

HECKE ALGEBRA CHARACTERS AND IMMANANT CONJECTURES

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INTRODUCTION

The main purpose of this article is to announce and provide supporting evidence for two conjectures about the characters of the Hecke algebra $H_n(q)$ of type A_{n-1} , evaluated at elements of its Kazhdan–Lusztig basis. In addition, we prove a conjectured *immanant* inequality for Jacobi–Trudi matrices (definitions below) and show how our conjectures would imply stronger inequalities of a similar kind.

The immanant inequalities belong to the combinatorial theory of symmetric functions and consequently have gained considerable attention in algebraic combinatorics since their introduction by Goulden and Jackson [9]; see [10], [29], [30]. The Hecke algebra conjectures presented here are, however, independent of the application which led to their discovery, and because of their striking and unexpected nature they should be of interest to a broader audience. In particular, they appear to reflect aspects of the geometry of the flag variety that cannot yet be understood using available geometric machinery. It has also been discovered that Hecke algebras of type A_{n-1} arise naturally in the study of knots [7], [14], quantum groups [13], and Von Neumann algebras [15], [34]. Their character theory in particular plays an important role, via the Ocneanu trace and the commutant relationship between $H_n(q)$ and the quantum group $UGL_n(q)$. Thus there are important reasons to seek a better understanding of the characters.

The first of our conjectures asserts that certain virtual characters, *i.e.*, integral linear combinations of irreducible characters, take values on the Kazhdan–Lusztig basis which are polynomials in q with non-negative, symmetric and unimodal integer coefficients. A corresponding assertion for the *irreducible* characters follows from the theory of intersection homology and perverse sheaves for Schubert varieties [3], [27], together with the fact that the Kazhdan–Lusztig cell representations [17] are irreducible for type A_{n-1} . This fact is a weaker statement than our conjecture, however, since the irreducible characters are non-negative combinations of the virtual characters we consider. In fact, the conjecture for these virtual characters is best possible: any virtual character for which the assertion holds must be a non-negative combination of these.

The second conjecture asserts that for purposes of character evaluation, most Kazhdan–Lusztig basis elements are reducible to sums of certain of the simplest possible ones. To be precise, we pick out certain permutations, called co-dominant, whose corresponding Schubert varieties are smooth and very simple to describe.

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Consequently, the corresponding Kazhdan–Lusztig basis elements also have an extremely simple description. The conjecture then states that for each Kazhdan–Lusztig basis element C'_w , there is a sum $C'_{w_1} + \cdots + C'_{w_k}$ of basis elements with w_i co-dominant, such that

$$\chi(C'_w) = \chi(C'_{w_1} + \cdots + C'_{w_k})$$

for every Hecke algebra character χ . In particular, the second conjecture would reduce the first conjecture to its special case for co-dominant elements, thus eliminating Kazhdan–Lusztig polynomials from the problem.

We will make clear in our later discussion why there are good reasons to expect deep connections between these conjectures and the geometry of the flag variety. Unfortunately, even the powerful machinery of perverse sheaves and intersection homology appears, at least for the moment, to provide inadequate information for a solution of the conjectures.

We are able to prove various special cases of the conjectures. Several authors [19], [24], [33] have considered the problem of evaluating characters of $H_n(q)$ at elements of its natural basis, a problem which is the q -analog of determining the character table of the symmetric group S_n . We have made use of these results in order to prove a special case of our first conjecture in Section 4. It is conceivable that a similar approach could be used to resolve the first conjecture for co-dominant elements. To do so, one would have to prove a q -analog of Proposition 5.1, without the restriction assumed there on the co-dominant permutation w_f . The problem of evaluating characters at arbitrary elements of the Kazhdan–Lusztig basis is in general much more difficult since little explicit information is known about the Kazhdan–Lusztig polynomials, which appear as the coefficients expressing this basis in terms of the natural one.

In addition to the two aforementioned conjectures, this paper contains some theorems, whose relevance we now explain, along with the overall organization of the paper.

The present investigation originated with the author’s discovery that Kazhdan–Lusztig theory for the Hecke algebra $H_n(q)$ is the natural setting for the immanant inequalities conjectured by Goulden and Jackson [9] and extended by Stembridge, Stanley, and Greene [10], [29], [30]. It turns out that all these immanant inequalities would follow from our first Hecke algebra conjecture, combined with a theorem proved here (Theorem 1.5) relating a generating function for immanants to the Kazhdan–Lusztig basis for $H_n(q)$. Even without the Hecke algebra conjecture, Theorem 1.5 implies an immanant inequality stronger than the one proved by Greene.

Since many readers will not be familiar with the subject of immanant inequalities (or even the word ‘immanant’), we give a brief historical review of the topic in Section 1, and explain how our results and conjectures apply. As the main theorem of the section, Theorem 1.5, has a rather technical proof, using the Kazhdan–Lusztig conjecture on composition series of Verma modules, we defer its proof to the end of the paper, in Section 7.

We then turn to the presentation of our two central conjectures and some general remarks about them, followed by sections dealing with computational and special case evidence for the conjectures with q general, with $q = 1$, and with $q = 0$, respectively. If desired, the paper may be read beginning with Section 2, except

for the definition (1.2) of monomial characters, since the Hecke algebra conjectures and related results do not depend on the application to immanants.

We have included an appendix covering definitions and results from Kazhdan–Lusztig theory, most of which are standard by now, but which may not be familiar to all readers, especially from the combinatorial audience. Otherwise unexplained notation and facts stated without citation refer either to the appendix, or, where symmetric functions and Young tableaux are concerned, to the book of Macdonald [23].

1. IMMANANTS

We begin with the definition of immanants and a review of conjectured inequalities involving them. A much more complete survey may be found in [30].

Let χ be an irreducible character of S_n , or more generally any class function, and let A be an $n \times n$ matrix. Littlewood’s *immanants* [22] are matrix functionals generalizing the determinant and the permanent, defined by

$$\text{Imm}_\chi(A) = \sum_{w \in S_n} \chi(w) a_{1,w(1)} \cdots a_{n,w(n)}. \quad (1.1)$$

When $\chi = \chi_\lambda$ is the irreducible character indexed by the partition λ of n , we abbreviate $\text{Imm}_{\chi_\lambda}(A)$ to $\text{Imm}_\lambda(A)$. In particular, the determinant and the permanent are the immanants $\text{Imm}_{(1^n)}(A)$ and $\text{Imm}_{(n)}(A)$ corresponding to the sign character and the trivial character, respectively.

The rule of thumb for immanant inequalities is that if a matrix has non-negative minors in some suitable sense, then its irreducible immanants, or certain linear combinations of them, ought to be non-negative.

The oldest branch of the subject, going back to Schur, involves immanants of positive definite Hermitian matrices, typified by the ‘Schur dominance theorem’ and ‘Lieb permanental dominance conjecture,’ which state the following.

Theorem 1.1. (Schur [25]) *If A is positive definite, then $\text{Imm}_\lambda(A) - f_\lambda \det(A) \geq 0$ for all λ .*

Conjecture 1.1. (Lieb [21]) *If A is positive definite, then $f_\lambda \text{per}(A) - \text{Imm}_\lambda(A) \geq 0$.*

In each of these statements $f_\lambda = \chi_\lambda(1)$ is the degree of the irreducible character χ_λ .

The more recent branches concern immanants of *totally positive* matrices (A is totally positive if every minor of A is non-negative) and *Jacobi–Trudi* matrices (defined below). In their strongest forms the conjectured inequalities concern *monomial* immanants, which we now define.

From the theory of symmetric functions, the irreducible characters of S_n are given by $\chi_\lambda(w) = \langle s_\lambda, p_{\tau(w)} \rangle$, where s_λ denotes a Schur function, $p_{\tau(w)}$ is the power-sum symmetric function indexed by the partition $\tau(w)$ corresponding to the decomposition of w into cycles, and $\langle s_\lambda, s_\mu \rangle = \langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$. Here h_μ and m_λ denote the complete and the monomial symmetric functions, respectively.

Thus we have a natural linear correspondence mapping the Schur functions of degree n to the irreducible characters of S_n . Under this correspondence the monomial symmetric function m_λ is carried to a ‘monomial’ virtual character ϕ_λ given by

$$\phi_\lambda(w) = \langle m_\lambda, p_{\tau(w)} \rangle. \quad (1.2)$$

The corresponding *monomial immanants* are the immanants $\text{Imm}_{\phi_\lambda}(A)$. Since the Schur functions are non-negative linear combinations of monomial symmetric functions, non-negativity statements for monomial immanants are in general stronger than their counterparts for irreducible immanants.

For totally positive matrices we have the following analogs of Schur dominance and permanent dominance. Actually, Conjecture 1.2 is a stronger statement which implies both permanent dominance and Theorem 1.2. No correspondingly strong statement holds for positive definite matrices.

Theorem 1.2. (Stembridge [32]) *If A is totally positive, then $\text{Imm}_\lambda(A) - f_\lambda \det(A) \geq 0$ for all λ .*

Conjecture 1.2. (Stembridge [32]) *If A is totally positive, then $\text{Imm}_{\phi_\lambda}(A) \geq 0$ for all λ .*

Our main concern will be with the third and most combinatorial branch of the subject, immanant inequalities for *Jacobi–Trudi* matrices. Let h_n denote the complete homogeneous symmetric function of degree n in countably many variables $X = (x_1, x_2, \dots)$. By convention, $h_n = 0$ for $n < 0$ and $h_0 = 1$. Let μ and ν be partitions with n parts, written in descending order as $(\mu_1 \geq \mu_2 \geq \dots \geq \mu_n)$. We allow 0 as a part. Jacobi–Trudi matrices are of the form

$$H_{\mu/\nu}(X) = [h_{\mu_i - \nu_j}(X)]_{i,j=1}^n. \quad (1.3)$$

The classical Jacobi–Trudi matrix would be $H_{\mu+\delta/\nu+\delta}$ in this notation, where $\delta = (n-1, \dots, 1, 0)$. The *Jacobi–Trudi identity* expresses the (skew) Schur functions in terms of homogeneous symmetric functions as

$$s_{\mu/\nu} = \det(H_{\mu+\delta/\nu+\delta}). \quad (1.4)$$

We have altered the notation in order to include the possibility that $H_{\mu/\nu}$ may have repeated rows or columns, which is a relevant case for immanants other than the determinant.

Since the minors of a Jacobi–Trudi matrix are skew Schur functions, they are ‘non-negative’ in the sense either of being non-negative linear combinations of monomial symmetric functions, or of Schur functions.

In their survey of applications of immanants to matrices having combinatorial significance, Goulden and Jackson [9] conjectured the following result, which has since been proved by Greene.

Theorem 1.3. (Greene [10]) *The irreducible immanants $\text{Imm}_\lambda(H_{\mu/\nu})$ of Jacobi–Trudi matrices are non-negative linear combinations of monomial symmetric functions for all λ .*

This theorem can be strengthened in either or both of two ways: we can ask for non-negativity in terms of Schur functions rather than monomials, or we can pass to monomial immanants. The three conjectures so obtained are due to Stembridge [30]. Our main application of Kazhdan–Lusztig theory to immanants is the proof of one of these conjectures. Thus we state the first conjecture as a theorem and the other two as conjectures, as follows. Note that Conjecture 1.4 implies both of the others.

Theorem 1.4. *If $H_{\mu/\nu}$ is a Jacobi–Trudi matrix, then $\text{Imm}_\lambda(H_{\mu/\nu}) - f_\lambda \det(H_{\mu/\nu})$ is a non-negative linear combination of Schur functions.*

Conjecture 1.3. (Stembridge [30]) *The monomial immanants $\text{Imm}_{\phi_\lambda}(H_{\mu/\nu})$ of Jacobi–Trudi matrices are non-negative linear combinations of monomial symmetric functions for all λ .*

Conjecture 1.4. (Stembridge [30]) *The monomial immanants $\text{Imm}_{\phi_\lambda}(H_{\mu/\nu})$ of Jacobi–Trudi matrices are non-negative linear combinations of Schur functions for all λ .*

To conclude this section, we shall explain how Theorem 1.4 can be proven using Kazhdan–Lusztig theory, as well as how Conjectures 1.2 through 1.4 would follow from the conjecture on Hecke algebra characters developed in the next section.

It is convenient to introduce the following formal sum, which serves as a kind of generating function for the immanants of $H_{\mu/\nu}$. Let

$$I_{\mu/\nu} = \sum_{w \in S_n} T_w h_{\mu-w(\nu)}(X). \quad (1.5)$$

This is an element of $H_n(1) \otimes \Lambda(X)$, the group algebra of S_n with coefficients extended to the ring of symmetric functions $\Lambda(X)$. Extending S_n characters χ in the obvious way to $\Lambda(X)$ -linear functionals on $H_n(1) \otimes \Lambda(X)$, we may express the immanants of Jacobi–Trudi matrices as

$$\text{Imm}_\chi(H_{\mu/\nu}(X)) = \chi(I_{\mu/\nu}). \quad (1.6)$$

With this notation, Theorem 1.4 is an immediate consequence of the following two statements.

Theorem 1.5. *$I_{\mu/\nu}$ is a non-negative linear combination of the basis elements $C'_w(1) s_\kappa(X)$ for $H_n(1) \otimes \Lambda(X)$, where $\{C'_w(q)\}$ is the Kazhdan–Lusztig basis of the Hecke algebra $H_n(q)$.*

Lemma 1.1. *Let χ_λ be an irreducible character of $H_n(q)$, of degree f_λ , and let C'_w be a Kazhdan–Lusztig basis element. Put $\chi = \chi_\lambda - f_\lambda \chi_{(1^n)}$. Then $\chi(q^{\ell(w)/2} C'_w)$ is a polynomial with non-negative integer coefficients. In fact, these coefficients are unimodal and symmetric about $q^{\ell(w)/2}$.*

Theorem 1.5 is proven in Section 7; the coefficients which appear have an explicit interpretation as multiplicities of irreducible composition factors in a certain infinite-dimensional tensor product of \mathfrak{sl}_n modules.

Let us now prove Lemma 1.1.

Proof. First of all, the character $\chi_{(1^n)}$, which is the sign representation of $H_n(q)$, vanishes on C'_w unless $w = 1$, because the Kazhdan–Lusztig polynomials $P_{v,w}$ obey $P_{v,w} = P_{sv,w}$ for any simple reflection s such that $sw < s$. Since $C'_1 = T_1 = 1$, we have $\chi_\lambda(C'_1) = f_\lambda$ and $\chi_{(1^n)}(C'_1) = 1$. Thus $\chi(C'_w)$ is zero if $w = 1$ and we may replace χ with χ_λ otherwise.

Recall that the irreducible $H_n(q)$ module V_λ can be constructed as a cell representation: a quotient E_λ/F_λ of submodules $F_\lambda \subseteq E_\lambda \subseteq H_n(q)$, where each of E_λ and F_λ is the linear span of a subset of the Kazhdan–Lusztig basis $\{C'_w\}$. Thus V_λ has a Kazhdan–Lusztig basis given by those elements C'_u which are in E_λ but not F_λ , and $a \in H_n(q)$ acts by a matrix whose (u, v) entry is the coefficient of C'_u in aC'_v . In particular, when $a = q^{\ell(w)/2} C'_w$ the diagonal entries (whose sum is the trace $\chi_\lambda(q^{\ell(w)/2} C'_w)$) are the structure coefficients $q^{\ell(w)/2} f_{w,u}^u(q)$, which are polynomials with the required non-negativity, symmetry and unimodality properties. \square

In the same way that Theorem 1.4 follows from Theorem 1.5 and Lemma 1.1, it is clear that Conjecture 1.4 (and the weaker Conjecture 1.3) would follow from Theorem 1.5 and a stronger version of Lemma 1.1 in which monomial characters replace the characters χ . Such a strengthening of Lemma 1.1 is precisely our first Hecke algebra conjecture, Conjecture 2.1.

In fact, not only the conjectures on Jacobi–Trudi matrices but also Conjecture 1.2 on totally positive matrices would follow from Conjecture 2.1—see the remarks at the end of the next section.

2. A FIRST CONJECTURE

Since the irreducible characters of $H_n(q)$ correspond to those of $\mathbb{C}S_n = H_n(1)$, the same is true of virtual characters. In particular, the monomial characters ϕ_λ of S_n defined by (1.2) have unique analogs for $H_n(q)$, specializing to (1.2) for $q = 1$. Our first conjecture, which is natural to make in view of the proof of Theorem 1.4 via Theorem 1.5 and Lemma 1.1, concerns evaluations of these monomial characters.

Conjecture 2.1. *For every monomial character ϕ_λ of $H_n(q)$ and every Kazhdan–Lusztig basis element C'_w , $\phi_\lambda(q^{\ell(w)/2}C'_w)$ is a polynomial with non-negative integer coefficients. Moreover, these coefficients are unimodal and symmetric about $q^{\ell(w)/2}$.*

Note that the symmetry of the coefficients about $q^{\ell(w)/2}$ holds for every virtual character. This follows from Lemma 1.1, or, to give a more elementary reason, from the invariance of C'_w under the Hecke algebra involution defined in (8.5).

We shall discuss four broad aspects of Conjecture 2.1 here. More detailed information can be found with the supporting evidence in Sections 4–6.

Geometry. By definition, the homogeneous symmetric function h_n is the sum $\sum_{|\lambda|=n} m_\lambda$ of all monomials. Hence the sum $\sum_{|\lambda|=n} \phi_\lambda$ is the character corresponding to $h_n = s_{(n)}$, i.e., the ‘trivial’ character $\chi_{(n)}(T_w) = q^w$. Applying this to C'_w we obtain

$$\sum_{|\lambda|=n} \phi_\lambda(q^{\ell(w)/2}C'_w) = \sum_{v \leq w} P_{v,w}(q)q^v. \quad (2.3)$$

The expression on the right is the Poincaré series for the global intersection homology of the Schubert variety Γ_w associated to w .

The intersection homology decomposition theorem of [3] gives, for a projective algebraic map $f: Y \rightarrow X$ of complex projective varieties, a natural direct sum decomposition of the intersection homology of Y . The Poincaré series of the summands have the symmetry and unimodality properties we want for $\phi_\lambda(q^{\ell(w)/2}C'_w)$. It is to be expected that (2.3) reflects some such decomposition, where Y is either the Schubert variety Γ_w or some related variety.

Best possible statement. Conjectures 1.2 and 1.4 are best possible, in that by appropriate choices of totally positive matrix A or Jacobi–Trudi matrix $H_{\mu/\nu}$, one can see that any immanant with the required non-negativity property must be a non-negative linear combination of monomial immanants. It follows that any virtual character with the properties asserted for irreducible characters in Lemma 1.1 and for monomial characters in Conjecture 2.1 is necessarily a non-negative linear combination of monomial characters. This can also be seen from the evaluation of $\phi_\lambda(q^{\ell(w)/2}C'_w)$ given by Proposition 4.1 for w the longest element of a parabolic subgroup of S_n .

Total positivity. Stembridge [32] proves Theorem 1.2 by associating to a matrix A an element $[A]$ of the group algebra $\mathbb{C}S_n$, in a manner analogous to the definition (1.5) of $I_{\mu/\nu}$. The main part of his proof is a demonstration that for totally positive A , $[A]$ is a non-negative linear combination of certain group algebra elements on which the irreducible characters take non-negative values. In fact, if we let $\sum H$ denote the sum of all elements in a subgroup $H \subseteq S_n$, his proof shows that $[A]$ is a sum of products of elements of the form $\sum W_J$ for various parabolic subgroups W_J . It follows that $[A]$ is a non-negative linear combination of Kazhdan–Lusztig basis elements $C'_w(1)$, and therefore Conjecture 2.1 implies Conjecture 1.2.

Other Coxeter groups. There is probably an analog of Conjecture 2.1 for other Coxeter groups, or at least Weyl groups. At present, however, it is not apparent what the analog of a monomial character should be. Nor do we even know that the analog of Lemma 1.1 for irreducible characters holds, since in general the cell representations are reducible.

3. A SECOND CONJECTURE

Whereas Conjecture 2.1 appears natural in view of the proof of Theorem 1.4, our second conjecture arose completely unexpectedly from examining data computed to verify Conjecture 2.1. The conjecture applies to all characters and could in principle have been discovered independently of any computation of monomial characters. It is a bit of good fortune, however, that because Conjecture 2.1 is best possible, tables of monomial characters are full of zeroes and small, highly recognizable polynomials, making linear relationships stand out. This, together with the fact that the co-dominant permutations defined below seem to play a special role in immanant inequalities, is what made it possible to detect the second conjecture by ‘inspection’.

To state the conjecture, we must define a certain class of permutations which we call *co-dominant* because they are of the form w_0v , where v is one of Lascoux’s *dominant vexillary* permutations [20], and w_0 is the longest permutation in S_n .

Definition . Let $f(1) \leq f(2) \leq \dots \leq f(n)$ be a non-decreasing sequence of integers with $f(n) = n$ and $f(i) \geq i$ for all i . Let w_f be the lexicographically greatest permutation satisfying $w_f(i) \leq f(i)$ for all i . Such a permutation w_f is called *co-dominant*.

Proposition 3.1. *Let w_f be co-dominant. Then w_f determines f by the rule $f(i) = \max\{w_f(j) \mid j \leq i\}$. We have $\{v \mid v(i) \leq f(i) \forall i\} = \{v \mid v \leq w_f\}$, where \leq is Bruhat order. The permutations $v \leq w_f$ may be described as those whose permutation matrices are zero in a pattern of entries forming a Ferrers diagram in the lower-left corner of the matrix. With respect to a fixed base flag $0 \subset F_1 \subset F_2 \subset \dots \subset F_n = \mathbb{C}^n$ in the flag variety \mathcal{F}_n , the Schubert variety Γ_{w_f} consists of all flags $0 \subset G_1 \subset G_2 \subset \dots \subset G_n = \mathbb{C}^n$ satisfying $G_i \subseteq F_{f(i)}$ for all i . This variety is smooth. In particular, $P_{v,w_f} = 1$ for all $v \leq w_f$, or equivalently*

$$q^{\ell(w_f)/2} C'_{w_f} = \sum_{v \leq w_f} T_v. \quad (3.1)$$

Proof. Clearly $w_f(1), w_f(2), \dots$ can be obtained writing down in turn for each i the greatest number between 1 and $f(i)$ which has not yet been used. In particular, for i such that $f(i-1) < f(i)$ (and for $i=1$) we have $w_f(i) = f(i)$. The rule for recovering f follows immediately.

Suppose $v(i) \leq f(i)$ for all i . Suppose $v(j) > v(k)$ for some $j < k$, and let $v' = v \cdot (j\ k)$. Then clearly $v'(i) \leq f(i)$ for all i . This shows that the set $\{v \mid v(i) \leq f(i) \forall i\}$ is closed going downward in Bruhat order, hence contains $\{v \mid v \leq w_f\}$.

Again suppose $v(i) \leq f(i)$ for all i , and suppose $v \neq w_f$. Let j be the least index for which $v(j) \neq w_f(j)$. Let $k = v^{-1}(w_f(j))$, so $v(k) = w_f(j)$. Since $v(i) = w_f(i) \neq w_f(j)$ for $i < j$, we have $k > j$. Since w_f is lexicographically greater than v , $v(j) < w_f(j) = v(k)$. Hence $v' = v \cdot (j\ k) > v$ in Bruhat order. Again we have $v'(i) \leq f(i)$ for all i . We may repeatedly replace v by v' until we reach w_f , showing $v \leq w_f$ and proving $\{v \mid v(i) \leq f(i) \forall i\} = \{v \mid v \leq w_f\}$.

The permutation matrix description merely restates the condition $v(i) \leq f(i)$ for all i .

The description of Γ_{w_f} is a direct translation of the condition $v(i) \leq f(i)$ for all i into Schubert conditions on the flag. Consider the variety Γ^k of partial flags $0 \subset G_1 \subset G_2 \subset \cdots \subset G_k$ with $\dim(G_i) = i$ satisfying $G_i \subseteq F_{f(i)}$ for all i . We see that Γ^{k+1} is a bundle over Γ_k with fiber $\mathbb{P}^{f(k+1)-k-1}(\mathbb{C})$. Hence each Γ^k is smooth, including $\Gamma^n = \Gamma_{w_f}$. The remaining statements follow immediately. \square

We remark that w is co-dominant if and only if there are no $i < j < k$ with $w(j) < w(k) < w(i)$. The number of these in S_n is well known to be the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Now we may state our conjecture.

Conjecture 3.1. *Write $a \sim b$ if $a, b \in H_n(q)$ are such that $\chi(a) = \chi(b)$ for every Hecke algebra character χ . Then for every $w \in S_n$ there exist co-dominant permutations w_1, \dots, w_k such that $C'_w \sim C'_{w_1} + \cdots + C'_{w_k}$. The number k of these is given by $P_{1,w}(1)$, and more precisely,*

$$P_{1,w}(q) = \sum_{i=1}^k q^{(\ell(w) - \ell(w_i))/2}. \quad (3.2)$$

Proof of (3.2), assuming what goes before. Consider the character χ corresponding to the action of $H_n(q)$ on the complex vector space with basis consisting of the points of the finite ‘flag variety’ over $GL_n(q)$, for q a prime power. We have $\chi(T_1) = [n]_q! = 1(1+q) \cdots (1+q+\cdots+q^{n-1})$ and $\chi(T_v) = 0$ for $v \neq 1$. Hence $a \sim b$ implies that a and b contain T_1 with the same coefficient. Applying this to $q^{\ell(w)/2} C'_w \sim \sum_i q^{(\ell(w) - \ell(w_i))/2} q^{\ell(w_i)/2} C'_{w_i}$, using $P_{1,w_i} = 1$, yields (3.2). \square

We have several remarks to make about Conjecture 3.1.

Reduction. If Conjecture 3.1 holds, it clearly reduces Conjecture 2.1 to the case that w is co-dominant.

Geometry. Like Conjecture 2.1, Conjecture 3.1 seems to suggest some sort of direct sum decomposition of perverse sheaf theoretic objects. However, the picture is much murkier. For Conjecture 2.1, one can envision the kind of set-up that might account for the character values, even if it is hard to guess exactly what varieties and maps are involved. It is not at all evident how to account for the decomposition given by Conjecture 3.1. Even how to give a geometric interpretation to the relation \sim is far from clear. Our feeling is that Conjecture 3.1 reflects deeper phenomena than Conjecture 2.1.

In particular, there are many smooth Schubert varieties corresponding to non-co-dominant permutations. Yet the conjecture suggests that each such variety should

be ‘equivalent’ in some undetermined character theoretic sense to a co-dominant one. In particular its cohomology ring must have the same Poincaré series as that of a co-dominant one. It seems to the author rather amazing that anything like this should be so.

Combinatorics of Kazhdan–Lusztig polynomials. Any rule for explicitly listing the co-dominant permutations w_1, \dots, w_k provided by the conjecture affords a combinatorial interpretation of the coefficients of $P_{1,w}$ for all w . Instances of the conjecture may be computed to seek clues to such a rule. Unfortunately the resulting information is incomplete, since the expressions $C'_w \sim C'_{w_1} + \dots + C'_{w_k}$ are in general not unique.

Since $P_{1,w}(0) = 1$, the conjectured expansion $C'_w \sim C'_{w_1} + \dots + C'_{w_k}$ must have a unique ‘leading term,’ which we may take to be C'_{w_1} , with $\ell(w_1) = \ell(w)$. It seems likely that any rule for the expansion will obey certain additional conditions (which still don’t determine it uniquely): (1) if w is co-dominant, then $w_1 = w$; (2) more generally, if $w \leq w'$ with w' co-dominant, then $w_1 \leq w'$; and (3) $w_i \leq w_1$ for all i .

One might expect the leading w_1 to depend on w in a Bruhat-order-preserving fashion, but that is incompatible with (1). Consider $w = 3412$ in S_4 . For this w , w_1 must be 3241 or 2431. But 3214 and 1432 are co-dominants less than w and neither candidate for w_1 is greater than both of these.

More immanant conjectures. Stanley and Stembridge [29] offered conjectures which reduce Conjecture 1.4 to special cases associated with co-dominant permutations. Restated in the language used here, their [29, Conjecture 5.1] becomes:

Conjecture 3.2. ([29]) *Define $E_{\mu/\nu}^\kappa \in H_n(1)$ by expanding $I_{\mu/\nu}$ with respect to Schur functions:*

$$I_{\mu/\nu} = \sum_{|\kappa|=|\mu|+|\nu|} E_{\mu/\nu}^\kappa s_\kappa(X). \quad (3.3)$$

Then there exist co-dominant permutations w_1, \dots, w_k such that $E_{\mu/\nu}^\kappa \sim C'_{w_1}(1) + \dots + C'_{w_k}(1)$.

A further conjecture in the same paper ([29, Conjecture 5.4]) refines the above statement with some conditions relating the terms in the expansion $C'_{w_1}(1) + \dots + C'_{w_k}(1)$ to Young tableaux.

Conjecture 3.2 would follow from Conjecture 3.1, since by Theorem 1.5, $E_{\mu/\nu}^\kappa$ is a non-negative linear combination of Kazhdan–Lusztig basis elements. One of the conditions which the more refined conjecture [29, Conjecture 5.4] imposes on the terms is that each $w_i \leq w_f$, where w_f is the largest co-dominant avoiding the pattern of zeros in the Jacobi–Trudi matrix $H_{\mu/\nu}$. This would follow from the conditions proposed above on the rule for the expansion $C'_w \sim C'_{w_1} + \dots + C'_{w_k}$. We do not see how to account for the other conditions in their conjecture.

4. THE GENERIC CASE

In this section we discuss special cases of the conjectures which hold when q is regarded as an indeterminate.

Both Conjectures 2.1 and 3.1 have been verified by computer for $n \leq 7$. Table 1 gives the monomial character values $\phi_\lambda(q^{\ell(w)/2} C'_w)$ and expansions $C'_w \sim C'_{w_1} + C'_{w_2}$ with w_i co-dominant for $w \in S_4$. Note that $a \sim b$ is equivalent to $\phi_\lambda(a) = \phi_\lambda(b)$ for all λ . Thus equivalences such as $C'_{1243} \sim C'_{1324} \sim C'_{2134}$ (showing that the expansion by co-dominants is non-unique) can be read off at once from the table.

In the following proposition we collect those evaluations $\phi_\lambda(q^{\ell(w)/2}C'_w)$ which can be obtained readily from well known properties of cell representations and Hecke algebra characters.

Proposition 4.1. (1) Let $W_J = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ be a parabolic subgroup of $W = S_n$. Let $w_J \in W_J$ be its longest element. Then

$$\phi_\mu(q^{\ell(w_J)/2}C'_{w_J}) = \begin{cases} \sum_{w \in W_J} q^w = [\lambda_1]_q! \cdots [\lambda_k]_q!, & \mu = \lambda \\ 0, & \mu \neq \lambda. \end{cases} \quad (4.1)$$

(2) Let w be an arbitrary element of $W_J = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$, and write it as $w = w_1 w_2 \cdots w_k$ with $w_i \in S_{\lambda_i}$. Then

$$\phi_\mu(q^{\ell(w)/2}C'_w) = \sum_{\substack{\mu^{(1)} \cup \cdots \cup \mu^{(k)} = \mu \\ |\mu^{(i)}| = \lambda_i}} \prod_{i=1}^k \phi_{\mu^{(i)}}(q^{\ell(w_i)/2}C'_{w_i}). \quad (4.2)$$

(3) Let λ be the shape of the tableaux $P(w), Q(w)$ associated with w by the Schensted correspondence; λ' the conjugate partition. Then $\phi_\mu(q^{\ell(w)/2}C'_w) = 0$ unless $\mu \geq \lambda'$ in dominance order.

Proof. We begin with (2). Formula (4.2) follows from the fact that $q^{\ell(w)/2}C'_w = q^{\ell(w_1)/2}C'_{w_1} \cdots q^{\ell(w_k)/2}C'_{w_k}$, together with the rule for restriction of $H_n(q)$ characters to $H_{\lambda_1}(q) \otimes \cdots \otimes H_{\lambda_k}(q)$ (the branching coefficients do not involve q so are the same as for S_n characters).

Given (2), (1) reduces to the case $W_J = S_n$, $w_J = w_0$. The single element C'_{w_0} spans a two-sided ideal in $H_n(q)$, the Kazhdan–Lusztig cell affording the ‘trivial’ one-dimensional representation $V_{(n)}$ in which T_w maps to q^w . Hence C'_{w_0} acts as 0 in every cell representation except $V_{(n)}$, where it acts as $\sum_{w \in S_n} q^w = [n]_q!$. This is (4.1) for $W_J = S_n$.

For (3), recall that the Kazhdan–Lusztig cells for S_n are given by the Schensted correspondence, so that C'_w belongs to the two-sided cell indexed by the partition λ' . Since the ordering $<_{LR}$ among two-sided cells is opposite to dominance order among partitions, the two-sided ideal HC'_wH in $H = H_n(q)$ is spanned by elements C'_v belonging to cells indexed by $\mu \geq \lambda'$. In particular, C'_w acts as 0 in the cell representation V_μ unless $\mu \geq \lambda'$. Since the monomial characters and the irreducible characters are dominance-lower-triangular with respect to one another, this implies (3). \square

Next we shall prove Conjecture 2.1 in the case $w = s_1 \cdots s_k$, where s_1, \dots, s_k are distinct simple reflections (*i.e.*, adjacent transpositions in S_n). By (4.2) we may assume that s_1, \dots, s_k are *all* the simple reflections.

Proposition 4.2. Let

$$\psi(a) = \sum_{|\lambda|=n} \phi_\lambda(a) h_\lambda(X) = \sum_{|\lambda|=n} \chi_\lambda(a) s_\lambda(X) \quad (4.3)$$

for any $a \in H_n(q)$. Let w_n be a Coxeter element of S_n , that is, the product of all $n - 1$ simple reflections, in any order. Then $\psi(q^{(n-1)/2}C'_{w_n})$ depends only on n , and we have the generating function

$$1 + \sum_{n \geq 1} \psi(q^{(n-1)/2}C'_{w_n}) = \frac{\sum_{n \geq 0} h_n(X)}{1 - q \sum_{n \geq 2} (1 + q + \cdots + q^{n-2}) h_n(X)}. \quad (4.4)$$

Proof. Write w_n as $s_1 \cdots s_{n-1}$. The elements $v \leq w_n$ in Bruhat order correspond to subsets $I = \{i_1 < \cdots < i_j\} \subseteq \{1, \dots, n-1\}$, with $v = s_{i_1} \cdots s_{i_j}$. For all such v , $P_{v, w_n} = 1$, or

$$q^{(n-1)/2} C'_{w_n} = \sum_I T_{s_{i_1} \cdots s_{i_j}}. \quad (4.5)$$

This follows by induction from the formula (8.8) for $C'_s C'_w$.

Ram [24] has evaluated $\psi(T_v)$ for T_v of the form appearing in (4.5), obtaining

$$\psi(T_{s_{i_1} \cdots s_{i_j}}) = \frac{q^n}{(q-1)^{\ell(\mu)}} h_\mu((1-q^{-1})X). \quad (4.6)$$

Here $\mu = (j_1, j_2 - j_1, \dots, n - j_k)$ is the composition of n determined by $\{j_1 < \cdots < j_k\} = \{1, \dots, n\} \setminus I$; equivalently it describes the cycle structure of the permutation $s_{i_1} \cdots s_{i_j}$. The expression $h_\mu((1-q^{-1})X)$ on the right hand side means $\prod_{i=1}^{k+1} h_{\mu_i}((1-q^{-1})X)$, where $h_m((1-q^{-1})X)$ is the symmetric polynomial, homogeneous of degree m in X , defined by

$$\sum_{m \geq 0} h_m((1-q^{-1})X) = \prod_i \frac{1-q^{-1}x_i}{1-x_i}. \quad (4.7)$$

Combining (4.5) and (4.6) leads to the generating function

$$\begin{aligned} 1 + \sum_{n \geq 1} \psi(q^{(n-1)/2} C'_{w_n}) &= \frac{1}{1 - \frac{1}{q-1} \sum_{k \geq 1} q^k h_k((1-q^{-1})X)} \\ &= \frac{1}{1 - \frac{1}{q-1} (H(qX)/H(X) - 1)} \\ &= \frac{H(X)}{H(X) - \frac{1}{q-1} (H(qX) - H(X))}, \end{aligned} \quad (4.8)$$

where $H(X) = \sum_m h_m(X) = \prod_i (1-x_i)^{-1}$. The final formula in (4.8) is just another way of writing (4.4), so the proof is complete. \square

Proposition 4.2 establishes Conjecture 2.1 for $w = w_n$. To see this, rewrite (4.4) as

$$1 + \frac{\sum_{n \geq 1} (1+q+\cdots+q^{n-1})h_n(X)}{1 - \sum_{n \geq 2} (q+q^2+\cdots+q^{n-1})h_n(X)}. \quad (4.9)$$

In this form it is easy to see that the coefficient of h_λ is a polynomial in q with non-negative coefficients, unimodal and symmetric about $q^{(|\lambda|-1)/2}$.

The generating function (4.4) is an interesting one which has arisen in other contexts [5], [28], [31]. Specifically, as explained in [28], associated with the Coxeter complex of S_n is a smooth toric variety whose homology groups carry an S_n action. Let $P_\lambda(q)$ be the polynomial whose coefficients are the multiplicities of V_λ in these homology groups. Then $\sum_\lambda P_\lambda(q) s_\lambda$ turns out to be given by (4.4). In other words,

$$P_\lambda(q) = \chi_\lambda(q^{\ell(w_n)/2} C'_{w_n}). \quad (4.10)$$

No direct explanation of (4.10) seems to be known, nor whether the corresponding result holds for every Coxeter group. It is known that $\sum_\lambda f_\lambda P_\lambda(q) = \sum_{w \in W} q^{|D(w)|}$ for every Coxeter group W , where $|D(w)|$ is the number of descents of w , *i.e.*, the number of simple reflections s such that $sw < w$. This follows from the fact that there is a W -equivariant isomorphism between the homology of the toric variety

and a suitable quotient of the Stanley–Reisner ring of the Coxeter complex. It is not hard to show that (4.10) agrees with this.

We close this section with a conjectured exact value for certain $\phi_\lambda(w)$.

Conjecture 4.1. *Let w_f be co-dominant (f as in Proposition 3.1). Then*

$$\phi_{(n)}(q^{\ell(w_f)/2} C'_{w_f}) = [n]_q [f(1) - 1]_q [f(2) - 2]_q \cdots [f(n-1) - (n-1)]_q, \quad (4.11)$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$.

This conjecture agrees with Propositions 4.1 and 4.2 in the cases where they overlap. Its $q = 1$ specialization will be proven in the next section.

5. THE CASE $q = 1$

The case of Conjecture 2.1 for $q = 1$ and $w = w_f$ co-dominant is equivalent to a special case of Conjecture 1.4. To see this, choose μ and ν such that the 0 entries of $H_{\mu/\nu}$ form the forbidden pattern describing permutations $v \leq w_f$ (such μ and ν can always be found). Then the term $T_v h_{\mu-\nu(\nu)}$ in (2.1) is non-zero exactly for $v \leq w_f$, and the coefficient of $s_{(N)}$ in $I_{\mu/\nu}$ is $C'_{w_f}(1)$, where $N = |\mu| - |\nu|$. Hence Conjecture 1.4 implies Conjecture 2.1.

Stanley and Stembridge [29] proved Conjecture 1.4 in two cases. First, they proved it when μ/ν is a ‘rim hook’, corresponding to the n -cycle $w_f = (1\ 2\ \cdots\ n)$. Their result in this case is equivalent to the $q = 1$ specialization of Proposition 4.1.

Second, they proved the part we need concerning the coefficient of $s_{(N)}$ in the case that the forbidden pattern is not too big, which leads to the following corollary.

Proposition 5.1. ([29]) *Let w_f be co-dominant, let k be the least value for which $f(k) = n$, and assume $f(1) \geq k - 1$. Then $\phi_\lambda(C'_{w_f}(1)) \geq 0$ for all λ . \square*

Conjecture 4.1 can also be proven for $q = 1$, as follows.

Proposition 5.2. *Let w_f be co-dominant. Then*

$$\phi_{(n)}(C'_{w_f}(1)) = n(f(1) - 1) \cdots (f(n-1) - (n-1)). \quad (5.1)$$

Proof. Let V be the pattern avoided by ones in permutation matrices corresponding to $v \leq w_f$. This pattern V consists of the last $n - f(i)$ entries in column i for each i . On S_n , the monomial character $\phi_{(n)}(w)$ is n if w is an n -cycle and 0 otherwise. What (5.1) asserts, therefore, is that the number of n -cycles whose permutation matrix avoids V is given by $(f(1) - 1)(f(2) - 2) \cdots (f(n-1) - (n-1))$.

Let r_k be the number of ways to place k mutually non-attacking rooks on a chess board of shape V , that is, the number of subsets of V in which no two elements share a row or a column. Consider a fixed such ‘rook placement.’ Permutations in S_n whose matrix is 1 at the positions of all the rooks correspond in a fairly obvious bijective way to permutations in S_{n-k} , with n -cycles corresponding to $(n-k)$ -cycles. It follows by the sieve principle that the number of n -cycles avoiding V is $\sum_k (-1)^k r_k (n-k-1)!$.

By the sieve principle we also find that the total number of permutations avoiding V is $\sum_k (-1)^k r_k (n-k)!$. Thus the number of n -cycles avoiding V is the same as the number of all permutations avoiding a pattern of the same shape as V in an $(n-1) \times (n-1)$ matrix, which is easily seen to be $(f(1) - 1)(f(2) - 2) \cdots (f(n-1) - (n-1))$. \square

6. THE CASE $q = 0$

In this section we calculate $\phi_\lambda(q^{\ell(w)/2}C'_w)$ for all λ and w , with $q = 0$. The calculation confirms Conjectures 2.1, 3.1 and 4.1 for the specialization $q = 0$.

The Kazhdan–Lusztig basis elements C'_w are not well-defined for $q = 0$, but $q^{\ell(w)/2}C'_w$ is well-defined and equal to

$$C_w^0 = \sum_{v \leq w} T_v, \quad (6.1)$$

since $P_{v,w}(0) = 1$ for all $v \leq w$.

The irreducible and monomial ‘characters’ aren’t properly named for $q = 0$, since $H_n(0)$ is not a semi-simple algebra. Nevertheless they are well-defined by specialization from $H_n(q)$, and in particular they may be computed from the W -graph form of the cell representations by setting $q = 0$.

Proposition 6.1. *The monomial characters ϕ_λ of $H_n(0)$ take the values*

$$\phi_\lambda(C_w^0) = \begin{cases} 1, & W_{J(w)} \cong S_\lambda \\ 0, & \text{otherwise.} \end{cases} \quad (6.2)$$

Here $J(w)$ is the set of simple reflections appearing in a reduced expression for w , $W_{J(w)}$ is the corresponding parabolic subgroup of S_n (the smallest parabolic subgroup containing w), and $W_{J(w)} \cong S_\lambda$ means $W_{J(w)}$ is a subgroup of the form $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ with the factors in any order.

Remark. Conjecture 2.1 requires that $\phi_\lambda(C_w^0) = 1$ for one λ and $\phi_\lambda(C_w^0) = 0$ for all others, since by (2.3) the sum over all λ is 1. Conjecture 4.1, taken together with Proposition 4.1, requires that for co-dominant w , the λ with $\phi_\lambda(C_w^0) = 1$ is the one given by Proposition 6.1. This is compatible with Conjecture 3.1, and indicates that the leading co-dominant w_1 in the expansion of C'_w should have $J(w_1) = J(w)$.

We need a simple lemma for the proof of Proposition 6.1.

Lemma 6.1. *Let I be a set of simple reflections such that $I \cap J(w) \neq \emptyset$. Then*

$$\sum_{\substack{v \leq w \\ v \in W_I}} (-1)^{\ell(v)} = 0. \quad (6.3)$$

Proof. Since w cannot be 1, the left descent set $D(w)$ is non-empty. Let $t \in D(w)$. If $t \in I$, then $v \leq w \Leftrightarrow tv \leq w$ and $v \in W_I \Leftrightarrow tv \in W_I$. This implies (6.3).

If $t \notin I$, we proceed by induction on $\ell(w)$. We have $\ell(tw) < \ell(w)$ and $J(w) \subseteq J(tw) \cup \{t\}$, which implies $J(tw) \cap I \neq \emptyset$. By induction, the sum of $(-1)^{\ell(v)}$ over $W_I \cap \{v \mid v \leq tw\}$ is zero. But $W_I \cap \{v \mid v \leq tw\} = W_I \cap \{v \mid v \leq w\}$, for if we fix a reduced expression tU for w then either set consists of all v having reduced expressions which use only simple reflections in I and are subwords of U . \square

Proof of Proposition 6.1. By Möbius inversion on the Bruhat order,

$$T_w = \sum_{v \leq w} (-1)^{\ell(vw)} C_w^0. \quad (6.4)$$

Hence the proposition is equivalent to

$$\phi_\lambda(T_w) = \sum_{\substack{v \leq w \\ W_{J(v)} \cong S_\lambda}} (-1)^{\ell(vw)}, \quad (6.5)$$

or

$$\begin{aligned} \chi_\lambda(T_w) &= \sum_{\mu} K_{\lambda,\mu} \sum_{\substack{v \leq w \\ W_{J(v)} \cong S_\mu}} (-1)^{\ell(vw)} \\ &= \sum_{v \leq w} (-1)^{\ell(vw)} K_{\lambda,\mu(v)}, \end{aligned} \tag{6.6}$$

where $\mu(v)$ is the type of the parabolic subgroup $W_{J(v)}$, and $s_\lambda = \sum_{\mu} K_{\lambda,\mu} m_\mu$.

The ‘Kostka number’ $K_{\lambda,\mu(v)} = \langle s_\lambda, h_{\mu(v)} \rangle$ has a combinatorial description: it is the number of standard Young tableaux T of shape λ whose descent set $D(T)$ is disjoint from $J(v)$. Hence the last expression in (6.6) is equal to

$$\begin{aligned} \sum_{T \in \text{SYT}(\lambda)} \sum_{\substack{v \leq w \\ v \in W_{S \setminus D(T)}}} (-1)^{\ell(vw)} &= (-1)^{\ell(w)} |\{T \in \text{SYT}(\lambda) \mid D(T) \supseteq J(w)\}| \\ &= (-1)^{\ell(w)} K_{\lambda',\mu(w)} \end{aligned} \tag{6.7}$$

by Lemma 6.1 and the fact that the inner sum has only the term $v = 1$ for $D(T) \supseteq J(w)$. $\text{SYT}(\lambda)$ stands for the set of standard Young tableaux of shape λ .

Now we prove $\chi_\lambda(T_w) = (-1)^{\ell(w)} K_{\lambda',\mu(w)}$ for $q = 0$. Recall that the cell representation V_λ can be regarded as spanned by the vertices of a W -graph G_λ . These vertices are indexed by tableaux $T \in \text{SYT}(\lambda)$, and each is marked with the descent set $D(T)$. The edges are labeled with certain integers $\tilde{\mu}(S, T)$ which are irrelevant for us.

The element $T_s \in H_n(q)$ acts on V_λ as follows. If $s \in D(T)$, then $T_s(T) = -T + q^{1/2} \sum_{s \notin D(S)} \tilde{\mu}(S, T) S$. If $s \notin D(T)$, then $T_s(T) = qT$. Thus for $q = 0$, the matrix of T_w is diagonal, with (T, T) entry $(-1)^{\ell(w)}$ if $w \in W_{D(T)}$ and zero otherwise. It is obvious that the trace of this matrix is given by (6.7). \square

7. PROOF OF THE MAIN THEOREM OF SECTION 1

In this section we prove Theorem 1.5, using a generalization of the Steinberg tensor product theorem to infinite-dimensional irreducible highest weight modules over a simple complex Lie algebra (Proposition 7.2). The derivation of this generalized Steinberg theorem is parallel to the usual derivation in the finite-dimensional case (see, *e.g.*, [11]), with the role of the Weyl character formula played by the Kazhdan–Lusztig conjecture.

We require the following notation. Let \mathfrak{g} be a finite-dimensional simple complex Lie algebra with Weyl group W (for our application, $\mathfrak{g} = \mathfrak{sl}_n$ and $W = S_n$). As usual, δ denotes one-half the sum of the positive roots. Let M_λ denote the Verma module with highest weight λ , L_λ the corresponding irreducible module (infinite-dimensional for non-dominant λ).

Because we must consider tensor products of infinite-dimensional modules, we shall work with a class of \mathfrak{g} -modules somewhat larger than the commonly used Bernstein–Gel’fand–Gel’fand category \mathcal{O} . Thus we define the class \mathcal{C} to contain a \mathfrak{g} -module V if:

- (1) V is a direct sum $V = \bigoplus_{\lambda} V_{\lambda}$ of weight spaces belonging to integral weights λ .
- (2) For every integer N , $\bigoplus_{\langle \lambda, \delta \rangle \geq N} V_{\lambda}$ is finite-dimensional.

The class \mathcal{C} contains the Verma modules M_λ and the irreducible modules L_λ . The *formal character* of $V \in \mathcal{C}$ we define by $\text{ch}(V) = \sum_\lambda \dim(V_\lambda)x^\lambda$, where the symbols x^λ denote formal exponentials, multiplying by the rule $x^\lambda x^\mu = x^{\lambda+\mu}$.

Lemma 7.1. *Let $W \subseteq V$ be a submodule of $V \in \mathcal{C}$. Then W and V/W belong to \mathcal{C} , and we have $\text{ch}(V) = \text{ch}(W) + \text{ch}(V/W)$. If in addition $U \in \mathcal{C}$, then $U \otimes V \in \mathcal{C}$, and we have $\text{ch}(U \otimes V) = \text{ch}(U)\text{ch}(V)$.*

Proof. For the first part, everything is clear, provided we know that W is the direct sum of its weight spaces. But every element $a \in W$ belongs to V , and hence $a \in V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}$ for a finite set of weights $\{\lambda_i\}$. This space is a finite-dimensional module for the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, and so is its intersection with W . Since every such module is a direct sum of weight spaces, each weight space component of a is again in W , so $W = \bigoplus_\lambda W_\lambda$.

For the second part, if we let $\{a_\alpha\}$ and $\{b_\beta\}$ be bases of weight vectors for U and V respectively, then $\{a_\alpha \otimes b_\beta\}$ is a basis of weight vectors for $U \otimes V$. The properties asserted for $U \otimes V$ follow easily. \square

Modules in \mathcal{C} (even in \mathcal{O}) are not necessarily direct sums of irreducibles, but for formal computations with multiplicities we can get by with the weaker notion of ‘composition series,’ which we define as follows.

Definition . A *composition series* of a module $V \in \mathcal{C}$ is a finite or infinite sequence

$$0 = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \quad (7.1)$$

of submodules $W_i \subseteq V$, with $\bigcup_i W_i = V$ and W_{i+1}/W_i an irreducible highest weight module L_{λ_i} for each i .

Note that the weight spaces $(W_i)_\lambda$ form a filtration of the finite-dimensional space V_λ , so only a finite number of the quotients $(W_{i+1})_\lambda/(W_i)_\lambda$ are non-zero for any given λ . Thus $\sum_{i=0}^\infty \text{ch}(W_{i+1}/W_i)$ is well-defined and equal to $\text{ch}(V)$.

Lemma 7.2. *Every $V \in \mathcal{C}$ has a composition series. The number $m_\lambda(V)$ of indices i for which $W_i/W_{i+1} \cong L_\lambda$ is finite and is uniquely determined by the equation*

$$\sum_\lambda m_\lambda(V) \text{ch}(L_\lambda) = \text{ch}(V). \quad (7.2)$$

In particular $m_\lambda(V)$ does not depend on the choice of composition series.

Proof. Obviously we may assume $V \neq 0$.

Let λ be a weight for which $V_\lambda \neq 0$ and $\langle \lambda, \delta \rangle$ is maximal. Such a weight clearly exists for $V \in \mathcal{C}$. Choose a nonzero $a \in V_\lambda$ and let V_a be the submodule generated by a . Since a is a highest weight vector, V_a is a quotient of the Verma module M_λ and therefore has a finite composition series

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_i = V_a.$$

If we have constructed a partial composition series

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k \neq V$$

for V , we may apply the above construction to a maximal weight vector a in V/W_k to obtain an extension

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k \subseteq \cdots \subseteq W_l = W_k + V_a.$$

Repeating the process, we either reach $W_l = V$ at some stage, or we produce an infinite sequence

$$0 = W_0 \subseteq W_1 \subseteq \cdots$$

At each stage, we reduce the dimension of a non-zero weight space $(V/W_l)_\lambda$ for which $\langle \lambda, \delta \rangle$ is maximal. By condition (2) in the definition of \mathcal{C} , we must eventually exhaust every weight space, *i.e.*, $(V/W_l)_\lambda = 0$ for sufficiently large l . Hence our sequence is a composition series.

As for the multiplicities, clearly L_λ cannot occur as a quotient W_{i+1}/W_i more than $\dim(V_\lambda)$ times, so $m_\lambda(V)$ is finite. Equation (7.2) is thus well-defined and valid. The formal characters $\text{ch}(L_\lambda)$ are linearly independent, since each has a distinct leading term x^λ . Therefore (7.2) determines the multiplicities $m_\lambda(V)$. \square

Remark. The rest of our work in this section consists entirely of computations upon formal power series $f(x) = \sum_\lambda c_\lambda x^\lambda$ with integer coefficients obeying

$$(2') \text{ For every integer } N, \text{ there are only finitely many weights } \lambda \text{ with } \langle \lambda, \delta \rangle \geq N \text{ and } c_\lambda \neq 0.$$

Formal series obeying (2') form a ring, and they have unique expansions $f(x) = \sum_\lambda d_\lambda \text{ch}(M_\lambda) = \sum_\lambda e_\lambda \text{ch}(L_\lambda)$, where the coefficients d_λ and e_λ obey the same restriction (2') as c_λ . The formal series $f(x)$ is actually the formal character of some module in \mathcal{C} if and only if the coefficients e_λ are non-negative. Our need for the underlying representation theory is confined to a single consequence of it: the product of two series which are actually characters is again a character.

We now proceed to the derivation of the generalized Steinberg tensor product theorem from the Kazhdan–Lusztig conjecture.

It will be convenient to define some notation in connection with the action of W on weights. If λ is a weight and w is an element of W , define $w \cdot \lambda = w(\lambda + \delta) - \delta$. If $\lambda + \delta$ is dominant, then $W^\lambda = \{w \mid w \cdot \lambda = \lambda\}$ denotes the stabilizer of $\lambda + \delta$, a parabolic subgroup of W . Each coset wW^λ contains a unique longest element; let D^λ denote the set of these.

Definition . The expression $w \cdot \lambda$ is in *standard form* if (1) $\lambda + \delta$ is dominant, and (2) $w \in D^\lambda$.

Every weight has a unique expression in standard form.

The formal characters of Verma modules are given by

$$\text{ch}(M_\lambda) = x^\lambda / \Delta(x), \quad \text{where} \quad \Delta(x) = \prod_\alpha (1 - x^{-\alpha}), \quad (7.3)$$

the product taken over all positive roots α .

By Harish–Chandra’s theorem, $M_{v \cdot \lambda}$ has a finite composition series whose composition factors are of the form $L_{w \cdot \lambda}$. Their multiplicities are given by the Kazhdan–Lusztig conjecture:

Proposition 7.1. ([2], [6], [12]) *For $\lambda + \delta$ dominant and integral we have*

$$\text{ch}(M_{v \cdot \lambda}) = \sum_{w \in D^\lambda} P_{v,w}(1) \text{ch}(L_{w \cdot \lambda}) \quad (7.4)$$

or equivalently

$$\sum_{w \in W} (-1)^{\ell(vw)} Q_{v,w}(1) \text{ch}(M_{w \cdot \lambda}) = \begin{cases} \text{ch}(L_{v \cdot \lambda}), & v \in D^\lambda \\ 0, & v \notin D^\lambda, \end{cases} \quad (7.5)$$

where $P_{v,w}$ and $Q_{v,w} = P_{w_0v,w_0v}$ are Kazhdan–Lusztig polynomials for W .

The proofs in [2] and [6] treat the case $\lambda = 0$. The reduction of the general case to this one had already been found in [12]. The equivalence of (7.4) with (7.5) is a consequence of the fact that $[P_{v,w}]$ and $[(-1)^{\ell(vw)}Q_{v,w}]$ are inverse $|W| \times |W|$ matrices.

Proposition 7.2. (Generalized Steinberg theorem) *Let $s \cdot \alpha$, $t \cdot \beta$ and $u \cdot \gamma$ be integral weights expressed in standard form. Then the multiplicity of $L_{u \cdot \gamma}$ as a composition factor of $L_{s \cdot \alpha} \otimes L_{t \cdot \beta}$ is given by*

$$\sum_{\substack{r \in D^\gamma \\ v, w \in W}} (-1)^{\ell(stvw)} P_{r,u}(1) Q_{s,v}(1) Q_{t,w}(1) \mathbf{p}(v \cdot \alpha + w \cdot \beta - r \cdot \gamma), \quad (7.6)$$

where $\mathbf{p}(\lambda)$ is the partition function, defined by

$$\frac{1}{\Delta(x)} = \sum_{\lambda} \mathbf{p}(\lambda) x^{-\lambda}. \quad (7.7)$$

Proof. If f is a formal character, let $\langle \text{ch}(L_\lambda) \rangle f$ denote the coefficient of $\text{ch}(L_\lambda)$ in the expansion of f by irreducible characters, and let $\langle \text{ch}(M_\lambda) \rangle f$ denote the coefficient of $\text{ch}(M_\lambda)$ in its expansion by Verma module characters. By (7.4) we have, for $u \cdot \gamma$ standard,

$$\langle \text{ch}(L_{u \cdot \gamma}) \rangle f = \sum_{r \in D^\gamma} P_{r,u}(1) \langle \text{ch}(M_{r \cdot \gamma}) \rangle f. \quad (7.8)$$

We are to show that $\langle \text{ch}(L_{u \cdot \gamma}) \rangle \text{ch}(L_{s \cdot \alpha}) \text{ch}(L_{t \cdot \beta})$ is equal to (7.6). By (7.5),

$$\text{ch}(L_{s \cdot \alpha}) \text{ch}(L_{t \cdot \beta}) = \sum_{v, w \in W} (-1)^{\ell(stvw)} Q_{s,v}(1) Q_{t,w}(1) \text{ch}(M_{v \cdot \alpha}) \text{ch}(M_{w \cdot \beta}). \quad (7.9)$$

Note that $\langle \text{ch}(M_{r \cdot \gamma}) \rangle \text{ch}(M_{v \cdot \alpha}) \text{ch}(M_{w \cdot \beta}) = \langle x^{r \cdot \gamma} \rangle (x^{v \cdot \alpha + w \cdot \beta} / \Delta(x))$, by (7.3). Applying this to (7.8) with f given by (7.9) yields

$$\begin{aligned} & \langle \text{ch}(L_{u \cdot \gamma}) \rangle \text{ch}(L_{s \cdot \alpha}) \text{ch}(L_{t \cdot \beta}) \\ &= \sum_{\substack{r \in D^\gamma \\ v, w \in W}} (-1)^{\ell(stvw)} P_{r,u}(1) Q_{s,v}(1) Q_{t,w}(1) \langle x^{r \cdot \gamma} \rangle (x^{v \cdot \alpha + w \cdot \beta} / \Delta(x)) \\ &= \sum_{\substack{r \in D^\gamma \\ v, w \in W}} (-1)^{\ell(stvw)} P_{r,u}(1) Q_{s,v}(1) Q_{t,w}(1) \mathbf{p}(v \cdot \alpha + w \cdot \beta - r \cdot \gamma) \quad \square \end{aligned} \quad (7.10)$$

Observe that (7.6) reduces to zero when $s \notin D^\alpha$ or $t \notin D^\beta$. This is clear from the proof, since (7.9) reduces to zero in these cases, by (7.5).

Finally, we apply this to immanants.

Proof of Theorem 1.5. The coefficient of $C'_t(1) s_\kappa(X)$ in $I_{\mu/\nu}$ is given by

$$m_{\mu/\nu}^{t, \kappa} = \sum_{w \in S_n} (-1)^{\ell(tw)} Q_{t,w}(1) K_{\kappa, \mu - w(\nu)}, \quad (7.11)$$

where the Kostka number $K_{\kappa, \lambda}$ is by definition the coefficient of s_κ in h_λ . It is well-known that

$$K_{\kappa, \lambda} = \sum_{v \in S_n} (-1)^{\ell(v)} \mathbf{p}(v \cdot \kappa - \lambda), \quad (7.12)$$

where κ is a partition, regarded as a dominant weight for \mathfrak{sl}_n , and λ is any sequence $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $|\lambda| = |\kappa|$. Indeed, (7.12) is equivalent to the fact that $\text{ch}(L_\kappa)$ is the Schur function s_κ for κ dominant.

Combining (7.11) and (7.12) we obtain

$$m_{\mu/\nu}^{t,\kappa} = \sum_{v,w \in S_n} (-1)^{\ell(tv w)} Q_{t,w}(1) \mathfrak{p}(v \cdot \kappa + w \cdot (\nu - \delta) - (\mu - \delta)). \quad (7.13)$$

By Proposition 7.2, this is a non-negative integer, since it is (7.6) with $\alpha = \kappa$, $s = 1$, $\beta = \nu - \delta$, $t = t$, $\gamma = \mu - \delta$, and $u \in D^\gamma$ representing the coset $1W^\gamma$. Note that $\alpha + \delta$, $\beta + \delta$ and $\gamma + \delta$ are dominant as required, since they are partitions, and $u \cdot \gamma$ and $s \cdot \kappa$ are in standard form. For $t \notin D^\beta$, $t \cdot \beta$ is not standard, but then, as we observed above, (7.6) reduces to zero. \square

It is instructive to examine the case $t = 1$. Extracting the coefficient of $C'_1(1)s_\kappa(X)$ in $I_{\mu/\nu}$ amounts to finding the coefficient of $s_\kappa(X)$ in $\det(H_{\mu/\nu}) = \det(H_{\gamma+\delta/\beta+\delta}) = s_{\gamma/\beta}(X)$. The equality of this with $\langle s_\gamma, s_\kappa s_\beta \rangle$ is the defining identity for skew Schur functions. But of course $\langle s_\gamma, s_\kappa s_\beta \rangle$ is the multiplicity of L_γ in $L_\kappa \otimes L_\beta$, since $\text{ch}(L_\lambda) = s_\lambda$ for dominant λ . Thus the symmetric functions $\langle C'_w(1) \rangle_{I_{\gamma+\delta/\beta+\delta}}$ appear as rather natural generalizations of the skew Schur function $s_{\gamma/\beta}$.

It may also be worth noting that the coefficient of s_κ in $\langle C'_w(1) \rangle_{I_{\gamma+\delta/\beta+\delta}}$ is the multiplicity of L_γ in $L_\kappa \otimes L_{w \cdot \beta}$ for $w \cdot \beta$ standard, and zero otherwise. Since L_κ is finite-dimensional, this particular tensor product does belong to the category \mathcal{O} , not just to \mathcal{C} .

APPENDIX: HECKE ALGEBRA AND KAZHDAN–LUSZTIG POLYNOMIALS

Here we state the definitions and theorems we use from Kazhdan–Lusztig theory. Since this paper is meant for a varied audience, this material has been included for completeness, despite the risk that it may be unnecessary to specialists and incomprehensible to everyone else. Standard references are [4] and [17].

The symmetric group S_n is an example of a *Coxeter group*, with its distinguished generating set of *simple reflections* S consisting of the adjacent transpositions $(i \ i + 1)$. A minimum-length expression for $w \in S_n$ as a product of simple reflections is called a *reduced expression*; its length $\ell(w)$ is the number of inversions in w . The *sign* of w is $(-1)^{\ell(w)}$.

The *Bruhat order* is a partial order on S_n with the defining property that $v \leq w$ if for some, or equivalently for every, reduced expression $s_1 \cdots s_\ell$ for w , there is a subword $s_{i_1} \cdots s_{i_k}$ which is a reduced expression for v . The Bruhat order is the transitive closure of the relation $v < w$ if $v = w \cdot (i \ j)$ for some $i < j$ such that $w(i) > w(j)$.

For each $s \in S$, either $sw > w$ or $sw < w$ in Bruhat order. Those s with $sw < w$ are the (left) *descents* of w . The descent set is denoted $D(w)$. There is a reduced expression for w beginning with s if and only if $s \in D(w)$.

A subgroup $W_J \subseteq S_n$ generated by a subset $J \subseteq S$ is called *parabolic*. If $(i_1 \ i_1 + 1), (i_2 \ i_2 + 1), \dots, (i_k \ i_k + 1)$ are the simple reflections not in J , with $i_1 < \dots < i_k$, then W_J has the form $S_{\lambda_1} \times \cdots \times S_{\lambda_{k+1}}$, where $\lambda = (i_1, i_2 - i_1, \dots, n - i_k)$. Each coset wW_J contains a unique longest and Bruhat greatest element, which can be obtained by sorting each of the successive segments of length $\lambda_1, \lambda_2, \dots, \lambda_{k+1}$ in the sequence $w(1), \dots, w(n)$ into decreasing order. The longest element of S_n is denoted w_0 .

A presentation of S_n is given by the generators S and relations

$$s^2 = 1, \quad (8.1a)$$

$$\begin{aligned} rs &= sr & \text{if } |i - j| > 1, \\ rsr &= srs & \text{if } |i - j| = 1, \end{aligned} \quad (8.1b)$$

where $r = (i \ i + 1)$ and $s = (j \ j + 1)$. The *Hecke algebra* $H_n(q)$ associated to S_n is the algebra with unit over $\mathbb{C}[q^{-1/2}, q^{1/2}]$ generated by elements T_s for $s \in S$, subject to the relations

$$T_s^2 = (q - 1)T_s + q, \quad (8.2a)$$

$$\begin{aligned} T_r T_s &= T_s T_r & \text{if } |i - j| > 1, \\ T_r T_s T_r &= T_s T_r T_s & \text{if } |i - j| = 1. \end{aligned} \quad (8.2b)$$

For $w \in S_n$, we define

$$T_w = T_{s_1} \cdots T_{s_\ell}, \quad (8.3)$$

where $s_1 \cdots s_\ell$ is a reduced expression for w . This is well-defined because all reduced expressions for w are connected by relations of the form (8.1b). The elements T_w form a basis for the Hecke algebra and we have

$$T_s T_w = \begin{cases} T_{sw}, & sw > w \\ (q - 1)T_w + qT_{sw}, & sw < w. \end{cases} \quad (8.4)$$

In view of (8.1), the specialization $H_n(1)$ is the group algebra $\mathbb{C}S_n$, with $\{T_w\}$ as the natural basis. For q an indeterminate, $H_n(q)$ is a semi-simple algebra. Specializing them at $q = 1$ sets up a natural correspondence between its irreducible representations and those of S_n .

There is a \mathbb{C} -linear involution of $H_n(q)$ defined by

$$\bar{q} = q^{-1}; \quad \overline{T_w} = (T_{w^{-1}})^{-1}. \quad (8.5)$$

The *Kazhdan–Lusztig basis* $\{C'_w\}$ of $H_n(q)$ is uniquely defined by the conditions

$$\overline{C'_w} = C'_w,$$

$$q^{\ell(w)/2} C'_w = \sum_{v \leq w} P_{v,w}(q) T_v, \quad (8.6)$$

$$P_{v,w}(q) \in \mathbb{Z}[q], \quad P_{w,w}(q) = 1, \quad \deg(P_{v,w}) < (\ell(w) - \ell(v))/2 \quad \text{for } v \neq w.$$

The $P_{v,w}(q)$ are *Kazhdan–Lusztig polynomials*. They have constant term $P_{v,w}(0) = 1$ for all $v \leq w$. The inverse of (8.6) is

$$T_w = \sum_{v \leq w} (-1)^{\ell(vw)} Q_{v,w}(q) q^{\ell(v)/2} C'_v, \quad (8.7)$$

where $Q_{v,w} = P_{w_0 w, w_0 v}$.

The basis C'_w satisfies the two equivalent formulas

$$C'_s C'_w = \begin{cases} C'_{sw} + \sum_{sv < v} \mu(v, w) C'_v, & sw > w \\ (q^{-1/2} + q^{1/2}) C'_w, & sw < w, \end{cases} \quad (8.8)$$

$$T_s C'_w = \begin{cases} -C'_w + q^{1/2} \sum_{sv < v} \tilde{\mu}(v, w) C'_v, & sw > w \\ q C'_w, & sw < w, \end{cases} \quad (8.9)$$

where $\mu(v, w)$ is the coefficient of $q^{(\ell(w) - \ell(v) - 1)/2}$ in $P_{v,w}(q)$ if $\ell(w) - \ell(v)$ is odd, zero otherwise, and $\tilde{\mu}(v, w)$ is $\mu(v, w)$ or $\mu(w, v)$, accordingly as $v < w$ or $w < v$.

A reflexive and transitive relation \leq is called a *preorder*. To a preorder corresponds an equivalence relation defined by $x \approx y$ if $x \leq y \leq x$. The preorder induces a partial order on \approx -equivalence classes.

Preorders \leq_L and \leq_R are defined on S_n as follows: \leq_L (resp. \leq_R) is the weakest relation such that for all w , the linear span of $\{C'_v \mid v \leq_L w\}$ (resp. $\{C'_v \mid v \leq_R w\}$) is a left (resp. right) ideal in $H_n(q)$. The transitive closure of $\leq_L \cup \leq_R$ is denoted \leq_{LR} ; thus $\{C'_v \mid v \leq_{LR} w\}$ spans a two-sided ideal in $H_n(q)$.

The equivalence classes of \approx_L , \approx_R , and \approx_{LR} are the *left*, *right*, and *two-sided cells*, respectively. Regarding the span $\mathbb{C}[q^{-1/2}, q^{1/2}]\{C'_v \mid v \approx_L w\}$ as a quotient $\mathbb{C}[q^{-1/2}, q^{1/2}]\{C'_v \mid v \leq_L w\}/\mathbb{C}[q^{-1/2}, q^{1/2}]\{C'_v \mid v <_L w\}$ makes it a left $H_n(q)$ -module, called a *cell representation*.

Let $(P(w), Q(w))$ be the pair of standard Young tableaux (of shape $\lambda(w)$) corresponding to w under the Schensted correspondence. Then $v \approx_L w$ if and only if $Q(v) = Q(w)$, $v \approx_R w$ if and only if $P(v) = P(w)$, and $v \approx_{LR} w$ if and only if $\lambda(v) = \lambda(w)$. It is convenient to index cells by the transposed tableaux $P(v)'$, $Q(v)'$ of shape λ' . With this indexing the left cells indexed by tableaux of shape λ afford the irreducible representation V_λ , according to the usual indexing for S_n modules.

The ordering $<_{LR}$ on two-sided cells corresponds to the opposite of the dominance order on their indexing partitions. No direct statement of this fact seems to appear in the literature, but it can be deduced without much difficulty from the results of [1] and [16] on primitive ideals in $U(\mathfrak{gl}_n)$.

Using (8.9), the action of T_s in a cell representation can be described by a *W-graph*. The vertices correspond to basis elements C'_v for v in a left cell, hence to the tableaux $P(v)'$ of a fixed shape. Vertices v, w are connected by an edge labeled $\tilde{\mu}(v, w) = \tilde{\mu}(S, T)$, which depends only on the tableaux $S = P(v)'$, $T = P(w)'$. Since $D(v)$ is the complement of the descent set $D(P(v)')$, we have for $s \in D(T)$, $T_s(T) = -T + q^{1/2} \sum_{s \notin D(S)} \tilde{\mu}(S, T)S$, and for $s \notin D(T)$, $T_s(T) = qT$.

The elements C'_w are closely connected with the geometry of the *flag variety*. Let V be an n -dimensional complex vector space. The flag variety \mathcal{F}_n consists of all *flags* $0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = V$ of subspaces with $\dim(F_i) = i$. Fix a flag F^0 and let B be its stabilizer in GL_n , *i.e.*, the Borel subgroup of all upper triangular matrices. Each flag $F = gF^0$ corresponds to a coset gB of B . We have the *Bruhat decomposition*

$$GL_n = \bigcup_{w \in S_n} BwB. \quad (8.10)$$

(Regard w as a permutation matrix.) This induces a decomposition of the flag variety into *Schubert cells* $\gamma_w = \{gF^0 \mid gB \subseteq BwB\}$ with dimension $\dim(\gamma_w) = \ell(w)$. Their closures $\Gamma_w = \overline{\gamma_w}$ are the *Schubert varieties*. We have $\Gamma_v \subseteq \Gamma_w$ if and only if $v \leq w$ in Bruhat order.

The flags $F \in \Gamma_w$ can be described by *Schubert conditions* with respect to the base flag F^0 . Namely, $F \in \Gamma_w$ if and only if for all i, j , $\dim(F_i \cap F_j^0) \geq |w(\{1, \dots, i\}) \cap \{1, \dots, j\}|$.

Let $IH_v^*(\Gamma_w)$ be the local intersection homology of Γ_w at a point of Γ_v . This is non-zero in even (real) dimensions only and satisfies (by [18])

$$P_{v,w}(q) = \sum_i q^i \dim(IH_v^{2n-2i}(\Gamma_w)). \quad (8.11)$$

In particular, the Kazhdan–Lusztig polynomials have non-negative coefficients. Moreover, they are equal to 1 for all $v \leq w$ if Γ_w is smooth.

Consider a variety $\Gamma_{u,v}$ consisting of pairs (F^1, F^2) such that $F^1 \in \Gamma_u$, and, with respect to F^1 as a base flag, $F^2 \in \Gamma_v$. Projecting $\Gamma_{u,v}$ on F^2 induces a decomposition of the intersection homology of $\Gamma_{u,v}$ described by certain polynomials with non-negative, symmetric and unimodal coefficients [3]. By [27], these polynomials are the structure coefficients $q^{(\ell(u)+\ell(v)-\ell(w))/2} f_{u,v}^w(q)$ defined by

$$C'_u C'_v = \sum_w f_{u,v}^w(q) C'_w. \quad (8.12)$$

When q is a power of a prime we may consider the finite ‘flag variety’ over a field $\mathbb{F}(q)$. If we form a complex vector space by taking this finite set of flags as a basis, then the Hecke algebra $H_n(q)$ (specialized to this integer q) acts on it as the algebra of all operators commuting with the action of $GL_n(q)$. The operator $T_w \in H_n(q)$ maps a flag F to the sum of all flags F' in the Schubert cell $\gamma_{w^{-1}}$, constructed with respect to F as base flag. The trace of this action is therefore $\chi(T_w) = 0$ for $w \neq 1$, $\chi(T_1) = |\mathcal{F}_n(q)| = [n]_q! = (1)(1+q) \cdots (1+q+\cdots+q^{n-1})$. This character (really a different character for each q) is the one used in the proof following Conjecture 3.1.

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$C'_w \sim C'_{w_1} + C'_{w_2}$	$\phi_{(4)}$	$\phi_{(31)}$	$\phi_{(22)}$	$\phi_{(211)}$	$\phi_{(1^4)}$
$C'_{1234} \sim C'_{1234}$	0	0	0	0	1
$C'_{1243} \sim C'_{1243}$	0	0	0	$1+q$	0
$C'_{1324} \sim C'_{1324}$	0	0	0	$1+q$	0
$C'_{2134} \sim C'_{2134}$	0	0	0	$1+q$	0
$C'_{1423} \sim C'_{1342}$	0	$1+q+q^2$	0	q	0
$C'_{1342} \sim C'_{1342}$	0	$1+q+q^2$	0	q	0
$C'_{2143} \sim C'_{2143}$	0	0	$1+2q+q^2$	0	0
$C'_{3124} \sim C'_{2314}$	0	$1+q+q^2$	0	q	0
$C'_{2314} \sim C'_{2314}$	0	$1+q+q^2$	0	q	0
$C'_{1432} \sim C'_{1432}$	0	$1+2q+2q^2+q^3$	0	0	0
$C'_{2341} \sim C'_{2341}$	$1+q+q^2+q^3$	$q+q^2$	$q+q^2$	0	0
$C'_{2413} \sim C'_{2341}$	$1+q+q^2+q^3$	$q+q^2$	$q+q^2$	0	0
$C'_{3142} \sim C'_{2341}$	$1+q+q^2+q^3$	$q+q^2$	$q+q^2$	0	0
$C'_{3214} \sim C'_{3214}$	0	$1+2q+2q^2+q^3$	0	0	0
$C'_{4123} \sim C'_{2341}$	$1+q+q^2+q^3$	$q+q^2$	$q+q^2$	0	0
$C'_{2431} \sim C'_{2431}$	$1+2q+2q^2+2q^3+q^4$	$q+2q^2+q^3$	0	0	0
$C'_{3241} \sim C'_{3241}$	$1+2q+2q^2+2q^3+q^4$	$q+2q^2+q^3$	0	0	0
$C'_{3412} \sim C'_{3241} + C'_{2143}$	$1+2q+2q^2+2q^3+q^4$	$q+2q^2+q^3$	$q+2q^2+q^3$	0	0
$C'_{4132} \sim C'_{2431}$	$1+2q+2q^2+2q^3+q^4$	$q+2q^2+q^3$	0	0	0
$C'_{4213} \sim C'_{3241}$	$1+2q+2q^2+2q^3+q^4$	$q+2q^2+q^3$	0	0	0
$C'_{3421} \sim C'_{3421}$	$1+3q+4q^2+4q^3+3q^4+q^5$	q^2+q^3	0	0	0
$C'_{4231} \sim C'_{3421} + C'_{3214}$	$1+3q+4q^2+4q^3+3q^4+q^5$	$q+3q^2+3q^3+q^4$	0	0	0
$C'_{4312} \sim C'_{3421}$	$1+3q+4q^2+4q^3+3q^4+q^5$	q^2+q^3	0	0	0
$C'_{4321} \sim C'_{4321}$	$1+3q+5q^2+6q^3+5q^4+3q^5+q^6$	0	0	0	0

Table 1. Conjectures 2.1 and 3.1 for $n = 4$