\lambda)}, \quad \text{where } a, b \in \{\pm 1\} \text{ and } \langle \lambda, \alpha_i^\vee \rangle = (a+b)/2 \tag{7}$$

(the case $a = b = -1$ follows by taking inverses on both sides in (6)). The canonical homomorphism $\mathcal{B}(W) \to W$ extends to a homomorphism $\mathcal{B}(W, X) \to W \ltimes X$ which is the identity on X.

For clarity when dealing with double affine braid groups later on, we define separately the *right* affine braid group $\mathcal{B}(X, W)$, generated by W and X with additional relations

$$T_i x^\lambda = x^\lambda T_i \qquad \text{if } \langle \lambda, \alpha_i^\vee \rangle = 0\,; \tag{8}$$

$$T_i^{-1} x^\lambda T_i^{-1} = x^{s_i(\lambda)} \qquad \text{if } \langle \lambda, \alpha_i^\vee \rangle = 1. \tag{9}$$

There is an isomorphism $\mathcal{B}(X, W) \cong \mathcal{B}(W, X)$ which maps $T_i \mapsto T_i^{-1}$ and is the identity on X.

2.10. If $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$ is a non-degenerate root system, the root lattice Q is free with basis (α_i). Identify $Q^\vee$ with a quotient of the free abelian group $\widehat{Q}^\vee$ with basis $(\alpha_i^\vee)$, and set $P = \operatorname{Hom}(\widehat{Q}^\vee, \mathbb{Z})$. The roots and coroots in X are then given by homomorphisms $Q \to X \to P$, where the matrix of the composite $Q \to P$ is the Cartan matrix A. Suppose that $X \to P$ factors through a second lattice X' as

$$Q \to X \underset{j}{\to} X' \to P.$$

This induces a root system $\big(X', (\alpha_i'), (\alpha_i'^\vee)\big)$ in X' with the same Cartan matrix A and canonically isomorphic Weyl and braid groups $W' = W$, $\mathcal{B}(W') = \mathcal{B}(W)$. There is an induced homomorphism of affine braid groups $j_{\mathcal{B}} \colon \mathcal{B}(W, X) \to \mathcal{B}(W, X')$ which restricts to j on X and to the canonical isomorphism on $\mathcal{B}(W)$.

Theorem 2.11. *The image of* $j_{\mathcal{B}} \colon \mathcal{B}(W, X) \to \mathcal{B}(W, X')$ *is normal in* $\mathcal{B}(W, X')$, *and the induced maps* $\ker(j) \to \ker(j_{\mathcal{B}})$, $\operatorname{coker}(j) \to \operatorname{coker}(j_{\mathcal{B}})$ *are isomorphisms.*

Proof (outline). First suppose that $X' = X \oplus \mathbb{Z}\nu$, where $\langle \nu, \alpha_i^\vee \rangle \in \{0, 1\}$ for all i. One proves that there exists an automorphism η of $\mathcal{B}(W, X)$ which fixes X, such that $\eta(T_i) = T_i$ if $\langle \nu, \alpha_i^\vee \rangle = 0$, and $\eta(T_i) = T_i^{-1} x^{-\alpha_i}$ if $\langle \nu, \alpha_i^\vee \rangle = 1$. Then one checks that $\eta^{\mathbb{Z}} \ltimes \mathcal{B}(W, X) \cong \mathcal{B}(W, X')$, with $\eta \mapsto x^\nu$. Iterating this gives $\mathcal{B}(W, X \oplus P) \cong P \ltimes \mathcal{B}(W, X)$, and similarly, $\mathcal{B}(W, X' \oplus P) \cong P \ltimes \mathcal{B}(W, X')$. Replacing X, X' with $X \oplus P$, $X' \oplus P$, we may assume that $X \to P$ and $X' \to P$ are surjective.

Next one verifies that if $X \to X'$ is surjective, with kernel Z, then $\mathcal{B}(W, X') \cong \mathcal{B}(W, X)/Z$. Applying this to $0 \to Z \to X \to P \to 0$ and $0 \to Z' \to X' \to P \to 0$, we get surjections $\mathcal{B}(W, X) \to \mathcal{B}(W, P)$, $\mathcal{B}(W, X') \to \mathcal{B}(W, P)$ with kernels Z, Z'. The theorem then follows by some easy diagram chasing. $\square$

2.12. Let $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$ be a root system. It may happen that for one or more of the simple roots α_i, we have $\alpha_i^\vee \in 2X^\vee$. Then we can form another (degenerate) root system by adjoining a new simple root $2\alpha_i$ and coroot $\alpha_i^\vee/2$. Note that $s_{(2\alpha_i),(\alpha_i^\vee/2)} = s_i$, so this new root system has the same Weyl group as the original one, but a larger set of roots $R' = R \cup W(2\alpha_i)$.

If a root system contains two simple roots α_i, $\alpha_{i'}$ such that $s_i = s_{i'}$ and $\alpha_i \neq \pm\alpha_{i'}$, it is said to be *non-reduced*, otherwise it is *reduced*. We remark that $s_{i'} = s_i$ implies $\alpha_{i'} = d\alpha_i$, $\alpha_{i'}^\vee = (1/d)\alpha_i^\vee$, where $d \in \{\pm 1, \pm 2, \pm 1/2\}$. Hence every non-reduced root system is constructed by extensions as above from a reduced root system with the same Weyl group.

3. Affine root systems and affine Weyl groups

Definition 3.1. A root system $\bigl(X, (\alpha_i), (\alpha_i^\vee)\bigr)$, is *affine* if its Cartan matrix A is singular, and for every proper subset J of the indices, the root system $\bigl(X, (\alpha_i)_{i\in J}, (\alpha_i^\vee)_{i\in J}\bigr)$ is finite.

3.2. The definition implies that the nullspace of A is one-dimensional. If X is non-degenerate, then $\{\lambda \in Q : \langle \lambda, \alpha_i^\vee \rangle = 0 \text{ for all } i\}$ is a sublattice of rank 1. It always has a (unique) generator $\delta \in Q_+$, called the *nullroot*.

We index the simple roots by $i = 0, 1, \ldots, n$. We always assume that $i = 0$ is an *affine node*, meaning that $\alpha_0 \in \mathbb{Q}\alpha + \mathbb{Q}\delta$ for some root α of the finite root system $\bigl(X, (\alpha_1, \ldots, \alpha_n), (\alpha_1^\vee, \ldots, \alpha_n^\vee)\bigr)$. This condition is equivalent to $s_1, \ldots, s_n$ generating the finite Weyl group $W_0 = W/Q_0'$, where W is the Weyl group and Q_0' is the kernel of its induced action on $X/(X \cap \mathbb{Q}\delta)$. Every affine root system has at least one affine node.

3.3. The affine Cartan matrices are classified in Kac [8] and Macdonald [16]. They are symmetrizable and indecomposable. We refer to them using Macdonald's nomenclature, but with a tilde over the names to distinguish them from finite types. Those denoted $\widetilde{X}_n$, or $X_n^{(1)}$ in Kac, are the *untwisted types*, where $X_n = A_n$, B_n, C_n, D_n, $E_{6,7,8}$, F_4, or G_2 is a Cartan matrix of finite type. Their duals (if different) $\widetilde{B}_n^\vee$, $\widetilde{C}_n^\vee$, $\widetilde{F}_4^\vee$, $\widetilde{G}_2^\vee$ are the *dual untwisted types*, denoted $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $E_6^{(2)}$, and $D_4^{(3)}$ in Kac.

The remaining *mixed* types, denoted $A_{2n}^{(2)}$ in Kac, are exceptional in that they have three root lengths. Although the mixed types are isomorphic to their duals, we prefer to distinguish between them, denoting a mixed type as $\widetilde{BC}_n$ when the distinguished affine root α_0 is the longest simple root, and $\widetilde{BC}_n^\vee$ when α_0 is the shortest simple root.

Types $\widetilde{B}_n$, $\widetilde{C}_n^\vee$, $\widetilde{BC}_n$, $\widetilde{BC}_n^\vee$ contain one or more simple roots α_i such that $\langle \alpha_j, \alpha_i^\vee \rangle$ is even for all j. There exist affine root systems X of these types such that $\alpha_i^\vee \in 2X^\vee$. A *non-reduced affine root system* is a non-reduced extension (§2.12) of such a root system X.

3.4. The Weyl group W_a of any affine root system $\bigl(X, (\alpha_i), (\alpha_i^\vee)\bigr)$ is isomorphic to the affine Weyl group $W = Q_0' \rtimes W_0$ of some finite root system $\bigl(Y, (\alpha_i'), (\alpha_i'^\vee)\bigr)$.

Conversely, the affine Weyl group $Y \rtimes W_0$ of any finite root system is a semidirect extension $\Pi \ltimes W_a$ of the Weyl group of a corresponding affine root system. We now fix precise notation and explain how this correspondence comes about.

3.5. Let $\big(Y, (\alpha'_i), (\alpha'^\vee_i)\big)$, $i = 1, \ldots, n$, be a finite root system, with Weyl group W_0 and root lattice Q'_0. Let ϕ' be the (unique) dominant short root. Let $W_e = Y \rtimes W_0$ be the affine Weyl group of Y, and set $W_a = Q'_0 \rtimes W_0 \subseteq W_e$. Write y^λ for $\lambda \in Y$ regarded as an element of W_e. The orbit $W_0(\phi')$ consists of all the short roots, and spans Q_0. Defining $s_0 = y^{\phi'} s_{\phi'}$, it follows that s_0 and $s_1, \ldots, s_n \in W_0$ generate W_a. We will construct an affine root system whose Weyl group W is isomorphic to W_a, with simple reflections corresponding to the generators $s_0, \ldots, s_n$.

3.6. Let $X = Y^\vee \oplus \mathbb{Z}$, and fix a non-zero element δ in the second summand. We need not assume that δ is a generator, so in general we have $X = Y^\vee \oplus \mathbb{Z}\delta/m$ for some positive integer m. Define the pairing $\langle X, Y\rangle \to \mathbb{Z}$, extending the canonical pairing $\langle Y^\vee, Y\rangle \to \mathbb{Z}$, with $\langle \delta, Y\rangle = 0$.

Let $\theta = \phi'^\vee$ be the highest coroot. For $i \neq 0$, set $\alpha_i = \alpha'^\vee_i$ and $\alpha^\vee_i = \alpha'_i$ (regarded as a linear functional on X via $\langle\cdot\,,\cdot\rangle$). Put $\alpha_0 = \delta - \theta$ and $\alpha^\vee_0 = -\phi'$. The subgroup $W_0 \subseteq W_a$ acts via its original action on $Y^\vee$, fixing δ. The subgroup $Q'_0 \subseteq W_a$ acts by *translations*, given by the formula

$$y^{\beta'}(\mu^\vee) = \mu^\vee - \langle \mu^\vee, \beta'\rangle\delta, \tag{10}$$

One checks that the element $y^{\phi'} s_{\phi'} \in W_a$ acts as the simple reflection s_0, identifying W_a with the Weyl group W of X.

For Y of type Z_n ($Z = A, B, \ldots, G$), the affine root system X just constructed is of untwisted type $\widetilde{Z}_n$, with nullroot δ. In this case the affine roots are

$$R = R'^\vee_0 + \mathbb{Z}\delta, \tag{11}$$

and the positive roots are $R_+ = (R'^\vee_0 + \mathbb{Z}_{>0}\delta) \cup (R'^\vee_0)_+$.

3.7. Let $\big(X, (\alpha_0, \ldots, \alpha_n), (\alpha^\vee_0, \ldots, \alpha^\vee_n)\big)$ be any affine root system, W its Weyl group. Let Q_0, W_0 be the root lattice and Weyl group of the finite root system $\big(X, (\alpha_1, \ldots, \alpha_n), (\alpha^\vee_1, \ldots, \alpha^\vee_n)\big)$. If X is of untwisted type, we have just seen that $W \cong Q'_0 \rtimes W_0$, where $Q'_0 = Q^\vee_0$. If $X^\vee$ is of untwisted type, then $W \cong W(X^\vee) \cong Q'_0 \rtimes W_0$, where $Q'_0 = (Q^\vee_0)^\vee = Q_0$. If X is of mixed type, its Weyl group is of type $\widetilde{C}_n$, so $W \cong Q'_0 \rtimes W_0$ where Q'_0 is of type C_n, hence $Q'_0 = Q^\vee_0$ for $\widetilde{BC}_n$, and $Q'_0 = Q_0$ for $\widetilde{BC}^\vee_n$.

3.8. Twisted affine root systems can also be constructed in the manner of §3.6, by taking θ to be any dominant coroot of Y or of a non-reduced finite root system containing Y. This yields dual untwisted types when θ is short, and mixed types when θ is one-half of a long coroot or twice a short coroot. However, when $\theta \neq \phi'^\vee$, we no longer have $W = Q'_0 \rtimes W_0$.

3.9. We now return to the situation of §3.5, fixing the finite root system Y and untwisted affine root system $X = Y^\vee \oplus \mathbb{Z}\delta/m$ in what follows. The affine Weyl group $W_e = Y \rtimes W_0$ of Y is called the *extended affine Weyl group*. The action of Q_0' on X given by (10) extends to an action of Y, hence the action of $W_a = Q_0' \rtimes W_0$ extends to W_e. By (11), W_e preserves the set of affine roots R.

3.10. The further properties of W_a and W_e are best understood in terms of the following "alcove picture." Let $H = \{x \in X_{\mathbb{R}}^\vee : \langle \delta, x\rangle = 1\}$ be the *level 1 plane*, and let $\Lambda_0^\vee \in H$ be the linear functional $\Lambda_0^\vee(Y^\vee) = 0$, $\langle \delta, \Lambda_0^\vee\rangle = 1$. The group W_e fixes δ, hence acts on H. The translations $Y \subset W_e$ act on H by

$$y^\lambda(\mu) = \mu + \lambda, \tag{12}$$

and the finite Weyl group W_0 is generated by reflections fixing $\Lambda_0^\vee$. In particular, the map $y^\lambda \mapsto \Lambda_0^\vee + \lambda$ identifies $Y \cong W_e/W_0$ with the orbit $W_e(\Lambda_0^\vee) \subset H$, equivariantly with respect to the original action of W_0 on Y, and the action of $Q_0' \subseteq Y$ by translations.

Each affine root $\alpha \in R$ induces an affine-linear functional $\alpha(x) = \langle \alpha, x\rangle$ on H. Its zero set $h_\alpha = \{x \in H : \alpha(x) = 0\}$ is an affine hyperplane in H, and $s_\alpha \in W = W_a$ fixes h_α pointwise. The space H is tessellated by *affine alcoves* bounded by the root hyperplanes h_α. We distinguish the *dominant alcove* $A_0 = H \cap (\mathbb{R}_+ X_+^\vee) = \{x \in H : \alpha(x) \geq 0 \text{ for all } \alpha \in R_+\}$.

The alcove A_0 is a fundamental domain for the action of W_a on H. Its walls are the root hyperplanes h_{α_i} for the simple affine roots $\alpha_0, \ldots, \alpha_n$. Let $\Pi \subseteq W_e$ be the stabilizer of A_0, or equivalently, $\Pi = \{\pi \in W_e : \pi(R_+) = R_+\}$. Since Π preserves the set of simple roots, it normalizes the subgroup $W_a \subseteq W_e$ and the set of Coxeter generators $S = \{s_0, \ldots, s_n\} \subseteq W_a$. The following are immediate.

Corollary 3.11. *With the notation above, we have $W_e = \Pi \ltimes W_a$. Moreover, Π is the normalizer in W_e of the set of Coxeter generators $S = \{s_0, \ldots, s_n\}$.*

Corollary 3.12. *The canonical homomorphism $Y \subset W_e \to W_e/W_a = \Pi$ induces an isomorphism $Y/Q_0' \cong \Pi$. In particular, Π is abelian.*

To make this explicit, write $\pi \in \Pi$ uniquely as

$$\pi = y^{\lambda_\pi} \cdot v_\pi \in Y \rtimes W_0. \tag{13}$$

Then π maps to the coset of λ_π in Y/Q_0'. In the notation of §3.10, we have $\Lambda_0^\vee + \lambda_\pi = y^{\lambda_\pi}(\Lambda_0^\vee) = \pi(\Lambda_0^\vee) \in A_0$. Equivalently, $\lambda_\pi \in Y$ is a dominant weight such that $\langle \lambda, \phi'^\vee\rangle \leq 1$, or *minuscule weight*. Conversely, if $\lambda \in Y$ is minuscule, there is a unique $\pi \in \Pi$ such that $y^{\lambda - \lambda_\pi} \in W_a$. Then $\lambda = \lambda_\pi$, because both weights are minuscule and A_0 is a fundamental domain for W_a. The minuscule weights λ_π (including $\lambda_1 = 0$) are thereby in bijection with Π.

3.13. The distinguished elements

$$y^{\phi'} = s_0 s_{\phi'}, \qquad y^{\lambda_\pi} = \pi v_\pi^{-1}, \tag{14}$$

where ϕ' is the dominant short root and λ_π are the minuscule weights, are characterized as the unique translations such that $s_0 \in y^{\phi'} W_0$, $\pi \in y^{\lambda_\pi} W_0$, consistent with our having written $W_e = Y \rtimes W_0$. If we write $W_e = W_0 \ltimes Y$, we instead distinguish the translations

$$y^{-\phi'} = s_{\phi'} s_0, \qquad y^{-\lambda_\pi} = v_\pi \pi^{-1} \tag{15}$$

corresponding to the *anti-dominant* short root and the "anti-minuscule" weights. Of course (14) and (15) are equivalent, but the corresponding formulas for the left and right affine braid groups will not be (see Theorem 4.2, Corollary 4.3).

4. Double affine braid groups

4.1. Let $W_e = Y \rtimes W_0 = \Pi \ltimes W_a$ be an extended affine Weyl group (§3.9–3.13). By Corollary 3.11, Π acts on W_a by Coxeter group automorphisms. Hence Π also acts on $\mathcal{B}(W_a)$, and we can form the *extended affine braid group* $\mathcal{B}(W_e) = \Pi \ltimes \mathcal{B}(W_a)$.

Define the length function on $W_e = \Pi \ltimes W_a$ by $l(\pi w) = l(w)$. Note that $l(w\pi) = l(\pi w^\pi) = l(w^\pi) = l(w)$. The length of $v = \pi w$ is again equal to $|R_+ \cap v^{-1}(-R_+)|$, or to the number of affine hyperplanes h_α separating $v(A_0)$ from A_0 in the alcove picture (§3.10). Identity (3) continues to hold in $\mathcal{B}(W_e)$.

The counterpart to Corollary 3.11 is the following theorem of Bernstein (see [9, (4.4)]).

Theorem 4.2. *The identification $\Pi \ltimes W_a = Y \rtimes W_0$ lifts to an isomorphism $\mathcal{B}(W_e) \cong \mathcal{B}(Y, W_0)$ between the extended affine braid group defined above, and the (right) affine braid group (§2.9) of the finite root system Y. The isomorphism is the identity on $\mathcal{B}(W_0)$ and given on the remaining generators by $y^{\phi'} \leftrightarrow T_0 T_{s_{\phi'}}$, $y^{\lambda_\pi} \leftrightarrow \pi T_{v_\pi^{-1}}$, in the notation of §3.5 and* (13).

We describe the restriction of the isomorphism to $Y \subseteq \mathcal{B}(Y, W_0)$ more explicitly. If $\lambda, \mu \in Y_+$ are dominant, the alcove picture shows that $l(y^{\lambda+\mu}) = l(y^\lambda) + l(y^\mu)$. Hence $T_{y^{\lambda+\mu}} = T_{y^\lambda} T_{y^\mu}$ in $\mathcal{B}(W_e)$. It follows that there is a well-defined group homomorphism $\phi \colon Y \to \mathcal{B}(W_e)$ such that $y^{\lambda-\mu} \mapsto T_{y^\lambda} T_{y^\mu}^{-1}$ for $\lambda, \mu \in Y_+$. In particular, this yields the formulas $y^{\phi'} \mapsto T_{y^{\phi'}} = T_0 T_{s_{\phi'}}$, $y^{\lambda_\pi} \mapsto T_{y^{\lambda_\pi}} = \pi T_{v_\pi^{-1}}$.

One verifies using the alcove picture that the elements $\phi(y^\lambda)$ and the generators T_i of $\mathcal{B}(W_0)$ satisfy the defining relations of $\mathcal{B}(Y, W_0)$. Hence ϕ extends to a homomorphism $\mathcal{B}(Y, W_0) \to \mathcal{B}(W_e)$. Next one verifies (with the help of Lemma 4.20, below) that the element $y^{\phi'} T_{s_{\phi'}}^{-1} \in \mathcal{B}(Q'_0, W_0)$ satisfies braid relations with the generators T_i, giving a homomorphism $\mathcal{B}(W_a) \to \mathcal{B}(Q'_0, W_0)$ inverse to ϕ. Hence ϕ maps $\mathcal{B}(Q'_0, W_0)$ isomorphically onto $\mathcal{B}(W_a)$, and by Theorem 2.11, it follows that ϕ is an isomorphism.

Corollary 4.3. *For a (left) extended affine Weyl group $W_a \rtimes \Pi = W_0 \ltimes X$, there is an isomorphism $\mathcal{B}(W_e) \cong \mathcal{B}(W_0, X)$ between the extended affine braid group and the left affine braid group of X, which is the identity on $\mathcal{B}(W_0)$, and satisfies $x^{-\phi} \leftrightarrow T_{s_\phi} T_0$, $x^{-\lambda_\pi} \leftrightarrow T_{v_\pi} \pi^{-1}$.*

4.4. We come now to the key construction in the theory. Fix two finite root systems $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$, $\big(Y, (\alpha_i'), (\alpha_i'^\vee)\big)$ with the same Weyl group W_0. More accurately, assume given an isomorphism of Coxeter groups $W_0 = (W(X), S) \cong (W(Y), S')$, and label the simple roots so that s_i corresponds to s_i' for each $i = 1, \ldots, n$.

Let $\phi \in Q_0 \subseteq X$, $\phi' \in Q_0' \subseteq Y$ be the dominant short roots. Let $\theta \in Q_0$, $\theta' \in Q_0'$ be the dominant roots such that $s_\theta = s_{\phi'}$, $s_{\theta'} = s_\phi$. There are unique W_0-equivariant pairings $(X, Q_0') \to \mathbb{Z}$, $(Q_0, Y)' \to \mathbb{Z}$ such that $(\beta, \phi') = \langle \beta, \theta^\vee \rangle$ for all $\beta \in X$ and $(\phi, \beta')' = \langle \beta', \theta'^\vee \rangle$ for all $\beta' \in Y$. One checks that $(\phi, \phi') = (\phi, \phi')' = 2$ if $s_\phi = s_{\phi'}$, and $(\phi, \phi') = (\phi, \phi')' = 1$ if $s_\phi \neq s_{\phi'}$. By W_0-equivariance, the two pairings therefore agree on $Q_0 \times Q_0'$. Fix a W_0-invariant pairing $(X, Y) \to \mathbb{Q}$ extending the two pairings $(\cdot\,,\cdot)$ and $(\cdot\,,\cdot)'$, and choose m such that $(X, Y) \subseteq \mathbb{Z}/m$.

Remark 4.5. The Cartan matrices of X and Y are clearly either of the same type (Z_n, Z_n), or of dual types $(Z_n, Z_n^\vee)$. In the symmetric case (Z_n, Z_n), the roots $\theta = \phi$, $\theta' = \phi'$ are short, and the pairing $(\cdot\,,\cdot)$ restricts on $Q_0 = Q_0'$ to the W_0-equivariant pairing such that $(\alpha, \alpha) = 2$ for short roots α. In the dual case $(Z_n, Z_n^\vee)$, θ and θ' are long, and the pairing restricts to the canonical pairing between Q_0 and $Q_0' = Q_0^\vee$. Types G_2 and F_4 are isomorphic to their duals, but only after relabelling the simple roots. Thus there is a genuine difference between types (G_2, G_2) and $(G_2, G_2^\vee)$, for instance. In particular, $\theta = \phi$ in the first case, and $\theta \neq \phi$ in the second.

4.6. Given the data in §4.4, set $\widetilde{X} = X \oplus \mathbb{Z}\delta/m$, $\widetilde{Y} = Y \oplus \mathbb{Z}\delta'/m$. Extend the linear functionals $\alpha_i^\vee$ on X to $\widetilde{X}$ so that $\langle \delta, \alpha_i^\vee \rangle = 0$. Define $\alpha_0 = \delta - \theta$, and let $\alpha_0^\vee$ be the extension of $-\theta^\vee$ such that $\langle \delta, \alpha_0^\vee \rangle = 0$. Making similar definitions in $\widetilde{Y}$, we get two affine root systems

$$\big(\widetilde{X}, (\alpha_0, \ldots, \alpha_n), (\alpha_0^\vee, \ldots, \alpha_n^\vee)\big), \quad \big(\widetilde{Y}, (\alpha_0', \ldots, \alpha_n'), (\alpha_0'^\vee, \ldots, \alpha_n'^\vee)\big).$$

Let Y act on $\widetilde{X}$ and X on $\widetilde{Y}$ by

$$y^\lambda(\mu) = \mu - (\mu, \lambda)\delta, \quad x^\mu(\lambda) = \lambda - (\mu, \lambda)\delta'.$$

Since $(\cdot\,,\cdot)$ is W_0-invariant, this extends to actions of the extended affine Weyl groups

$$W_e = Y \rtimes W_0, \quad W_e' = W_0 \ltimes X$$

on $\widetilde{X}$ and $\widetilde{Y}$, respectively. The semidirect products $W_e \ltimes \widetilde{X}$, $\widetilde{Y} \rtimes W_e'$ are the (left, right) *extended double affine Weyl groups*. We have the following easy counterpart of Corollary 3.11.

Corollary 4.7. *There is a canonical isomorphism $W_e \ltimes \widetilde{X} \cong \widetilde{Y} \rtimes W_e'$, which is the identity on X, Y and W_0, and maps $q = x^\delta$ to $y^{-\delta'}$. In fact, both groups are identified with $W_0 \ltimes (X \star Y)$, where $X \star Y$ is the Heisenberg group generated by X, Y and central element $q^{1/m}$, with relations*

$$x^\mu y^\lambda = q^{(\mu,\lambda)} y^\lambda x^\mu.$$

Remarks 4.8. (a) For consistency, set $q = y^{-\delta'}$ in the "right" double affine Weyl group $\widetilde{Y} \rtimes W'_e$. Then the isomorphism maps q to q.

(b) When X and Y are of dual types, the affine root systems $\widetilde{X}$, $\widetilde{Y}$ are of untwisted type (§3.6). When X and Y are of the same type, then $\widetilde{X}$, $\widetilde{Y}$ are of dual untwisted type (§3.8).

(c) The requirement that $(\cdot\,,\cdot)$ extend the pairings $(X, Q'_0) \to \mathbb{Z}$ and $(Q_0, Y')' \to \mathbb{Z}$ in §4.4 ensures that

$$W_e \ni y^{\phi'} s_{\phi'} = s_0 \in W(\widetilde{X}), \quad W'_e \ni s_\phi x^{-\phi} = s'_0 \in W(\widetilde{Y}).$$

Under the action of $W_e = Y \rtimes W_0 = \Pi \ltimes W_a$ on $\widetilde{X}$, the subgroup $W_a = Q'_0 \rtimes W_0$ is therefore identified with the Weyl group of $\widetilde{X}$. By Corollary 3.11, $\Pi \subset W_e$ acts on $\widetilde{X}$ by automorphisms of the root system, *i.e.* it permutes the affine simple roots and coroots. So W_e acts on $\widetilde{X}$ as the semi-direct product of the Weyl group W_a and the group of automorphisms Π. In particular, the extended double affine Weyl group $W_e \ltimes \widetilde{X}$ is the semidirect product

$$\Pi \ltimes (W_a \ltimes \widetilde{X})$$

of Π with the affine Weyl group (§2.8) of the affine root system $\widetilde{X}$. Similar remarks apply to $\widetilde{Y} \rtimes W'_e$.

4.9. Since Π acts by automorphisms of the affine root system $\widetilde{X}$, it also acts naturally on $\mathcal{B}(W_a, \widetilde{X})$ (§2.9), and we can form the semidirect product $\Pi \ltimes \mathcal{B}(W_a, \widetilde{X})$, which we may regard as an extended (left) affine braid group $\mathcal{B}(W_e, \widetilde{X})$ of the affine root system $\widetilde{X}$. Similarly, we can define $\mathcal{B}(\widetilde{Y}, W'_e) = \mathcal{B}(\widetilde{Y}, W'_a) \rtimes \Pi'$. Define $q = x^\delta$ in $\mathcal{B}(W_e, \widetilde{X})$, and $q = y^{-\delta'} \in \mathcal{B}(\widetilde{Y}, W_e)$, as in Remark 4.8(a). We come now to the fundamental theorem.

Theorem 4.10. *The isomorphism $W_e \ltimes \widetilde{X} \cong \widetilde{Y} \rtimes W'_e$ lifts to an isomorphism $\mathcal{B}(W_e, \widetilde{X}) \cong \mathcal{B}(\widetilde{Y}, W'_e)$, which is the identity on X, Y, and $\mathcal{B}(W_0)$, and maps $q = x^\delta$ to $q = y^{-\delta'}$. (Here X, Y are identified with their images under $\mathcal{B}(W_0, X) \cong \mathcal{B}(W'_e) \to \mathcal{B}(\widetilde{Y}, W'_e)$ and $\mathcal{B}(Y, W_0) \cong \mathcal{B}(W_e) \to \mathcal{B}(W_e, \widetilde{X})$, using Theorem 4.2 and Corollary 4.3)*

The group $\mathcal{B}(W_e, \widetilde{X}) = \mathcal{B}(\widetilde{Y}, W'_e)$ is the (extended) *double affine braid group.*

4.11. By §2.9, there is an isomorphism $\Phi\colon \mathcal{B}(\widetilde{Y}, W'_e) \to \mathcal{B}(W'_e, \widetilde{Y})$ given by

$$\Phi(y^\lambda) = y^\lambda \qquad \Phi(T'_0) = T_0'^{-1}$$
$$\Phi(\pi) = \pi' \quad (\pi' \in \Pi') \qquad \Phi(T_i) = T_i^{-1} \quad (i = 1, \dots, n).$$

The element $T_0 \in \mathcal{B}(\widetilde{Y}, W'_e)$ is defined by $T_0 = y^{\phi'} T^{-1}_{s_{\phi'}}$, whereas $T_0 \in \mathcal{B}(W'_e, \widetilde{Y})$ is defined by $T_0 = T^{-1}_{s_{\phi'}} y^{-\phi'}$. Similarly, $\Pi \to \mathcal{B}(\widetilde{Y}, W'_e)$ is given by $\pi = y^{\lambda_\pi} v_\pi \mapsto y^{\lambda_\pi} T^{-1}_{v_\pi^{-1}}$, whereas $\Pi \to \mathcal{B}(W'_e, \widetilde{Y})$ is given by $\pi^{-1} \mapsto T^{-1}_{v_\pi} y^{-\lambda_\pi}$, hence $\pi \mapsto y^{\lambda_\pi} T_{v_\pi}$. Moreover, X is embedded in $\mathcal{B}(\widetilde{Y}, W'_e)$ via the identification $\mathcal{B}(W'_e) = \Pi' \ltimes \mathcal{B}(W_0, X)$, which is characterized by $x^{-\phi} \mapsto T_{s_\phi} T_0$ and $x^{-\lambda_{\pi'}} \mapsto T_{v_{\pi'}} \pi'^{-1}$, whereas $X \subset \mathcal{B}(W'_e, \widetilde{Y})$ is given via $\mathcal{B}(W'_e) = \mathcal{B}(X, W_0) \rtimes \Pi'$ by $x^\phi \mapsto T_0 T_{s_\phi}$, $x^{\lambda_{\pi'}} \mapsto \pi' T_{v_{\pi'}^{-1}}$. In $\mathcal{B}(W'_e, \widetilde{Y})$, finally, q denotes $y^{\delta'}$. Taking into account that $\Phi(T_w) = T^{-1}_{w^{-1}}$ for all $w \in W_0$, all this implies

$$\Phi(x^\mu) = x^\mu \qquad\qquad \Phi(T_0) = T_0^{-1}$$
$$\Phi(\pi) = \pi \quad (\pi \in \Pi) \qquad\qquad \Phi(q) = q^{-1}.$$

Theorem 4.10 therefore has the following equivalent alternate formulation.

Corollary 4.12. *There is an isomorphism $\mathcal{B}(W_e, \widetilde{X}) \cong \mathcal{B}(W'_e, \widetilde{Y})$, which is the identity on X, Y, Π and Π', maps $q = x^\delta$ to $q^{-1} = y^{-\delta'}$, and maps the generators T_i of $\mathcal{B}(W_0)$ to T_i^{-1}.*

4.13. Cherednik [1] announced Theorem 4.10 in the case $X = Y$, and suggested a possible topological proof, which was completed by Ion [7]. Macdonald [16, 3.5–3.7] gave an elementary proof, which however involves quite a bit of case-checking and only applies when $X = \mathrm{Hom}(Q_0^\vee, \mathbb{Z})$, $Y = \mathrm{Hom}(Q_0'^\vee, \mathbb{Z})$. We now outline a different elementary proof. First assume that the theorem holds in the "unextended" case, $X = Q_0$, $Y = Q'_0$, $W_e = W_a$, $W'_e = W'_a$. We will deduce the general case.

By Theorem 2.11, $\mathcal{B}(\widetilde{Q}'_0, W'_a)$ embeds in $\mathcal{B}(\widetilde{Y}, W'_a)$ as a normal subgroup, with quotient $\widetilde{Y}/\widetilde{Q}'_0 = Y/Q'_0 \cong \Pi$. Moreover, $\Pi \subseteq \mathcal{B}(W_e) = \mathcal{B}(Y, W_0)$ is a subgroup of $\mathcal{B}(\widetilde{Y}, W'_a)$, giving the semidirect decomposition $\mathcal{B}(\widetilde{Y}, W'_a) \cong \Pi \ltimes \mathcal{B}(\widetilde{Q}'_0, W'_a)$. By assumption, we have $\mathcal{B}(W_a, \widetilde{Q}_0) \cong \mathcal{B}(\widetilde{Q}'_0, W'_a)$, hence $\mathcal{B}(\widetilde{Y}, W'_a) \cong \Pi \ltimes \mathcal{B}(W_a, \widetilde{Q}_0) = \mathcal{B}(W_e, Q_0)$. This establishes the case where $X = Q_0$ and Y is general. Exchanging X and Y, we also get the case $Y = Q'_0$, $W_e = W_a$, where now X and W'_e are general.

By definition, $\mathcal{B}(\widetilde{Q}'_0, W'_e) = \mathcal{B}(\widetilde{Q}'_0, W'_a) \rtimes \Pi'$ and $\mathcal{B}(\widetilde{Y}, W'_e) = \mathcal{B}(\widetilde{Y}, W'_a) \rtimes \Pi'$, with $\Pi' \cong X/Q_0$ the same for both groups. Again, Theorem 2.11 implies that the first group is a normal subgroup of the second, with quotient Π. So we can repeat the preceding argument to get the general case.

4.14. Now fix $X = Q_0$, $Y = Q'_0$, so $W_e = W_a$, $W'_e = W'_a$. Using Theorem 4.2 and Corollary 4.3, we identify $\mathcal{B}(W'_a) = \mathcal{B}(W_0, X)$, $\mathcal{B}(W_a) = \mathcal{B}(Y, W_0)$. Then each group $\mathcal{B}(W_a, \widetilde{X})$, $\mathcal{B}(\widetilde{Y}, W'_a)$ has generators T_0, T'_0, $T_1, \ldots, T_n$, $q^{1/m}$. In both groups, $q^{1/m}$ is central, the generators $T_0, T_1, \ldots, T_n$ satisfy the braid relations of $\mathcal{B}(W_a)$, and $T'_0, T_1, \ldots, T_n$ satisfy those of $\mathcal{B}(W'_a)$.

The additional relations (7) for $\lambda \in Q_0$ and $i = 0$ complete a presentation of $\mathcal{B}(W_a, \widetilde{X})$, since those for $i \neq 0$ already hold in $\mathcal{B}(W_0, X) = \mathcal{B}(W'_a)$. For convenience, we write down these extra relations again here, after applying the identity $\langle \lambda, \alpha_0^\vee \rangle = -\langle \lambda, \theta^\vee \rangle$:

$$T_0^a x^\lambda T_0^b = x^{s_0(\lambda)}, \quad \text{where } a, b \in \{\pm 1\} \text{ and } -\langle \lambda, \theta^\vee \rangle = (a+b)/2. \tag{16}$$

In view of Corollary 4.12, to prove the theorem it suffices to express (16) in a "self-dual" form, in the sense that the substitutions $T_0 \leftrightarrow T_0'^{-1}$, $T_i \leftrightarrow T_i^{-1}$, $q \leftrightarrow q^{-1}$ ($i \neq 0$) should transform (16) into its counterpart with the roles of X and Y interchanged.

Lemma 4.15. *Relations* (16) *reduce to the case when λ is a short positive root $\alpha \neq \theta$ (i.e., $\alpha \neq \phi$ if $\theta = \phi$ is short).*

Proof. The short roots $\beta \neq \pm\theta$ span Q_0. Hence we can always write $\lambda = \beta_1 + \cdots + \beta_m$, where $\beta_i \in (R_0)_{\text{short}} \setminus \{\pm\theta\}$. In particular, $\langle \beta_i, \theta^\vee \rangle \in \{0, \pm 1\}$ for all i. Given that $\langle \lambda, \theta^\vee \rangle \in \{0, \pm 1\}$, we can always order the β_i so that those with $\langle \beta_i, \theta^\vee \rangle = 1$ and those with $\langle \beta_i, \theta^\vee \rangle = -1$ alternate. Writing (16) in the form $T_0^a x^\lambda = x^{s_0(\lambda)} T_0^{-b}$, it is easy to see that it follows from the same relation for each β_i. This reduces us to the case that $\alpha \neq \pm\theta$ is a short root. The case of (16) for $\langle \lambda, \theta^\vee \rangle = 1$ implies the case for $\langle \lambda, \theta^\vee \rangle = -1$, so positive roots α suffice. $\square$

4.16. A *parabolic subgroup* of W_0 is a subgroup of the form $W_J = \langle s_i : i \in J \rangle$, where $J \subseteq \{1, \ldots, n\}$. Since ϕ and ϕ' are dominant, their stabilizers are parabolic subgroups W_J, $W_{J'}$ respectively, where $J = \{i : \langle \phi, \alpha_i^\vee \rangle = 0\}$, and $J' = \{i : \langle \phi', \alpha_i'^\vee \rangle = 0\}$. Recall that each left, right and double coset vW_J, $W_{J'}v$, $W_{J'}vW_J$ has a unique representative of minimal length, which is also minimal in the Bruhat order.

Proposition 4.17. *Relations* (16) *for $\lambda = \alpha \neq \theta$ a short positive root reduce to the following.*

(a) For v such that $(v(\phi), \phi') = 0$ and v minimal in $W_{J'}vW_J$, the relation

$$T_0\, T_v T_0'^{-1} T_v^{-1} = T_v T_0'^{-1} T_v^{-1}\, T_0.$$

(b) For $v = v_1$ such that $(v(\phi), \phi') = 1$ and v minimal in $W_{J'}vW_J$, define v_2, v_3, v_4 minimal respectively in $W_{J'}vs_\phi W_J$, $W_{J'}s_\theta vs_\phi W_J$, $W_{J'}s_\theta vW_J$; this given, the relation

$$T_0^{-1} T_{v_1} T_0'^{-1} T_{v_2}^{-1} T_0^{-1} T_{v_3} T_0'^{-1} T_{v_4}^{-1} = q.$$

Proof. We can always write $\alpha = v(\phi)$ with v minimal in vW_J. If $i \in J'$, then T_i commutes with T_0. In $\mathcal{B}(W_a') = \mathcal{B}(W_0, Q_0)$ we have $x^{s_i(\alpha)} = T_i^\epsilon x^\alpha T_i^{\epsilon'}$, $\epsilon, \epsilon' = \pm 1$ for every positive short root α. These facts imply that relations (16) are invariant under replacement of α with $w(\alpha) \in W_{J'}\alpha$. Hence we can assume v minimal in $W_{J'}vW_J$.

We show that when $\langle \alpha, \theta^\vee \rangle = (v(\phi), \phi') = 0$, relation (16), which in this case reads $T_0 x^\alpha = x^\alpha T_0$, is equivalent to (a). The minimality of v in vW_J implies that if $v = s_{i_1} \cdots s_{i_l}$ is a reduced factorization, then $\langle s_{i_{k+1}} \cdots s_{i_l}(\phi), \alpha_{i_k}^\vee \rangle = 1$ for all k. Hence $x^\alpha = T_v x^\phi T_{v^{-1}} = T_v T_0'^{-1} T_{s_\phi}^{-1} T_{v^{-1}}$. The minimality also implies that $s_\phi = v^{-1} s_\alpha v$ is a reduced factorization. Therefore $T_{s_\phi}^{-1} T_{v^{-1}} = T_v^{-1} T_{s_\alpha}^{-1}$, and

$x^\alpha = T_v T_0'^{-1} T_v^{-1} T_{s_\alpha}^{-1}$. Now, since $\langle \alpha, \theta^\vee \rangle = 0$, we have $s_0 s_\alpha = s_\alpha s_0$, and both sides of this equation are reduced factorizations. Hence T_0 commutes with T_{s_α}, so (16) is equivalent to T_0 commuting with $T_v T_0'^{-1} T_v^{-1}$.

For $\langle \alpha, \theta^\vee \rangle = (v(\phi), \phi') = 1$, we have $s_0(\alpha) = \alpha + \alpha_0 = \alpha - \theta + \delta$, and thus relation (16) in this case reads $T_0^{-1} x^\alpha T_0^{-1} = q x^{-\beta}$, or $T_0^{-1} x^\alpha T_0^{-1} x^\beta = q$, where $\beta = -s_\theta(\alpha)$ satisfies $\alpha + \beta = \theta$. Let u be the minimal representative of $s_\theta v s_\phi W_J$. Then $\beta = u(\phi)$, and the same reasoning as in the previous paragraph gives $x^\alpha = T_v T_0'^{-1} T_{vs_\phi}^{-1}$, $x^\beta = T_u T_0'^{-1} T_{us_\phi}^{-1}$. Our relation now takes the form

$$T_0^{-1} T_v T_0'^{-1} T_{vs_\phi}^{-1} T_0^{-1} T_u T_0'^{-1} T_{us_\phi}^{-1} = q. \tag{17}$$

Using §2.6 and the fact that $s_\phi(\alpha_i) = \alpha_i$ for all $i \in J$, we deduce (for any J')

> (*) *if x, y are minimal in $W_{J'}x$, $W_{J'}y = W_{J'}xs_\phi$, respectively, and xw is minimal in $W_{J'}xW_J$, then yw is minimal in $W_{J'}yW_J$.*

By construction, u and v are minimal in their left W_J cosets, and (*) implies the same for us_ϕ and vs_ϕ. Hence the elements $v_1 = v$, v_2, v_3, v_4 defined in (b) are the minimal representatives of $W_{J'}v$, $W_{J'}vs_\phi$, $W_{J'}u$, $W_{J'}us_\phi$ respectively. By the analog of (*) for s_θ (operating on the left), we see that $v_1 = v$ implies $v_4 = us_\phi$, and if we set $v_2 = wvs_\phi$, then $v_3 = wu$. Now $w \in W_{J'}$ commutes with T_0, and the factorizations $vs_\phi = w^{-1}v_2$, $u = w^{-1}v_3$ are reduced, so (17) reduces to (b). □

Corollary 4.18. *The (unextended) double affine braid group $\mathcal{B}(W_a, \widetilde{Q}_0)$, where $\mathcal{B}(W_a) = \mathcal{B}(Q_0', W_0)$, has a presentation with generators T_0, T_0', $T_1, \ldots, T_n$, $q^{1/m}$ and the following (manifestly self-dual) relations: $q^{1/m}$ is central; braid relations for $T_0, T_1, \ldots, T_n \in \mathcal{B}(W_a)$ and for $T_0', T_1, \ldots, T_n \in \mathcal{B}(W_a')$; and the relations in Proposition 4.17.*

Example 4.19. Let $X = Y$ be of type A_{n-1}, with $\alpha_i = \alpha_i^\vee = e_i - e_{i+1}$ as in Example 2.5. Then $\phi = \theta = \phi' = \theta' = e_1 - e_n$, and $W_J = W_{J'} = \langle s_2, \ldots, s_{n-2} \rangle$. The presentation of $\mathcal{B}(W_a, \widetilde{Q}_0)$ is given by q central, braid relations and

(a) T_0 commutes with $T_1 T_{n-1} T_0'^{-1} (T_1 T_{n-1})^{-1}$,

(b) $T_0^{-1} T_1 T_0'^{-1} T_1^{-1} T_2^{-1} \cdots T_{n-1}^{-1} T_0^{-1} T_{n-1} T_0'^{-1} T_{n-1}^{-1} T_{n-2}^{-1} \cdots T_1^{-1} = q$.

There are seven double cosets $W_J v W_{J'}$. Two have $v(\phi) = \pm\phi$, one yields (a), and the other four provide the elements $v_1, \ldots, v_4$ in (b). In fact, in every type there turns out to be only one relation of type (b) and at most two of type (a), except for $\widetilde{D}_4$, which has three of type (a).

Lemma 4.20. *If ϕ is the dominant short root of a finite root system X, and $v \in W_0$ is such that $\alpha = v(\phi) \in (R_0)_+$, then in $\mathcal{B}(W_0, X)$ we have*

$$T_v x^\phi T_{s_\phi} T_v^{-1} = x^\alpha T_{s_\alpha}.$$

Proof. This reduces to the case that v is minimal in vW_J (in the notation of §4.16). As in the proof of Proposition 4.17 we then have $T_v x^\phi T_{s_\phi} = x^\alpha T_{v^{-1}}^{-1} T_{s_\phi} = x^\alpha T_{s_\alpha} T_v$. □

Lemma 4.20 will be used in the proof of Theorem 5.11. Its variant for $\mathcal{B}(Y, W_0)$ is $T_{v^{-1}}^{-1} y^\phi T_{s_\phi}^{-1} T_{v^{-1}} = y^\alpha T_{s_\alpha}^{-1}$, which is useful for verifying the braid relations in the proof of Theorem 4.2.

5. Hecke algebras and Cherednik algebras

5.1. Let $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$ be a non-degenerate root system, with Cartan matrix A, Weyl group W, and roots R. To each W-orbit in R we associate a parameter u_α, $u_\alpha = u_\beta$ if $\beta = w(\alpha)$. Set $u_i = u_{\alpha_i}$. The u_i are assumed to be invertible elements of some commutative ground ring $\mathfrak{A}$. If $\alpha_i^\vee \in 2X^\vee$, we also introduce a second parameter u_i'.

Lemma 5.2. *Let $\mathcal{H}$ be an $\mathfrak{A}$-algebra containing the group algebra $\mathfrak{A}X$, and $T_i \in \mathcal{H}$.*

(i) If $\alpha_i^\vee \notin 2X^\vee$, then commutation relations (5)–(6) *and the quadratic relation*

$$(T_i - u_i)(T_i + u_i^{-1}) = 0 \tag{18}$$

imply the more general commutation relations, for all $\lambda \in X$,

$$T_i x^\lambda - x^{s_i(\lambda)} T_i = \frac{(u_i - u_i^{-1})}{1 - x^{\alpha_i}} (x^\lambda - x^{s_i(\lambda)}). \tag{19}$$

(ii) If $\alpha_i^\vee \in 2X^\vee$, then (5)–(6), (18) *and the additional quadratic relation*

$$(T_i^{-1} x^{-\alpha_i} - u_i')(T_i^{-1} x^{-\alpha_i} + u_i'^{-1}) \tag{20}$$

imply

$$T_i x^\lambda - x^{s_i(\lambda)} T_i = \frac{(u_i - u_i^{-1}) + (u_i' - u_i'^{-1}) x^{\alpha_i}}{1 - x^{2\alpha_i}} (x^\lambda - x^{s_i(\lambda)}) \tag{21}$$

(iii) Given (18)*, relation* (21) *implies* (20)*, and* (19) *implies that* (20) *holds with $u_i' = u_i$.*

Note that the denominators in (19), (21) divide $x^\lambda - x^{s_i(\lambda_i)}$.

For the well-known proof, observe that each side of (19), (21), viewed as an operator on x^λ, satisfies $F(x^\lambda x^\mu) = F(x^\lambda) x^\mu + x^{s_i(\lambda)} F(x^\mu)$. Hence (19), (21) for x^λ, x^μ, imply the same for $x^{\lambda \pm \mu}$. This reduces (i) to the special cases $\langle \lambda, \alpha_i^\vee \rangle \in \{0, 1\}$, which in turn reduce to (5)–(6), using the identity $T_i^{-1} = T_i - u_i + u_i^{-1}$, which is equivalent to (18). Similarly, (ii) reduces to the special cases $\langle \lambda, \alpha_i^\vee \rangle = 0$, which is (5) ((6) is vacuous if $\alpha_i^\vee \in 2X^\vee$), and $\lambda = \alpha_i$ (since $\langle \alpha_i, \alpha_i^\vee \rangle = 2$). Modulo (18), this last case is equivalent to (20), which also gives (iii) in case (ii). For (iii) in case (i), observe that (19) is just (21) with $u_i' = u_i$.

Definition 5.3. The *affine Hecke algebra* $\mathcal{H}(W, X)$ is the quotient $(\mathfrak{A}\mathcal{B}(W, X))/\mathfrak{j}$, where $\mathfrak{j}$ is the 2-sided ideal generated by the quadratic relations (18) for all i, plus (20) for each i such that $\alpha_i^\vee \in 2X^\vee$.

Equivalently, $\mathcal{H}(W, X)$ is generated by elements x^λ ($\lambda \in X$) and T_i satisfying the braid relations of $\mathcal{B}(W)$, quadratic relations (18), and relations (19) or (21) depending on whether or not $\alpha_i^\vee \in 2X^\vee$.

Proposition 5.4. *The subalgebra of $\mathcal{H}(W,X)$ generated by the elements T_i is isomorphic to the ordinary Hecke algebra $\mathcal{H}(W)$, with basis $\{T_w : w \in W\}$, and $\mathcal{H}(W,X)$ has basis $\{T_w x^\lambda\}$.*

Proof. The commutation relations (19), (21) imply that the elements $T_w x^\lambda$ span; they are independent because the specialization $u_i = u_i' = 1$ collapses $\mathcal{H}(W,X)$ to the group algebra $\mathfrak{A}\cdot(W \ltimes X)$. (More precisely, specialization implies the result for $\mathfrak{A} = \mathbb{Z}[u_i^{\pm 1}, u_i'^{\pm 1}]$, and the general case follows by extension of scalars.) $\square$

5.5. Let Π be a group acting by automorphisms of the root system $\big(X, (\alpha_i), (\alpha_i^\vee)\big)$, and assume that $u_i = u_j$, $u_i' = u_j'$ for $\alpha_j \in \Pi(\alpha_i)$. Then Π acts on $\mathcal{H}(W,X)$, and we define the *extended affine Hecke algebra* to be the twisted group algebra $\Pi\cdot\mathcal{H}(W,X)$ generated by Π and $\mathcal{H}(W,X)$ with relations $\pi f = \pi(f)\pi$ for $\pi \in \Pi$, $f \in \mathcal{H}(W,X)$.

Up to now the root system X was arbitrary. If X is finite, with $W_0 \ltimes X = W_a \rtimes \Pi$ as in Corollary 4.3, then $\mathcal{H}(W_0, X)$ is isomorphic to the twisted group algebra $\mathcal{H}(W_a)\cdot\Pi$ of the ordinary Hecke algebra of W_a. The most interesting case is when X is affine; specifically when $X = \widetilde{X}$ as constructed in §4.6.

Definition 5.6. Given X, Y, $(\cdot\,,\cdot)$, $\widetilde{X}$, $\widetilde{Y}$, $W_e = \Pi \ltimes W_a$, $W_e' = W_a' \rtimes \Pi'$ as in §4.4–4.9, the (left) *Cherednik algebra* $\mathcal{H}(W_e, \widetilde{X})$ is the extended affine Hecke algebra $\Pi\cdot\mathcal{H}(W_a, \widetilde{X})$.

Equivalently, $\mathcal{H}(W_e, \widetilde{X})$ is generated by $x^\lambda \in X$, $\pi \in \Pi$, $T_0, \ldots, T_n$ and $q^{\pm 1/m}$, satisfying the relations of the double affine braid group $\mathcal{B}(W_e, \widetilde{X})$ and the quadratic relations (18), plus (20) if $\alpha_i^\vee \in 2\widetilde{X}^\vee$.

5.7. We will also define a *right* Cherednik algebra $\mathcal{H}(\widetilde{Y}, W_e')$, but first we must re-index the parameters. For convenience, we define $u_j' = u_j$ if $\alpha_j^\vee \notin 2\widetilde{X}^\vee$. Define $u_{i'} = u_i$ for $i \neq 0$, and set $u_{0'} = u_j'$, where α_j is a short simple root of the finite root system X. If $\alpha_i'^\vee \in 2\widetilde{Y}^\vee$ for $i \neq 0$ (there is at most one such index i), set $u_{i'}' = u_0$. If $\alpha_0'^\vee \in 2\widetilde{Y}^\vee$, set $u_{0'}' = u_0'$.

We now define $\mathcal{H}(\widetilde{Y}, W_e')$ to be the algebra with generators y^μ ($\mu \in Y$), $\pi' \in \Pi'$, $T_0', T_1, \ldots, T_n$, $q^{\pm 1/m}$ satisfying the relations of the right affine braid group $\mathcal{B}(\widetilde{Y}, W_e')$, relations (18) with $u_{i'}$ in place of u_i, and for $\alpha_i'^\vee \in 2\widetilde{Y}^\vee$, the relations

$$(T_i'^{-1} y^{\alpha_i'} - u_{i'}')(T_i'^{-1} y^{\alpha_i'} + u_{i'}'^{-1}), \tag{22}$$

where we define $T_i' = T_i$ if $i \neq 0$.

Corollary 5.8. *The elements $\{y^\mu T_w x^\lambda\}$ ($\mu \in Y$, $\lambda \in X$, $w \in W_0$) form an $\mathfrak{A}[q^{\pm 1/m}]$-basis of the Cherednik algebras $\mathcal{H}(W_e, \widetilde{X})$, $\mathcal{H}(\widetilde{Y}, W_e')$.*

This follows easily from Proposition 5.4 for $\mathcal{H}(W_e, \widetilde{X})$ and by symmetry for $\mathcal{H}(\widetilde{Y}, W_e')$. We remark that the factors $y^\mu T_w x^\lambda$ can be taken in any order.

Lemma 5.9. *We have $\alpha_0'^\vee \in 2\widetilde{Y}^\vee$ if and only if X, Y are both of type B_n and Π acts trivially on the simple roots of $\widetilde{X}$.*

Proof. By definition, $\alpha_0'^\vee = -\theta'^\vee$. We can only have $\theta'^\vee \in 2Y^\vee$ if Y is of type B_n and $\theta = \phi$ is short, hence X is also of type B_n. Let P_0' be the image of the canonical homomorphism $Y \to \mathrm{Hom}(Q_0'^\vee, \mathbb{Z})$. For type B_n we have either $Q_0' = P_0'$ or $P_0'/Q_0' \cong \mathbb{Z}/2\mathbb{Z}$, with $Q_0' = P_0'$ iff the short roots α' satisfy $\alpha'^\vee \in 2Y^\vee$. The isomorphism $\Pi \cong Y/Q_0'$ (Corollary 3.12) identifies P_0'/Q_0' with the quotient of Π by the kernel of its action on the simple roots of $\widetilde{X}$. □

Remark 5.10. If X, Y are of type B_n, then $\widetilde{X}$, $\widetilde{Y}$ are of type $\widetilde{C}_n^\vee$. Label the Dynkin diagram

$$\begin{array}{c} \bullet\!\Leftarrow\!\bullet\!-\!\bullet\!-\cdots-\!\bullet\!\Rightarrow\!\bullet \\ 0\quad 1\quad 2 \qquad\quad n{-}1\;\; n \end{array}\;. \tag{23}$$

If $\alpha_0'^\vee \notin 2\widetilde{Y}^\vee$, then Π acts non-trivially, exchanging nodes 0 and n, and similarly for $\alpha_0^\vee$ and Π'. The four associated parameters are related by the diagram

$$\begin{array}{ccc} (u_0' = u_{0'}') & = & (u_n' = u_{0'}) \\ \| & & \| \\ (u_0 = u_{n'}') & = & (u_n = u_{n'}) \end{array}\;, \tag{24}$$

where the horizontal equalities hold if $\alpha'^\vee \notin 2Y^\vee$ for short roots $\alpha' \in Y$, and the vertical ones hold if $\alpha^\vee \notin 2X^\vee$ for short roots $\alpha \in X$.

Theorem 5.11. *There is an isomorphism $\mathcal{H}(W_e, \widetilde{X}) \cong \mathcal{H}(\widetilde{Y}, W_e')$, which is the identity on all the generators X, Y, q, T_i, T_0, T_0', π, π'.*

Proof. For the most part, this is Theorem 4.10, but we must prove that relations (22) and the case of (18) for T_0' hold in $\mathcal{H}(W_e, \widetilde{X})$. By definition, $T_0' = T_{s_\phi}^{-1} x^{-\phi}$. By Lemma 4.20, this is conjugate to $T_j^{-1} x^{-\alpha_j}$ for a short simple root α_j. Then (20) for T_j implies (18) for T_0'. Similarly, if $i \neq 0$ in (22), then α_i' is short, and $T_i'^{-1} y^{\alpha_i}$ is conjugate to $y^{\alpha_i} T_i'^{-1}$ and in turn to $T_0 = y^{\phi'} T_{s_{\phi'}}^{-1}$. By Lemma 5.9, we only have $i = 0$ in (22) when X, Y are both of type B_n, so $\theta = \phi$, $\theta' = \phi'$. Then $T_0'^{-1} y^{\alpha_0'} = q^{-1} x^{\phi} T_{s_\phi} y^{-\phi'} = x^{-\alpha_0} T_0^{-1}$, which is conjugate to $T_0^{-1} x^{-\alpha_0}$. □

Corollary 5.12. *Assume given an automorphism $\varepsilon\colon \mathfrak{A} \to \mathfrak{A}$ such that $\varepsilon(u_i) = u_i^{-1}$, $\varepsilon(u_i') = u_i'^{-1}$. Then there is an ε-linear isomorphism $\mathcal{H}(W_e, \widetilde{X}) \cong \mathcal{H}(W_e', \widetilde{Y})$ which is the identity on X, Y, Π, Π', maps q to q^{-1}, and maps T_i to T_i^{-1} for all $i = 0', 0, 1, \ldots, n$, where the parameters $u_{i'}$, $u_{i'}'$ for $\mathcal{H}(W_e', \widetilde{Y})$ are as in §5.7.*

Proof. The map Φ in §4.11, composed with ε, preserves (18) and interchanges (22) with the version of (20) for $\widetilde{Y}$ in place of $\widetilde{X}$. □

5.13. Let $\mathcal{H} = \Pi \cdot \mathcal{H}(W, X)$ be an extended affine Hecke algebra. The ordinary (extended) Hecke algebra $\Pi \cdot \mathcal{H}(W)$ has a one-dimensional representation $\mathbf{1} = \mathfrak{A}e$ such that $\pi e = e$, $T_i e = u_i e$. The induced representation $\mathrm{Ind}_{\Pi\mathcal{H}(W)}^{\mathcal{H}}(\mathbf{1})$ is the *polynomial representation*. Proposition 5.4 implies that it is isomorphic to the left regular representation $\mathfrak{A}X$ of X, with Π acting via its action on X, and $T_0, \ldots, T_n$ acting as the operators

$$
\begin{aligned}
T_i &= u_i s_i + \frac{(u_i - u_i^{-1})}{1 - x^{\alpha_i}}(1 - s_i) && (25)\\
&= u_i - u_i^{-1}\frac{1 - u_i^2 x^{\alpha_i}}{1 - x^{\alpha_i}}(1 - s_i) && (26)\\
&= -u_i^{-1} + u_i(1 + s_i)\frac{1 - u_i^{-2} x^{\alpha_i}}{1 - x^{\alpha_i}} && (27)
\end{aligned}
$$

or, if $\alpha_i^\vee \in 2X^\vee$,

$$
\begin{aligned}
T_i &= u_i s_i + \frac{(u_i - u_i^{-1}) + (u'_i - u_i'^{-1})x^{\alpha_i}}{1 - x^{2\alpha_i}}(1 - s_i) && (28)\\
&= u_i - u_i^{-1}\frac{(1 - u_i u'_i x^{\alpha_i})(1 + (u_i/u'_i)x^{\alpha_i})}{1 - x^{2\alpha_i}}(1 - s_i) && (29)\\
&= -u_i^{-1} + u_i(1 + s_i)\frac{(1 - (u_i u'_i)^{-1} x^{\alpha_i})(1 + (u'_i/u_i)x^{\alpha_i})}{1 - x^{2\alpha_i}}. && (30)
\end{aligned}
$$

In particular, these operators satisfy braid relations. The quadratic relations can be seen directly from (26)–(27) and (29)–(30). The polynomial representation specializes at $u_i = u'_i = 1$ to the $\mathfrak{A}$-linearization of the action of $\Pi \ltimes (W \ltimes X)$ on X. It is faithful if Π acts faithfully.

5.14. For any root $\alpha \in R$, define a partial ordering on X by $\mu <_\alpha \lambda$ if $\lambda - \mu \in \mathbb{Z}\alpha$ and $|\langle \mu, \alpha^\vee \rangle| < |\langle \lambda, \alpha^\vee \rangle|$, or $\langle \mu, \alpha^\vee \rangle = -\langle \lambda, \alpha^\vee \rangle > 0$. Each *root string* $\lambda + \mathbb{Z}\alpha$ is totally ordered by $<_\alpha$. Explicitly,

$$
\begin{aligned}
&\lambda <_\alpha \lambda + \alpha <_\alpha \lambda - \alpha <_\alpha \lambda + 2\alpha <_\alpha \lambda - 2\alpha <_\alpha \cdots \quad \text{if } \langle \lambda, \alpha^\vee \rangle = 0\\
&\lambda <_\alpha \lambda - \alpha <_\alpha \lambda + \alpha <_\alpha \lambda - 2\alpha <_\alpha \lambda + 2\alpha <_\alpha \cdots \quad \text{if } \langle \lambda, \alpha^\vee \rangle = 1.
\end{aligned}
$$

If $B \subseteq R$, define $<_B$ to be the transitive closure of the union $\bigcup_{\alpha \in B} <_\alpha$. In general $<_B$ is not a partial order; we may have $\lambda <_B \lambda$.

Proposition 5.15. *Let $w \in W$, $B = R_+ \cap w^{-1}(-R_+)$. In the polynomial representation, we have*

$$
T_w(x^\lambda) = u^{\rho_B(\lambda)} x^{w(\lambda)} + \sum_{\mu <_{w(B)} w(\lambda)} a_\mu x^\mu,
$$

where $a_\mu \in \mathfrak{A}$ and $u^{\rho_B(\lambda)} = \prod_{\alpha \in B} u_\alpha^{\sigma(-\langle \lambda, \alpha^\vee \rangle)}$, $\sigma(k) = \pm 1$ as $k \geq 0$ or $k < 0$.

Proof. The case $w = s_i$ follows from formulas (25), (28), and the general case by induction on $l(w)$, using the fact that if $w = s_i v > v$ and $B' = R_+ \cap v^{-1}(-R_+)$, then $B = B' \cup \{v^{-1}(\alpha_i)\}$. $\square$

6. Macdonald polynomials

6.1. Let $\big(\widetilde{X},(\alpha_i),(\alpha_i^\vee)\big)$ be a non-degenerate reduced affine root system (§3). As always, we take $i=0$ to be an affine node, denote the Weyl group, roots, etc. by W, R, R_+, Q, Q_+, and let W_0, R_0, Q_0, etc. denote the same for the finite root system with simple roots $\alpha_1,\ldots,\alpha_n$. We also allow non-reduced affine root systems, regarded as extensions (§2.12) of a reduced affine root system $\widetilde{X}$, with a larger set of roots R. In the non-reduced case, we do not give the extra simple roots their own symbols, but designate them simply as $2\alpha_i$.

Let δ be the nullroot, and assume that the dual of $\widetilde{X}$ is degenerate, *i.e.*, $\delta^\vee=0$. Possibly after adjoining a fractional multiple of δ, we can always assume that $\widetilde{X}=X\oplus\mathbb{Z}\delta/m$, where $Q_0\subseteq X$. Fix such a decomposition.

To each i such that $2\alpha_i\notin R$, we associate a parameter u_i and put $t_i=u_i^2$. To each i such that $2\alpha_i\in R$ we associate two parameters u_i, u_i' and put $t_i=u_iu_i'$, $t_i'=u_i/u_i'$. We require that simple roots in the same W-orbit have the same parameters, and put $t_\alpha=t_{\alpha_i}$, $t_\alpha'=t_{\alpha_i}'$ if $\alpha\in W(\alpha_i)$. We denote by $\mathbb{Q}(t)$ the field of rational functions in the parameters. The group algebra $\mathbb{Q}(t)\widetilde{X}$ is the ring of Laurent polynomials $\mathbb{Q}(t)[x^{\pm\varepsilon_1},\ldots,x^{\pm\varepsilon_N}]$, where $\{\varepsilon_1,\ldots,\varepsilon_N\}$ is a basis of $\widetilde{X}$. As in §4, we let $q=x^\delta$. Then $\mathbb{Q}(t)\widetilde{X}=\mathbb{Q}(t)[q^{\pm1/m}]X$, and we identify it with a subring of $\mathbb{Q}(q,t)X$.

As in §3.7, let $W=Q_0'\rtimes W_0$, where $Q_0'=Q_0^\vee$ if $\widetilde{X}$ is of untwisted type or $\widetilde{BC}_n$, and $Q_0'=Q_0$ otherwise. In either case, Q_0' acts on $\mathbb{Q}(q,t)X$ by the formula

$$y^\mu(x^\lambda)=q^{-(\lambda,\mu)}x^\lambda, \tag{31}$$

in terms of the W_0-invariant pairing $(Q_0,Q_0')\to\mathbb{Z}$ in §4.4 (see also §4.6).

6.2. Let $\mathbb{Q}(q,t)X\widehat{\ }$ denote the $\mathbb{Q}(q,t)$-vector space of possibly infinite formal linear combinations $f=\sum_{\lambda\in X}a_\lambda x^\lambda$. The space $\mathbb{Q}(q,t)X\widehat{\ }$ is a $\mathbb{Q}(q,t)X$-module—*i.e.*, it makes sense to multiply $f\in\mathbb{Q}(q,t)X\widehat{\ }$ by $p\in\mathbb{Q}(q,t)X$. We regard $\mathbb{Q}(q,t)X$ as a submodule of $\mathbb{Q}(q,t)X\widehat{\ }$. Write

$$[x^\lambda]f=a_\lambda$$

for the coefficient of x^λ in f. Let $\overline{\cdot}$ denote the involution on $\mathbb{Q}(q,t)$ and $\mathbb{Q}(q,t)X$ such that

$$\overline{u_i}=u_i^{-1},\quad \overline{u_i'}=u_i'^{-1},\quad \overline{t_\alpha}=t_\alpha^{-1},\quad \overline{t_\alpha'}=t_\alpha'^{-1},\quad \overline{q}=q^{-1},\quad \overline{x^\lambda}=x^{-\lambda}.$$

It extends to $\mathbb{Q}(q,t)X\widehat{\ }$ by the rule $\overline{\sum_\lambda a_\lambda x^\lambda}=\sum_\lambda\overline{a_\lambda}x^{-\lambda}$. The following theorem is due to Cherednik.

Theorem 6.3. *There is a unique element $\Delta_0=\overline{\Delta_0}\in\mathbb{Q}(q,t)Q_0\widehat{\ }\subseteq\mathbb{Q}(q,t)X\widehat{\ }$ with constant term $[1]\Delta_0=1$, such that for each Coxeter generator s_i of W,*

$$s_i(\Delta_0)=\frac{1-t_ix^{\alpha_i}}{t_i-x^{\alpha_i}}\Delta_0,\quad \text{or}\quad s_i(\Delta_0)=\frac{(1-t_ix^{\alpha_i})(1+t_i'x^{\alpha_i})}{(t_i-x^{\alpha_i})(t_i'+x^{\alpha_i})}\Delta_0, \tag{32}$$

where the second formula applies if $2\alpha_i\in R$.

Proof. Define a formal series $\Delta \in \mathbb{Q}[\![q,t]\!]Q_0\widehat{\ }$ by

$$\Delta = \prod_{\substack{\alpha\in R_+\\ 2\alpha\notin R,\,\alpha\notin 2R}} \frac{1-x^\alpha}{1-t_\alpha x^\alpha} \prod_{\substack{\alpha\in R_+\\ 2\alpha\in R}} \frac{1-x^{2\alpha}}{(1-t_\alpha x^\alpha)(1+t'_\alpha x^\alpha)}.$$

The coefficients $[x^\lambda]\Delta \in \mathbb{Q}[\![q,t]\!]$ are not rational functions. Define $\Delta_0 = \Delta/([1]\Delta)$. Since s_i leaves the set $R_+\setminus\{\alpha_i, 2\alpha_i\}$ invariant, it follows that Δ and Δ_0 satisfy (32). By construction, Δ_0 has constant term 1. These conditions can be expressed as a system of linear equations over $\mathbb{Q}(q,t)$ in the coefficients $[x^\lambda]\Delta_0$, which therefore have a solution Δ_0' with coefficients in $\mathbb{Q}(q,t)$.

Now, Δ_0'/Δ_0 is W-invariant. For $0 \neq \lambda \in Q_0$, choose $\mu \in Q_0'$ such that $(\lambda,\mu) \neq 0$. Then (31) implies that $[x^\lambda](\Delta_0'/\Delta_0) = 0$, *i.e.*, Δ_0'/Δ_0 is a constant. Hence $\Delta_0' = \Delta_0$, since they both have constant term 1. This shows that Δ_0 has coefficients in $\mathbb{Q}(q,t)$ and is unique. One checks that (32) is $\overline{\cdot}$-invariant, which implies $\Delta_0 = \overline{\Delta_0}$ by uniqueness. □

The *Macdonald constant term identity* [16, (5.8.20)] provides an explicit infinite product expansion for $[1]\Delta$, but it is not practicable to compute the coefficients of Δ_0 directly from the formula $\Delta_0 = \Delta/([1]\Delta)$. A better procedure is to equate the coefficients of $y^{\phi'}(\Delta_0) = s_0 s_{\phi'}(\Delta_0)$, as given by (31) on the one hand, and by (32) on the other. This leads to a recurrence which determines the coefficients.

Definition 6.4. *Cherednik's inner product* on $\mathbb{Q}(q,t)X$ is defined by the formula

$$\langle f, g\rangle_0 = [1](f\,\overline{g}\,\Delta_0).$$

It is linear in f and $\overline{\cdot}$-hermitian by Theorem 6.3, *i.e.*, $\langle g, f\rangle_0 = \overline{\langle f, g\rangle_0}$.

Lemma 6.5. *Let $B = (R_0)_+$. Under the identification of X with the set W_e'/W_0 of minimal left coset representatives in $W_e' = W_0 \ltimes X$, the ordering $<_B$ defined in §5.14 coincides with the Bruhat order $<$ in W_e'.*

Proof. Let w_λ be minimal in $x^\lambda W_0$. If $s_\beta w_\lambda < w_\lambda$ for a reflection $s_\beta \in W_a'$, then clearly $w_{s_\beta(\lambda)} < w_\lambda$. The Bruhat order on W_e'/W_0 is the transitive closure of these relations. In the alcove picture (§3.10), s_β belongs to a root $\beta = \alpha^\vee + k\delta'$ of the affine root system $X^\vee \oplus \mathbb{Z}\delta'$, where we can assume that $\alpha \in (R_0)_+$. The condition $s_\beta w_\lambda < w_\lambda$ means that h_β separates $\Lambda_0^\vee + \lambda$ from the dominant alcove A_0. This is equivalent to $s_\beta(\lambda) <_\alpha \lambda$, and $<_B$ is by definition the transitive closure of these relations. □

We fix the partial ordering $<_B$ on X, with $B = (R_0)_+$, and denote it by $<$.

Theorem 6.6. *There is a unique basis $\{E_\lambda : \lambda \in X\}$ of $\mathbb{Q}(q,t)X$ satisfying the orthogonality and triangularity conditions*

$$\begin{aligned} &(i) && \langle E_\lambda, E_\mu\rangle_0 = 0 \text{ for } \lambda \neq \mu.\\ &(ii) && E_\lambda = x^\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} x^\mu, \quad c_{\lambda\mu} \in \mathbb{Q}(q,t). \end{aligned} \tag{33}$$

The E_λ are the (non-symmetric) *Macdonald polynomials*. Let us review how their existence and other properties are established using Cherednik algebras.

6.7. If $\widetilde{X}$ is of untwisted or dual untwisted type, choose Y and $(X,Y) \to \mathbb{Z}/m$ as in §4.4. One can always take $Y = Q_0'$, but other choices may be more convenient—for instance, in type A_{n-1}, it is handy to let $X = Y = \mathbb{Z}^n$ be the weight lattice of GL_n (Example 2.5).

Non-reduced and mixed types are handled as follows. If $2\alpha_i \in R$, the specialization $u_i' = u_i$, hence $t_i' = 1$, collapses Δ_0 and $\langle \cdot\,, \cdot \rangle_0$ to their counterparts for the root system with $2\alpha_i$ omitted. Similarly, specializing $u_i' = 1$ omits α_i. The restriction of $<$ to cosets of the (possibly smaller) root lattice Q_0 in the resulting root system does not change. It follows that if Macdonald polynomials E_λ exist for the original root system, then they specialize at $u_i' = 1$ (resp. $u_i' = u_i$) to E_λ for the root system with α_i (resp. $2\alpha_i$) omitted. To be fully correct, we must also show that the coefficients of E_λ do not have poles at these specializations. This will follow from Corollary 6.15.

Every affine root system $\widetilde{X}$ of mixed or non-reduced type embeds as above (perhaps after adjoining $\delta/2$) into a root system of one of two maximally non-reduced types: (a) $\widetilde{X}$ of type $\widetilde{C}_n^\vee$ with $2\alpha_0$, $2\alpha_n$ adjoined (indexing the simple roots as in (23)), or (b) $\widetilde{X}$ of type $\widetilde{B}_n$, with $2\alpha_n$ adjoined. For these types, choose Y and $(\cdot\,,\cdot)$ as for $\widetilde{X}$ of *reduced* type $\widetilde{C}_n^\vee$ or $\widetilde{B}_n$, respectively. Specifically X, Y are of types (B_n, B_n) in (a), or (B_n, C_n) in (b), and we have $\alpha^\vee \in 2\widetilde{X}^\vee$ for all short roots α. In case (a) we also require Y to satisfy $\alpha'^\vee \in 2\widetilde{Y}^\vee$ for short roots α', so as not to force the parameters for $i = 0$ and $i = n$ to coincide (Remark 5.10).

Let $\mathcal{H} = \mathcal{H}(W_e, \widetilde{X})$ be the Cherednik algebra (Definition 5.6) attached to X, Y, $(\cdot\,,\cdot)$, with ground ring $\mathfrak{A} = \mathbb{Q}(t)$, and parameters u_i equated with those in §6.1, setting $u_i' = u_i$ in the reduced case. We identify $\mathbb{Q}(q,t)X$ with the underlying space of the polynomial representation (§5.13) of $\mathcal{H}$, after extension of scalars from $\mathbb{Q}(t)[q^{\pm 1}]$ to $\mathbb{Q}(q,t)$. Note that in formulas (25)–(30) for $i = 0$, we have $x^{\alpha_0} = qx^{-\theta}$, and $s_0(x^\lambda) = q^{\langle \lambda, \theta^\vee \rangle} s_\theta(x^\lambda)$, where $\delta = \alpha_0 + \theta$.

Proposition 6.8. *The operators T_i (§5.13) are unitary with respect to $\langle \cdot\,, \cdot \rangle_0$.*

Proof. For any operator T, let T^* denote its adjoint, $\langle T^* f, g \rangle_0 = \langle f, Tg \rangle_0$. We are to show that $T_i^* = T_i^{-1} = T_i - u_i + u_i^{-1}$, or equivalently, since $u_i^* = \overline{u_i} = u_i^{-1}$, that

$$(T_i - u_i)^* = (T_i - u_i).$$

From (32), we deduce that

$$s_i^* = \frac{1 - t_i x^{\alpha_i}}{t_i - x^{\alpha_i}} s_i = s_i \frac{t_i - x^{\alpha_i}}{1 - t_i x^{\alpha_i}}$$

if $2\alpha_i \notin R$, or

$$s_i^* = \frac{(1 - t_i x^{\alpha_i})(1 + t_i' x^{\alpha_i})}{(t_i - x^{\alpha_i})(t_i' + x^{\alpha_i})} s_i = s_i \frac{(t_i - x^{\alpha_i})(t_i' + x^{\alpha_i})}{(1 - t_i x^{\alpha_i})(1 + t_i' x^{\alpha_i})}$$

if $2\alpha_i \in R$. The fractions appearing in these expressions are self-adjoint, since $s_i^* s_i$ and $s_i s_i^*$ are self-adjoint. The result now follows easily from (26)–(27) in the first case (where $t_i = u_i^2$), and (29)–(30) in the second (where $t_i = u_i u_i'$, $t_i' = u_i/u_i'$). $\square$

Proposition 6.9. *For $i \neq 0$, introduce formal "logarithms" k_i, $k_\alpha = k_i$ for $\alpha \in W_0(\alpha_i)$, with the convention that $q^{k_i} = u_i$. Set*

$$\rho^\vee = \sum_{\alpha\in(R_0)_+} k_\alpha \alpha^\vee, \qquad \rho'^\vee = \sum_{\alpha\in(R_0)_+} k_\alpha \alpha'^\vee,$$

where $\alpha' \in (R_0')_+$ is the positive root such that $s_{\alpha'} = s_\alpha$. Then the Cherednik operators *$y^\mu \in \mathcal{H}$, acting on $\mathbb{Q}(q,t)X$, satisfy*

$$y^\mu(x^\lambda) = q^{-(\lambda,\mu)+\langle \mu, w_\lambda(\rho'^\vee)\rangle} x^\lambda + \sum_{\mu<\lambda} b_{\lambda\mu} x^\mu, \quad b_{\lambda\mu} \in \mathbb{Q}(q,t), \tag{34}$$

where w_λ is the minimal representative of $x^\lambda W_0$ in W_e'.

Proof. It suffices to take $\mu \in Y_+$ dominant, so $y^\mu = T_{y^\mu}$. Bear in mind that $\widetilde{X}$ is now a reduced affine root system of untwisted or dual untwisted type (§6.7), not the root system we started with in §6.1. The affine roots are $\alpha + d\mathbb{Z}\delta$ for $\alpha \in R_0$, where $d = (\alpha,\alpha')/2$, both for untwisted types and their duals. We have $(\alpha,\mu) = d\langle \mu, \alpha'^\vee\rangle$ for all $\mu \in Y$.

If $\beta = \alpha + k\delta$ is a root, then $y^\mu(\beta) = \alpha + (k - (\alpha,\mu))\delta$, and the condition $\beta \in B = R_+ \cap y^{-\mu}(-R_+)$ holds if and only if $0 \le k < (\alpha,\mu)$ and $\alpha \in (R_0)_+$. It follows that for any $\alpha \in (R_0)_+$, the number of roots of the form $\alpha + k\delta \in B$ is equal to $\langle \mu, \alpha'^\vee\rangle$. We also have $x^{y^\mu(\lambda)} = q^{-(\lambda,\mu)} x^\lambda$ by (31). The form of (34) now follows from Proposition 5.15, with leading coefficient given by

$$q^{-(\lambda,\mu)} \prod_{\alpha\in(R_0)_+} u_\alpha^{\langle \mu,\alpha'^\vee\rangle \sigma(-\langle \lambda,\alpha^\vee\rangle)}.$$

This is equal to $q^{-(\lambda,\mu)+\langle \mu, w_\lambda(\rho'^\vee)\rangle}$ because $w_\lambda(\rho'^\vee) = \sum_{\alpha\in(R_0)_+} \pm k_\alpha \alpha'^\vee$, with a minus sign if $\alpha^\vee \in w_\lambda(-(R_0^\vee)_+)$, or equivalently, if $\langle \lambda, \alpha^\vee\rangle > 0$ (see the next remark). $\square$

Note that $\rho^\vee$, $\rho'^\vee$ are characterized by $\langle \alpha_i, \rho^\vee\rangle = \langle \alpha_i', \rho'^\vee\rangle = 2k_i$.

Remark 6.10. The action of $W_e' = W_0 \ltimes X$ on $Y^\vee$ factors through W_0. Hence, $w_\lambda(\rho'^\vee)$ in (34) depends only on the image of w_λ in W_0, which is the minimal element v_λ such that $v_\lambda^{-1}(\lambda) \in -X_+$. A better way to write (34) is as follows. Define $\Lambda_0^\vee \in \widetilde{Y}_{\mathbb{Q}}^\vee$ by $\Lambda_0^\vee(Y) = 0$, $\langle \delta', \Lambda_0^\vee\rangle = 1$. Let $\eta\colon X \to Y_{\mathbb{Q}}^\vee$ be the homomorphism induced by the pairing $(X,Y) \to \mathbb{Q}$, that is, $(\lambda,\mu) = \langle \mu, \eta(\lambda)\rangle$. Then $w_\lambda(\Lambda_0^\vee) = \Lambda_0^\vee + \eta(\lambda)$, and (34) takes the form

$$y^\mu(x^\lambda) = q^{-\langle \mu, w_\lambda(\Lambda_0^\vee - \rho'^\vee)\rangle} x^\lambda + \sum_{\mu<\lambda} b_{\lambda\mu} x^\mu, \quad b_{\lambda\mu} \in \mathbb{Q}(q,t), \tag{35}$$

valid for all $\mu \in \widetilde{Y}$.

Corollary 6.11. *Theorem 6.6 holds with $E_\lambda \in (\mathbb{Q}(q,t)Q_0)x^\lambda$ determined uniquely as the joint eigenfunction with eigenvalue $q^{-\langle \mu, w_\lambda(\Lambda_0^\vee - \rho'^\vee)\rangle}$ of the operators y^μ, normalized so that $[x^\lambda]E_\lambda = 1$.*

Proof. The y^μ act on $(\mathbb{Q}(q,t)Q_0)x^\lambda$ as commuting, lower-triangular operators without repeated joint eigenvalues. Since the y^μ are unitary by Proposition 6.8, their joint eigenfunctions E_λ are orthogonal. □

6.12. Relation (19) can be written $\phi_i x^\lambda = x^{s_i(\lambda)}\phi_i$, where $\phi_i = T_i - (u_i - u_i^{-1})/(1 - x^{\alpha_i})$. By Corollary 5.12, we also have $\psi_i y^\mu = y^{s_i(\mu)}\psi_i$ for $i = 0', 1, \ldots, n$, where $\psi_i = T_i^{-1} - (u_i^{-1} - u_i)/(1 - y^{\alpha_i'}) = T_i - (u_i - u_i^{-1})/(1 - y^{-\alpha_i'})$, and similarly for (21). It is advantageous to use $u_i\psi_i$ instead here. To this end, set

$$\begin{aligned}
&\widetilde{T}_i = u_i T_i \quad (i = 1, \ldots, n);\\
&\widetilde{T}_{0'} = u_{0'}T_{0'} = u_{0'}T_{s_\phi}^{-1}x^{-\phi} = u_j' T_v^{-1} T_j^{-1} x^{-\alpha_j} T_v = t_\phi \widetilde{T}_v^{-1}\widetilde{T}_j^{-1}x^{-\alpha_j}\widetilde{T}_v,
\end{aligned}$$

where $s_\phi = v^{-1}s_j v$ is a reduced factorization (Lemma 4.20). These operators depend only on the parameters t_i, t_i'. The intertwining relations $u_i\psi_i y^\mu = y^{s_i(\mu)}u_i\psi_i$, along with $\pi' y^\mu = y^{\pi'(\mu)}\pi'$ for $\pi \in \Pi'$ imply the following proposition.

Proposition 6.13. *If E_λ is a joint eigenfunction of the operators y^μ, $\mu \in \widetilde{Y}$ with eigenvalue $q^{\langle \mu, \Lambda\rangle}$, then $\Psi_i(E_\lambda)$ is a joint eigenfunction with eigenvalue $q^{\langle \mu, s_i(\Lambda)\rangle}$, where $i = 0', 1, \ldots, n$, and*

$$\Psi_i = \widetilde{T}_i + \frac{1 - t_i}{1 - q^{-\langle \alpha_i', \Lambda\rangle}}, \quad \text{or} \quad \Psi_i = \widetilde{T}_i + \frac{1 - t_i t_i' + (t_i' - t_i)q^{-\langle \alpha_i', \Lambda\rangle}}{1 - q^{-2\langle \alpha_i', \Lambda\rangle}},$$

the second formula applying in case $\alpha_i'^\vee \in 2\widetilde{Y}^\vee$. Similarly, $\pi'(E_\lambda)$ is a joint eigenfunction with eigenvalue $q^{\langle \mu, \pi'(\Lambda)\rangle}$, for any $\pi' \in \Pi'$.

Corollary 6.14. *For $i \neq 0$, if $s_i(\lambda) = \lambda$, then $s_i E_\lambda = E_\lambda$.*

Proof. Proposition 6.13 implies that $T_i E_\lambda$ is a scalar multiple of E_λ, and from the leading coefficient we deduce $T_i E_\lambda = u_i E_\lambda$, which is equivalent to $s_i E_\lambda = E_\lambda$. □

Corollary 6.15. *The Macdonald polynomials satisfy the recurrence*

$$E_{v_{\pi'}(\lambda) + \lambda_{\pi'}} = q^{-\langle \lambda_{(\pi'^{-1})}, w_\lambda(\rho^\vee)\rangle} x^{\lambda_{\pi'}} T_{v_{\pi'}}(E_\lambda), \quad \pi' = x^{\lambda_{\pi'}}v_{\pi'} \in \Pi', \tag{36}$$

$$E_{s_i(\lambda)} = \left(\widetilde{T}_i + \frac{1 - t_i}{1 - q^{(\lambda, \alpha_i') - \langle \alpha_i', w_\lambda(\rho'^\vee)\rangle}}\right) E_\lambda, \quad \langle \lambda, \alpha_i^\vee\rangle > 0,\, i \neq 0',\, t_i' = 1, \tag{37}$$

$$\begin{aligned}
E_{s_\phi(\lambda) + \phi} = t_\phi' q^{-\langle \phi, w_\lambda(\rho^\vee)\rangle}\left(\widetilde{T}_{0'} + \frac{1 - t_{0'}}{1 - q^{1 - (\lambda, \theta') + \langle \theta', w_\lambda(\rho'^\vee)\rangle}}\right) E_\lambda,&\\
\langle \lambda, \phi^\vee\rangle < 1,\, t_{0'}' = 1.&
\end{aligned} \tag{38}$$

If $t_i' \neq 1$, (37) becomes instead

$$E_{s_i(\lambda)} = \left(\widetilde{T}_i + \frac{1 - t_i t_i' + (t_i' - t_i)q^{(\lambda, \alpha_i') - \langle \alpha_i', w_\lambda(\rho'^\vee)\rangle}}{1 - q^{2((\lambda, \alpha_i') - \langle \alpha_i', w_\lambda(\rho'^\vee)\rangle)}}\right) E_\lambda, \tag{39}$$

with a corresponding modification to (38) if $t_{0'}' \neq 1$.

The base of the recurrence is $E_\lambda = x^\lambda$ for λ minuscule, *i.e.*, $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for $i \neq 0$ and $\langle \lambda, \phi^\vee \rangle \leq 1$. With this base, (36) is not essential to the recurrence, but it is often useful nevertheless.

To prove Corollary 6.15, first observe that the map $X \to \widetilde{Y}_{\mathbb{Q}}$, $\lambda \mapsto \Lambda_0^\vee + \lambda$ is equivariant with respect to the action of W'_e on $\widetilde{Y}$ and on $X = W'_e/W_0$. Then Proposition 6.13 and Corollary 6.11 imply that $\Psi_i(E_\lambda)$ (resp. $\pi'(E_\lambda)$) is a scalar multiple of $E_{s_i(\lambda)}$ (resp. $E_{\pi'(\lambda)} = E_{v_{\pi'}(\lambda)+\lambda_{\pi'}}$).

The action of Π' on $X = W'_e/W_0$ preserves the Bruhat order. Assuming by induction that (36) holds for $\nu < \lambda$, we conclude that $\pi' = x^{\lambda_{\pi'}} T_{v_{\pi'}}$ carries $\mathbb{Q}(q,t)\{x^\nu : \nu < \lambda\}$ into $\mathbb{Q}(q,t)\{x^\nu : \nu < v_{\pi'}(\lambda) + \lambda_{\pi'}\}$. Hence the coefficient of $x^{v_{\pi'}(\lambda)}$ in $T_{v_{\pi'}}(x^\lambda)$ determines the scalar factor in (36). For $\langle \lambda, \alpha_i^\vee \rangle > 0$ (resp. $\langle \lambda, \phi^\vee \rangle < 1$), we have $s_i w_\lambda > w_\lambda$ (resp. $s_{0'} w_\lambda > w_\lambda$). We may assume by induction that T_i (resp. $T_{0'}$) leaves invariant the space $\mathbb{Q}(q,t)\{x^\nu, x^{s_i(\nu)} : \nu < \lambda\}$. For $i \neq 0'$ and $s_i(\lambda) > \lambda$, we have $[x^{s_i(\lambda)}]\widetilde{T}_i(x^\lambda) = 1$, giving (37), and the coefficient of $x^{s_\phi(\lambda)+\phi}$ in $\widetilde{T}_{0'}(x^\lambda)$ determines the scalar factor in (38). The next lemma supplies the missing scalar factors.

Lemma 6.16. *(i) We have* $[x^{v_{\pi'}(\lambda)}]T_{v_{\pi'}}(x^\lambda) = q^{\langle \lambda_{(\pi'^{-1})}, w_\lambda(\rho^\vee) \rangle}$ *for any* $\pi' \in \Pi'$.
(ii) For $\langle \lambda, \phi^\vee \rangle < 1$, *we have* $[x^{s_\phi(\lambda)+\phi}]\widetilde{T}_{0'}(x^\lambda) = t_\phi'^{-1} q^{\langle \phi, w_\lambda(\rho^\vee) \rangle}$.

Proof. (i) Let $B = (R_0)_+ \cap v_{\pi'}^{-1}(-(R_0)_+)$. We claim that for any $\alpha \in (R_0)_+$, $\langle \lambda_{(\pi'^{-1})}, \alpha^\vee \rangle = 1$ if $\alpha \in B$, 0 otherwise. Then Proposition 5.15 gives

$$[x^{v_{\pi'}(\lambda)}]T_{v_{\pi'}}(x^\lambda) = \prod_{\alpha \in (R_0)_+} u_\alpha^{\langle \lambda_{(\pi'^{-1})}, \alpha^\vee \rangle \sigma(-\langle \lambda, \alpha^\vee \rangle)} = q^{\langle \lambda_{(\pi'^{-1})}, w_\lambda(\rho^\vee) \rangle}$$

by the argument in the proof of Proposition 6.9.

As to the claim, if $v_{\pi'} = 1$, then $B = \emptyset$ and $\langle \lambda_{(\pi'^{-1})}, \alpha^\vee \rangle = 0$ for all α. Otherwise, $\lambda_{(\pi'^{-1})} = -v_{\pi'}^{-1}(\lambda_{\pi'})$, and $\langle \lambda_{(\pi'^{-1})}, \alpha^\vee \rangle = -\langle \lambda_{\pi'}, v_{\pi'}(\alpha^\vee) \rangle \in \{0,1\}$ for all $\alpha \in (R_0)_+$, since $\lambda_{(\pi'^{-1})}$ is minuscule. Now, $v_{\pi'}(\alpha_j^\vee) = -\phi^\vee$, where $\pi'^{-1}(\alpha'_0) = \alpha'_j$, and $v_{\pi'}(\alpha_i^\vee)$ is a simple coroot for $i \neq j$. Since $v_{\pi'} \neq 1$, we have $\langle \lambda_{\pi'}, \phi^\vee \rangle = 1$, and it follows that $\langle \lambda_{(\pi'^{-1})}, \alpha_i^\vee \rangle = \delta_{ij}$. Given $\alpha \in (R_0)_+$, if $\langle \lambda_{(\pi'^{-1})}, \alpha^\vee \rangle = 1$, then $v_{\pi'}(\alpha) \in -(R_0)_+$ since $\lambda_{\pi'} \in X_+$. Conversely, if $\langle \lambda_{(\pi'^{-1})}, \alpha^\vee \rangle = 0$, the coefficient of $\alpha_j^\vee$ in $\alpha^\vee$ must be zero, hence $v_{\pi'}(\alpha) \in (R_0)_+$.

(ii) Let $B = (R_0)_+ \cap s_\phi(-(R_0)_+)$. The operator $s_\phi T_{s_\phi}$ is lower-triangular by Proposition 5.15, hence so is $T_{s_\phi}^{-1} s_\phi$, and $[x^{s_\phi(\lambda)+\phi}]T_{s_\phi}^{-1}(x^{\lambda-\phi})$ is inverse to

$$[x^{\lambda-\phi}]T_{s_\phi}(x^{s_\phi(\lambda)+\phi}) = \prod_{\alpha \in B} u_\alpha^{\sigma(-\langle s_\phi(\lambda)+\phi\,,\, \alpha^\vee \rangle)} = \prod_{\alpha \in B} u_\alpha^{\sigma(\langle \lambda-\phi\,,\, \alpha^\vee \rangle)},$$

using $s_\phi(B) = -B$ in the last equation. Now, ϕ is short and dominant, hence $\langle \phi, \alpha^\vee \rangle \in \{0,1\}$ for $\alpha \in (R_0)_+ \setminus \{\phi\}$. Moreover, $s_\phi(\alpha^\vee) = \alpha^\vee - \langle \phi, \alpha^\vee \rangle \phi^\vee$, and since $\phi^\vee$ is the highest coroot, this implies that $\langle \phi, \alpha^\vee \rangle > 0$ if and only if $s_\phi(\alpha) \in -(R_0)_+$. Thus for $\alpha \in (R_0)_+ \setminus \{\phi\}$, we have $\langle \phi, \alpha^\vee \rangle = 1$ if $\alpha \in B$, 0 otherwise. Since

$\langle \lambda, \phi^\vee \rangle \leq 0$, it follows that

$$[x^{s_\phi(\lambda)+\phi}]\widetilde{T}_{0'}(x^\lambda) = u'_\phi [x^{s_\phi(\lambda)+\phi}] T_{s_\phi}^{-1}(x^{\lambda-\phi}) = u'_\phi u_\phi \prod_{\alpha \in B\setminus\{\phi\}} u_\alpha^{-\sigma(\langle \lambda, \alpha^\vee\rangle - 1)}$$
$$= (u'_\phi/u_\phi) \prod_{\alpha\in (R_0)_+} u_\alpha^{\langle \phi,\alpha^\vee\rangle \sigma(-\langle\lambda,\alpha^\vee\rangle)} = t'^{-1}_\phi q^{\langle \phi, w_\lambda(\rho^\vee)\rangle}.$$

□

6.17. Suppose $\widetilde{X}$ is dual to an untwisted type. Then X, Y are of the same type, $\phi = \theta$, $\phi' = \theta'$, $s_\phi = s_{\phi'}$, and in $\mathcal{H}$ we have the identities $T_{0'} = x^\phi T_0^{-1} y^{\phi'} + u_{0'} - u_{0'}^{-1}$ and $\pi' = x^{\lambda_{\pi'}} \pi y^{\lambda_{(\pi^{-1})}}$ for $\pi' \in \Pi'$, $\pi \in \Pi$ such that $v_{\pi'} = v_\pi$. Using these identities, (36) and (39) for $i = 0'$ become

$$E_{v_{\pi'}(\lambda)+\lambda_{\pi'}} = q^{-(\lambda,\lambda_{(\pi^{-1})})} x^{\lambda_{\pi'}} \pi(E_\lambda), \quad \pi' \in \Pi',\ \pi\in\Pi,\ v_\pi = v_{\pi'}$$
$$E_{s_\theta(\lambda)+\theta} = q^{1-(\lambda,\theta')} \left(u_\theta x^{-\alpha_0} T_0^{-1} + \frac{(u_\theta/u'_0 - u_\theta u'_0) + (u_\theta/u_{0'} - u_\theta u_{0'}) q^r}{1-q^{2r}} \right) E_\lambda,$$

where $r = 1 - (\lambda, \theta') + \langle \theta', w_\lambda(\rho'^\vee)\rangle$. Note that the second formula simplifies to an analog of (37) if $u_{0'} = u'_0$.

6.18. Although our chief concern is with non-symmetric Macdonald polynomials, let us say a little about the symmetric version. Given $\lambda \in X_+$, let $V_\lambda = \mathbb{Q}(q,t)\{E_\nu : \nu \in W_0(\lambda)\}$. By Corollaries 6.14, 6.15, V_λ is an $\mathcal{H}(W_0)$-submodule of $\mathbb{Q}(q,t)X$. It follows that there is a unique W_0-invariant element $P_\lambda \in V_\lambda$ such that $[x^\lambda]P_\lambda = 1$. The P_λ are *symmetric Macdonald polynomials.* They are orthogonal and are joint eigenfunctions of all W_0-invariant operators $f(y) \in (\mathbb{Q}(q,t)Y)^{W_0}$. The coefficients of P_λ in terms of the E_ν can be determined explicitly using Corollary 6.15.

The P_λ are also orthogonal with respect to *Macdonald's inner product*, which is a symmetrization of $\langle \cdot, \cdot \rangle_0$. They were originally defined by Macdonald [14, 15] in terms of this orthogonality. When $t_i = q^{(\alpha_i,\alpha'_i)/2}$, they specialize to the irreducible characters of the algebraic group G with weight lattice X and root system Q_0. Other specializations yield Hall-Littlewood and Jack polynomials, and spherical functions for classical and p-adic symmetric spaces.

For GL_n, the P_λ are symmetric polynomials in $x_1, \ldots, x_n$, with coefficients in $\mathbb{Q}(q,t)$. As $n \to \infty$, they converge to symmetric functions $P_\lambda(x;q,t)$ in infinitely many variables x_i. A transformed and renormalized variant $\widetilde{H}_\lambda(x;q,t)$ of $P_\lambda(x;q,t)$ was the subject of Macdonald's *positivity conjecture*, proved in [6] by identifying $\widetilde{H}_\lambda(x;q,t)$ with the character of the fiber of a certain vector bundle on the Hilbert scheme H of 0-dimensional subschemes in $\mathbb{C}^2$, at a distinguished point of H corresponding to λ.

6.19. Macdonald polynomials for the maximally non-reduced extensions of affine root systems of type $\widetilde{C}_n^\vee$ are *Koornwinder polynomials.* Their coefficients belong to $\mathbb{Q}(t_0, t'_0, t_n, t'_n, t_1, q)$. Specializing the five t parameters in various ways yields most Macdonald polynomials for the infinite families of affine root systems.

7. A combinatorial formula

7.1. From Corollary 6.15 and the definition of the operators $\widetilde{T}_i$ it is clear that for a reduced affine root system, E_λ can be expressed as a sum of terms of the form

$$\pm x^\mu q^r t^s \prod_j \frac{1-t_{i_j}}{1-q^{a_j}t^{b_j}},$$

where t^s, t^{b_j} stand for monomials in the parameters t_i. It may be conjectured, at least for equal parameters $t_i = t$, that E_λ is a *positive* sum of such terms. With Haglund and Loehr [5], we proved this for type $\widetilde{A}_{n-1}$ by means of a combinatorial formula, which we will now present (referring the reader to [5] for the proof). Some of the combinatorial structure is the same as in Knop and Sahi's earlier formula [10] for non-symmetric Jack polynomials, but the lift to Macdonald polynomials requires more ingredients.

7.2. Take $X = Y = \mathbb{Z}^n$ the root system of GL_n, as in Example 2.5. The pairing $(X, Y) \to \mathbb{Z}$ (§4.4) is the standard inner product on $\mathbb{Z}^n$. We have $\phi = \theta = \phi' = \theta' = e_1 - e_n$, and $\Pi = \Pi'$ is cyclic, with generator π' acting on $X = W'_e/W_0$ by

$$\pi'(\lambda) = (\lambda_n + 1, \lambda_1, \dots, \lambda_{n-1}).$$

To π' corresponds an element $\pi \in \Pi$ such that $v_\pi = v_{\pi'}$, which acts on $\mathbb{Q}(q,t)\widetilde{X}$ by

$$\pi(x^\lambda) = q^{-\lambda_n} x^{(\lambda_n, \lambda_1, \dots, \lambda_{n-1})}, \quad \text{or} \quad \pi f(x_1, \dots, x_n) = f(x_2, \dots, x_n, x_1/q).$$

We have $\lambda_{\pi'} = \lambda_\pi = e_1$, $\lambda_{(\pi'^{-1})} = \lambda_{(\pi^{-1})} = -e_n$.

The simple roots are all W-conjugate, so there is a single parameter $t_i = t$ for all i. For $i \neq 0$, the operators $\widetilde{T}_i$ (§5.13, 6.12) are given by

$$\widetilde{T}_i = t s_i - \frac{1-t}{1 - x_i/x_{i+1}}(1 - s_i), \tag{40}$$

where s_i is the transposition $x_i \leftrightarrow x_{i+1}$. The analogous formula for $i = 0$ has $q x_n/x_1$ in place of x_i/x_{i+1}, and s_0 acts as $x_1 \mapsto q x_n$, $x_n \mapsto x_1/q$.

Let $\overline{\lambda}$ be the rearrangement of $(1, 2, \dots, n)$ such that $\overline{\lambda}_i > \overline{\lambda}_j$ if and only if $\lambda_i > \lambda_j$, for $i < j$. Then $w_\lambda(\rho^\vee) = -k\overline{\lambda}$, modulo a constant vector. From §6.17 and (37), we obtain *Knop's recurrence*, which determines E_λ for all $\lambda \in X$:

$$E_{(0,\dots,0)} = 1, \tag{41}$$

$$E_{(\lambda_n+1, \lambda_1, \dots, \lambda_{n-1})} = q^{\lambda_n} x_1 E_\lambda(x_2, \dots, x_n, x_1/q), \tag{42}$$

$$E_{s_i(\lambda)} = \left(\widetilde{T}_i + \frac{1-t}{1 - q^{\lambda_i - \lambda_{i+1}} t^{\overline{\lambda}_i - \overline{\lambda}_{i+1}}}\right) E_\lambda, \quad \lambda_i > \lambda_{i+1},\ i \neq 0. \tag{43}$$

7.3. By (42), we have $E_{\lambda + (r,r,\dots,r)} = (x_1 \cdots x_n)^r E_\lambda$. Without loss of generality, therefore, we restrict attention to *compositions* λ such that $\lambda_i \geq 0$ for all i. The *column diagram* of λ is

$$\mathrm{dg}(\lambda) = \{(i,j) \in \mathbb{N}^2 : 1 \leq i \leq n,\ 1 \leq j \leq \lambda_i\},$$

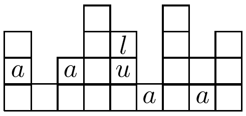
pictured as an array of n columns of boxes, with λ_i boxes in column i. The *augmented diagram* of λ,

$$\widehat{\mathrm{dg}}(\lambda) = \mathrm{dg}(\lambda) \cup \{(i,0) : 1 \leq i \leq n\},$$

is formed by adjoining an extra box in row 0 at the bottom of each column. The *arm* and *leg* of a box $u = (i,j) \in \mathrm{dg}(\lambda)$ are defined by

$$l(u) = \mu_i - j,$$
$$a(u) = |\{(i',j) \in \mathrm{dg}(\lambda) : i' < i,\ \lambda_{i'} \leq \lambda_i\} \cup \{(i',j-1) \in \widehat{\mathrm{dg}}(\lambda) : i' > i,\ \lambda_{i'} < \lambda_i\}|.$$

In the example below, the box marked l contributes to the leg of u, and those marked a contribute to the arm, giving $l(u) = 1$, $a(u) = 4$.

$\lambda = (2,0,1,3,2,0,3,1,2)$, $\widehat{\mathrm{dg}}(\lambda) =$ 

With these definitions, if $\lambda_i > \lambda_{i+1}$ and $u = (i,\ \mu_{i+1}+1)$, equation (43) can be written

$$E_{s_i(\lambda)} = \left(\widetilde{T}_i + \frac{1-t}{1-q^{l(u)+1}t^{a(u)}}\right) E_\lambda.$$

A *filling of* λ is a function $\sigma\colon \mathrm{dg}(\lambda) \to [n] = \{1,\dots,n\}$. Its associated *augmented filling* is the extension $\widehat{\sigma}\colon \widehat{\mathrm{dg}}(\lambda) \to [n]$ of σ such that $\widehat{\sigma}((j,0)) = j$ for $j = 1,\dots,n$. Distinct boxes $u = (i,j)$, $v = (i',j') \in \widehat{\mathrm{dg}}(\lambda)$ *attack* each other if either

(a) they are in the same row, $j = j'$, or

(b) they are in consecutive rows, and the lower box is to the right of the upper box: $j' = j-1$ and $i < i'$ or vice versa.

A filling σ is *non-attacking* if $\widehat{\sigma}(u) \neq \widehat{\sigma}(v)$ whenever u and v attack each other (non-attacking fillings are called *admissible* in [10]).

7.4. Let $d(u) = (i,j-1)$ denote the box directly below a box $u = (i,j)$. A box $u \in \mathrm{dg}(\lambda)$ is a *descent* of a filling σ if $\widehat{\sigma}(u) > \widehat{\sigma}(d(u))$. Define

$$\mathrm{Des}(\sigma) = \{\text{descents of } \sigma\}, \qquad \mathrm{maj}(\sigma) = \sum_{u \in \mathrm{Des}(\sigma)} (l(u)+1).$$

The *reading order* is the total ordering $<$ of the boxes in $\widehat{\mathrm{dg}}(\lambda)$ row by row, from top to bottom, and from *right to left* within each row. A *triple* consists of three boxes $u < v < w = d(u)$ in $\widehat{\mathrm{dg}}(\lambda)$, as shown:

Type I Type II

with the proviso that the column containing u, w is strictly taller than the column containing v in Type I, and weakly taller in Type II, *i.e.*, v contributes to the arm of u. A *co-inversion triple* of σ is a triple such that $\sigma(u) < \sigma(v) < \sigma(w)$ or $\sigma(v) < \sigma(w) < \sigma(u)$ or $\sigma(w) < \sigma(u) < \sigma(v)$. Define

$$\operatorname{coinv}(\sigma) = |\{\text{co-inversion triples of } \sigma\}|.$$

Example 7.5. The figure below shows the augmentation $\widehat{\sigma}$ of a non-attacking filling σ of $\lambda = (2, 1, 3, 0, 0, 2)$.

$$\widehat{\sigma} = \begin{array}{cccccc} & & 2 & & & \\ 6 & & 4 & & & 5 \\ 1 & 2 & 3 & & & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}.$$

The circled boxes are $\operatorname{Des}(\sigma)$, giving $\operatorname{maj}(\sigma) = 3$. Row 0 is the bottom row. There are two co-inversion triples, one of Type I formed by the 3 and the 5 in row 1 with the 4 in row 2, and one of Type II formed by the 6 and the 4 in row 2 with the 3 in row 1, giving $\operatorname{coinv}(\sigma) = 2$.

Theorem 7.6. *The Macdonald polynomials E_λ for GL_n are given by*

$$E_\lambda = \sum_{\substack{\sigma\colon \lambda \to [n] \\ \textit{non-attacking}}} x^\sigma q^{\operatorname{maj}(\sigma)} t^{\operatorname{coinv}(\sigma)} \prod_{\substack{u \in \operatorname{dg}(\lambda) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} \frac{1-t}{1 - q^{l(u)+1} t^{a(u)+1}}, \tag{44}$$

where $x^\sigma = \prod_{u \in \operatorname{dg}(\lambda)} x_{\sigma(u)}$.

7.7. Earlier, in [4], we gave a combinatorial formula for the symmetric Macdonald polynomials P_λ for GL_n, which had originally been conjectured by Haglund [3]. The combinatorial statistics $\operatorname{coinv}(\sigma)$ and $\operatorname{maj}(\sigma)$ first appeared in the formula for the symmetric case, which is expressed similarly as a sum over fillings of a diagram. Our work in the symmetric case relies heavily on the special theory of GL_n Macdonald polynomials in the $n \to \infty$ stable limit. It seems likely that the non-symmetric formula will provide better clues as to what we might expect for other root systems.

7.8. The proof of Theorem 7.6 is a direct verification that (44) satisfies Knop's recurrence (41)–(43). It is not difficult to check (42), and (41) is trivial. The hard part is to verify (43). In fact, we were only able to do it in the special case that $\lambda_{i+1} = 0$, which fortunately is enough. The difficulty lies in applying the operator $\widetilde{T}_i$ in (40) to (44), which is intractable if attempted head-on. To get around this, we recast (43) as asserting that certain expressions related to (44) are s_i-invariant. This is proved with the help of a symmetry lemma which originated in the theory of *LLT polynomials* [11, 12], and was also at the heart of our work in [4]. We invite the reader to consult [5] for more detail.

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