A GRADED REPRESENTATION MODEL FOR MACDONALD'S POLYNOMIALS.

Adriano M. Garsia and Mark Haiman

University of California, San Diego

Department of Mathematics, 0112

9500 Gilman Drive

La Jolla, CA 92093-0112

Phone (Haiman): (619) 534-2704

E-mail: mhaiman@macaulay.ucsd.edu

Sept. 21, 1992

ABSTRACT

We define doubly graded S_n-modules R_μ for which we conjecture that the multiplicities of irreducible representations in various bi-degrees are given by the Macdonald coefficients $K_{\lambda\mu}$. Assuming one fundamental conjecture, the modules R_μ can be given several equivalent definitions, which we discuss. We prove the conjectures in various special cases.

1. Introduction

Macdonald in (1, 2) introduced a remarkable new basis for the ring of symmetric polynomials. Macdonald's polynomials

$$P_\mu(X; t, q) \tag{1.1}$$

are symmetric polynomials in the variables $X = \{x_1, x_2, \ldots\}$, homogeneous of degree equal to the size of the indexing partition μ, with coefficients in the ring $\mathbb{Q}(t, q)$ of rational functions of the two parameters t and q. Under suitable specializations of these parameters, Macdonald's polynomials reduce to various important bases of the symmetric function ring, including the monomial and elementary symmetric functions, Schur functions, Hall-Littlewood functions, and Jack functions. Analogs of Macdonald's polynomials for arbitrary root systems arise in connection with q-analogs of the famous constant term identity conjectures of Macdonald (3).

The major outstanding problem on Macdonald's polynomials concerns coefficients $\tilde{K}_{\lambda\mu}(t, q)$ expanding a certain transformed variant of P_μ in terms of the Schur functions s_λ. As defined, $\tilde{K}_{\lambda\mu}(t, q)$ is a rational function of t and q; the problem is to show that it is in fact a polynomial in t and q, with non-negative integer coefficients. We refer to this assertion as the 'Macdonald positive K' conjecture, or *MPK*.

In this note we describe certain doubly graded S_n modules R_μ for which we expect that $\tilde{K}_{\lambda\mu}(t, q)$ is the generating function giving the multiplicities of the irreducible representation V_λ in the various bihomogeneous components, which would prove *MPK*. As we shall explain, the needed properties and even aspects of the construction of R_μ remain conjectural at present, but we have proven them (and with them *MPK*) in the following instances:

(1) For all μ a hook and all λ (Stembridge (4) also has a proof of MPK in this case).

(2) For all μ a partition with two rows or two columns and all λ.

(3) For all λ a hook and all μ (Macdonald (1) had already proven MPK in this case).

(4) By computer for all μ and λ of size at most 7 and for occasional cases of size 8.

Brief indications of the proof techniques for some of these cases will be given in the last section of this note.

We have also proven that the characters of our modules agree with the specializations $\tilde{K}_{\lambda\mu}(t,0)$ and $\tilde{K}_{\lambda\mu}(t,1)$ (whose values are known from (2)) and with the symmetries $\tilde{K}_{\lambda\mu'}(t,q) = \tilde{K}_{\lambda\mu}(q,t)$ and $\tilde{K}_{\lambda'\mu}(t,q) = t^{n(\mu)}q^{n(\mu')}\tilde{K}_{\lambda\mu}(t^{-1},q^{-1})$. The most interesting of these is the specialization

$$\tilde{K}_{\lambda\mu}(t,0) = t^{n(\mu)}K_{\lambda\mu}(t^{-1}), \tag{1.1}$$

where $K_{\lambda\mu}(t)$ denotes the t-Kostka coefficient. Our modules agree with (1.1) because their homogeneous components of degree $(\cdot,0)$ reduce to certain singly graded S_n modules known to model the coefficients $K_{\lambda\mu}(t)$. These modules, whose origins lie in the geometry of the flag variety, were first given an elementary treatment in the work of De Concini and Procesi (5), Kraft (6), and finally Garsia and Procesi (7). The present work was inspired primarily by the search for a 't,q-analog' of constructions introduced by these authors.

2. Definition and interpretation of $\tilde{K}_{\lambda\mu}(t,q)$

We shall work from the start with transformed versions of Macdonald's polynomials, as they are more convenient for us than the original P_μ.

Theorem 1. *There exists a unique basis $H_\mu(X;t,q)$ of the ring $\Lambda_{t,q}(X)$ of symmetric functions with coefficients in $\mathbb{Q}(t,q)$ having the following properties:*

(1) $H_\mu((1-q)X;t,q) = \sum_{\lambda\geq\mu} c_{\lambda\mu}(t,q)s_\lambda(X)$.

(2) $H_{\mu'}(X;t,q) = H_\mu(X;q,t)$.

(3) $\langle H_\mu(X;t,q), s_{(n)}(X)\rangle = 1$.

In *(1)*, the notation $H_\mu((1-q)X;t,q)$ indicates a 'plethystic substitution,' defined by requiring the transformation $f(X) \mapsto f((1-q)X)$ to be a ring homomorphism and putting $p_k((1-q)X) = (1-q^k)p_k(X)$ for the power sum $p_k(X) = \sum_i x_i^k$. The triangularity indicated by the sum is with respect to the dominance order on partitions. In *(2)*, μ' is the conjugate of the partition μ; in *(3)*, n is the size of μ and $\langle\cdot,\cdot\rangle$ is the usual inner product of symmetric functions in which the Schur functions are orthonormal.

The theorem can be proved by defining $t^{n(\mu)}H_\mu(X;t^{-1},q) = J_\mu(\frac{X}{1-t};t,q)$, where, as in (1), $J_\mu(X;t,q) = h_\mu(t,q)P_\mu(X;t,q)$ for a certain (t,q)-hook product $h_\mu(t,q)$. Then the theorem follows by straightforward calculations from Macdonald's existence and uniqueness theorem for the basis P_μ.

We now define the coefficients $\tilde{K}_{\lambda\mu}(t,q)$.

Definition. $\tilde{K}_{\lambda\mu}(t,q)$ is the coefficient of the Schur function $s_\lambda(X)$ in the expansion

$$H_\mu(X;t,q) = \sum_\lambda \tilde{K}_{\lambda\mu}(t,q)s_\lambda(X). \qquad (2.1)$$

These coefficients are related to Macdonald's $K_{\lambda\mu}$ by

$$\tilde{K}_{\lambda\mu}(t,q) = t^{n(\mu)}K_{\lambda\mu}(t^{-1},q). \qquad (2.2)$$

It is known that if *MPK* holds, then $K_{\lambda\mu}$ has degree $n(\mu)$ in t, so *MPK* for $\tilde{K}_{\lambda\mu}$ is equivalent to *MPK* for $K_{\lambda\mu}$.

Let

$$M = \bigoplus_{i,j\geq 0} (M)_{i,j} \tag{2.3}$$

be a doubly graded S_n module, $(M)_{i,j}$ denoting its homogeneous component of degree (i,j). It is useful to collect all information about the characters of the $(M)_{i,j}$ into a *Frobenius series*

$$F_M(X;t,q) = \sum_{i,j} \psi((M)_{i,j}) t^i q^j, \tag{2.4}$$

where ψ is the usual 'Frobenius map' assigning to an S_n module V the symmetric function $\psi(V) = \sum_\lambda m_\lambda(V) s_\lambda(X)$, where $m_\lambda(V)$ is the multiplicity of the irreducible V_λ in V.

Trivially, *MPK* is equivalent to the assertion that $H_\mu(X;t,q)$ is the Frobenius series of some finite dimensional doubly graded S_n module R_μ. Our contribution is to propose a simple construction of the required R_μ.

3. Constructions of R_μ

In the following subsections we give three candidates for a construction of R_μ. Assuming the fundamental '$n!$ conjecture' stated below, the three constructions yield the same object. Possible approaches to proving the $n!$ conjecture will be treated in Section 6.

3.1 Construction from Δ_μ

Throughout, μ will be a partition of n. Let the coordinates of the cells in the Ferrers diagram of μ be $\{(p_1,q_1),\ldots,(p_n,q_n)\}$, where p is the row coordinate and q is the column coordinate, indexed from zero, so that the corner cell is

$(0,0)$. Put

$$\Delta_\mu(x_1,\ldots,x_n,y_1,\ldots,y_n) = \det\begin{bmatrix} x_1^{p_1}y_1^{q_1} & x_2^{p_1}y_2^{q_1} & \cdots & x_n^{p_1}y_n^{q_1} \\ x_1^{p_2}y_1^{q_2} & x_2^{p_2}y_2^{q_2} & \cdots & x_n^{p_2}y_n^{q_2} \\ \vdots & \vdots & & \vdots \\ x_1^{p_n}y_1^{q_n} & x_2^{p_n}y_2^{q_n} & \cdots & x_n^{p_n}y_n^{q_n} \end{bmatrix}. \tag{3.1}$$

Since the bi-exponents (p_i,q_i) are distinct, Δ_μ is a non-zero S_n-alternating polynomial, homogeneous of degree $n(\mu)$ in x and $n(\mu')$ in y. Note that for $\mu=(1^n)$ and $\mu=(n)$, Δ_μ reduces to the Vandermonde determinant in x and in y, respectively.

Let $I_\mu \subseteq \mathbb{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n] = \mathbb{Q}[X,Y]$ be the ideal of all polynomials $p(x_1,\ldots,x_n,y_1,\ldots,y_n)$ with the property that the corresponding differential operator $p(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_n})$ annihilates Δ_μ. Clearly I_μ is an S_n-invariant doubly homogeneous ideal.

Definition. R^I_μ is the quotient ring $\mathbb{Q}[X,Y]/I_\mu$, with its natural structure of doubly graded S_n module.

Note that R^I_μ has additional structure: it is a doubly graded 0-dimensional Gorenstein algebra, generated by its elements of total degree 1.

The ring R^I_μ is isomorphic as a doubly graded S_n module to the space $\mathcal{H}_\mu$ of all partial derivatives of all orders of Δ_μ, so we are free if we wish to regard $\mathcal{H}_\mu$ rather than R^I_μ as our model for the Macdonald polynomial $H_\mu(X;t,q)$. The part of the space $\mathcal{H}_\mu$ homogeneous of degree zero in y is spanned by all derivatives of the *Garnir polynomials*

$$\Delta_T = \Delta(T_1)\Delta(T_2)\cdots\Delta(T_k), \tag{3.2}$$

where T is a tableau of shape μ and $\Delta(T_i)$ is the Vandermonde determinant in those variables x_j indexed by entries in the i-th column of T. By (8), this space is isomorphic to the graded S_n module studied in (7), showing that the Frobenius series of R^I_μ agrees with $H_\mu(X;t,q)$ for $q=0$.

It is known from the theory of Macdonald polynomials that $H_\mu(X;1,1)$ is the Frobenius characteristic of the regular representation of S_n (*i.e.*, $\tilde{K}_{\lambda\mu}(1,1) = f_\lambda$). In particular, our conjectures require that the dimension of R^I_μ and $\mathcal{H}_\mu$ is $n!$.

CONJECTURE ($n!$ conjecture). *The dimension of the space $\mathcal{H}_\mu$ of all partial derivatives of Δ_μ is $n!$.*

This conjecture will be discussed at length as we continue. In particular we will show that if it holds, then $\mathcal{H}_\mu$ affords the regular representation of S_n.

3.2 Construction by grading an orbit

Let $\alpha_0, \alpha_1, \ldots$ be a sequence of distinct rational numbers, and let $\beta_0, \beta_1, \ldots$ be another such sequence. As before, let the coordinates of cells in the Ferrers diagram of μ be $(p_1,q_1),\ldots,(p_n,q_n)$, in some order. Consider the following vector in $\mathbb{Q}^{2n}$:

$$a_\mu = (\alpha_{p_1},\ldots,a_{p_n},\beta_{q_1},\ldots,\beta_{q_n}). \tag{3.3}$$

We let S_n act on $\mathbb{Q}^{2n}$ by permuting the first n and the second n coordinates simultaneously and identically. We write $[a_\mu]$ for the orbit of a_μ under this action. Note that $[a_\mu]$ does not depend on the order in which the Ferrers diagram cells (p_i,q_i) are numbered. Note also that $[a_\mu]$ is a regular orbit, *i.e.*, a_μ has trivial stabilizer.

We regard $\mathbb{Q}[X,Y]$ as the coordinate ring of $\mathbb{Q}^{2n}$ with $x_1,\ldots,x_n,y_1,\ldots,y_n$ the coordinate functions. Then we have an ideal $J_\mu \subseteq \mathbb{Q}[X,Y]$ consisting of all polynomials vanishing on $[a_\mu]$. $\mathbb{Q}[X,Y]/J_\mu$ is the coordinate ring of the finite set $[a_\mu]$; it has a natural S_n action and obviously affords the regular representation.

Since J_μ is not a homogeneous ideal, we construct a graded S_n module by replacing it with its *associated graded ideal* gr J_μ, defined as the linear span of the homogeneneous components of highest degree in elements of J_μ. One shows that gr J_μ is a homogeneous ideal, it is S_n invariant, and $\mathbb{Q}[X,Y]/(\mathrm{gr}\ J_\mu)$ affords the regular representation of S_n.

DEFINITION. $R_\mu^{II} = \mathbb{Q}[X,Y]/(\mathrm{gr}\ J_\mu)$.

This definition is incomplete in two respects. First, we do not know that gr J_μ is doubly homogeneous, so R_μ^{II} is *a priori* only singly graded. Second, the construction might depend on the choice of the α_i and β_i. These defects disappear if we assume the $n!$ conjecture, by the following result.

PROPOSITION. *The ideal I_μ contains the ideal* gr J_μ*; hence if the $n!$ conjecture holds then* gr $J_\mu = I_\mu$*, $R_\mu^I = R_\mu^{II}$, and this ring affords the regular representation of S_n.*

There is a way to produce a doubly homogeneous 'associated graded ideal' from J_μ. Namely, we may grade with respect to a generalized degree in which the variables x_i are assigned a different weight from the variables y_i. In particular, taking the X weight very large, we define an ideal $\mathrm{gr}_Y\mathrm{gr}_X\ J_\mu$, and taking the Y weight large we define $\mathrm{gr}_X\mathrm{gr}_Y\ J_\mu$. In general, different choices of weights may yield different ideals, but in our situation we have the following.

PROPOSITION. *The $n!$ conjecture is equivalent to $\mathrm{gr}_Y\mathrm{gr}_X\ J_\mu = \mathrm{gr}_X\mathrm{gr}_Y\ J_\mu$, and if this holds then all weighted associated graded ideals of J_μ are the same.*

3.3 Construction by bigrading an orbit

Let $T_\mu = \mathbb{Q}[X,Y]/J_\mu$, the coordinate ring of the orbit $[a_\mu]$. We consider two filtrations of T_μ defined as follows.

$$F_d^X = \{\hat{p}(X,Y) \in T_\mu \mid \deg_X\, p(X,Y) \le d\},$$

$$F_d^Y = \{\hat{p}(X,Y) \in T_\mu \mid \deg_Y\, p(X,Y) \le d\}. \tag{3.4}$$

Here $p(X,Y)$ denotes a polynomial in $\mathbb{Q}[X,Y]$, $\deg_X\, p(X,Y)$ is its degree in X, and similarly for Y, and $\hat{p}(X,Y)$ is its image in T_μ. Clearly we have $F_0^{(-)} \subseteq F_1^{(-)} \subseteq \cdots$ and $F_i^{(-)} F_j^{(-)} \subseteq F_{i+j}^{(-)}$, as required for filtrations.

Given the two filtrations of the ring T_μ we construct the following doubly graded ring.

$$\operatorname{bigr} T_\mu = \bigoplus_{i,j} (F_i^X \cap F_j^Y)/(F_{i-1}^X \cap F_j^Y + F_i^X \cap F_{j-1}^Y). \tag{3.5}$$

This has a well-defined ring structure, owing to the fact that the $F_d^{(-)}$ are filtrations, and it is a doubly graded S_n module because these filtrations are S_n invariant. The next result makes precise the sense in which $\operatorname{bigr} T_\mu$ is a 'bigrading' of T_μ.

PROPOSITION. *As S_n modules,* $\operatorname{bigr} T_\mu$ *and T_μ are isomorphic.*

We remark that this proposition is deeper than the corresponding statement for a singly graded ring constructed from one filtration. With more than two filtrations, such a statement would be false—*i.e.*, there is no such thing as a 'trigr.'

Definition. $R^{III}_\mu = \operatorname{bigr} T_\mu$.

The ring R^{III}_μ is always a doubly graded Gorenstein algebra carrying the regular representation of S_n. Its potential defect is that it might not be generated by its elements of total degree 1, *i.e.* it may not be the quotient of $\mathbb{Q}[X,Y]$ by some ideal. Assuming the $n!$ conjecture corrects this flaw.

Proposition. *R^{III}_μ is generated by its elements of total degree 1 if and only if the $n!$ conjecture holds for μ, and in that case $R^I_\mu = R^{II}_\mu = R^{III}_\mu$.*

Considering only the part of R^{III}_μ homogeneous of degree zero in y, its construction reduces to one given in (7) for the module studied there. Consequently, the Frobenius series of R^{III}_μ, like that of R^I_μ, agrees with $H_\mu(X;t,q)$ for $q=0$.

4. Other conjectures

Let μ and ν be partitions of n obtained by deleting different corners from some partition of $n+1$. Let $\hat{I}_\nu$ be the image of the ideal I_ν in the ring $R^I_\mu = \mathbb{Q}[X,Y]/I_\mu$. Computational evidence supports the following curious conjecture.

Conjecture (Annihilator conjecture). *With μ and ν as above, the ideal $\hat{I}_\nu$ is its own annihilator in R^I_μ.*

The annihilator conjecture implies the $n!$ conjecture. It does so because assuming the annihilator conjecture, the dimension of $\mathbb{Q}[X,Y]/(I_\mu + I_\nu)$ is one half the dimension of R^I_μ and also of R^I_ν, so $\dim R^I_\mu = \dim R^I_\nu$. Since any partition can be reached from (n) by steps of the form $\mu \to \nu$ with μ and ν as in the conjecture, every R^I_μ must have the same dimension, $n!$.

The annihilator conjecture also implies that the Frobenius series of the modules R_μ agree with certain conjectures made by Butler, and apparently noticed also by Lascoux, on the relationship between $\tilde{K}_{\lambda\mu}$ and $\tilde{K}_{\lambda\nu}$ when μ and ν are as above. (Originally, these conjectures were only for the case when μ covers ν in dominance order.)

There is also computational evidence for an even stranger conjecture.

CONJECTURE. *The lattice of ideals in $\mathbb{Q}[X,Y]$ generated by all the ideals I_μ is distributive.*

Assuming both this conjecture and the annihilator conjecture, it is possible to work out inductive recurrences for the Hilbert series of the modules R_μ. These recurrences agree beautifully with what one predicts supposing the Frobenius series of R_μ is in fact $H_\mu(X;t,q)$.

5. Triangularity and Tor modules

Let $H'_\mu(X;t,q)$ denote the Frobenius series of the module R^I_μ (or R^{III}_μ, if desired). Defining properties *(2)* and *(3)* in Theorem 1 for $H_\mu(X;t,q)$ clearly hold for $H'_\mu(X;t,q)$. In principle, one might prove property *(1)*, and thus $H'_\mu = H_\mu$, implying both the $n!$ conjecture and *MPK*, via the following interpretation of $H'_\mu((1-q)X;t,q)$.

THEOREM 2. *Let $F_i(X;t,q)$ be the Frobenius series of the module*

$$\mathrm{Tor}_i^{\mathbb{Q}[X,Y]}(R^I_\mu, \mathbb{Q}[X]) = \mathrm{Tor}_i^{\mathbb{Q}[Y]}(R^I_\mu, \mathbb{Q}) \tag{5.1}$$

(which has a natural structure of doubly graded S_n module). Then

$$H'_\mu((1-q)X;t,q) = \sum_i (-1)^i F_i(X;t,q). \tag{5.2}$$

In particular, if the modules (5.1) contain only irreducible components V_λ with $\lambda \geq \mu$, then H'_μ satisfies property (1) of Theorem 1.

For those μ where we have carried out the computations, we find that the modules (5.1) do have only the desired irreducible components.

6. Toward proofs of the $n!$ conjecture and *MPK*

Let V be a finite set of points in $\mathbb{Q}^N$, $A = \mathbb{Q}[x_1, \ldots, x_N]$ the coordinate ring of $\mathbb{Q}^N$, and $J(V) \subseteq A$ the ideal of polynomials vanishing on V. We shall say that V is *Gorenstein* if the ring $A/(\mathrm{gr}\ J(V))$ is Gorenstein. Equivalently, if we embed V in an affine hyperplane in $(N+1)$-space and connect points of V by lines to the origin, then the local ring of this union of lines at the origin is Gorenstein.

If V is Gorenstein, then the Hilbert function $h_d = \dim(A/(\mathrm{gr}\ J(V)))_d$ is symmetric, meaning $h_d = h_{d_0-d}$, where d_0 is the largest degree in which $(A/(\mathrm{gr}\ J(V)))_{d_0} \neq 0$. We prove the following converse.

Theorem 3. *Assume V is an orbit of a finite linear group action. If $h_0+h_1+\cdots+h_d \geq h_{d_0}+h_{d_0-1}+\cdots+h_{d_0-d}$ for all $0 \leq d \leq d_0$, then V is Gorenstein.*

This theorem has direct application to the $n!$ conjecture, by the following result.

Proposition. *The $n!$ conjecture holds for μ if and only if $[a_\mu]$ is Gorenstein.*

There is even some evidence for the following stronger statement.

CONJECTURE. *Every regular orbit of S_n acting on $\mathbb{Q}^{2n}$ is Gorenstein.*

(The corresponding statement for the action on $\mathbb{Q}^n$ is classical; for the action on $\mathbb{Q}^{3n}$ it is false.)

In the case that μ is a hook we are able to prove the $n!$ conjecture by means of the preceding Theorem and Proposition. To establish the hypothesis of the Theorem it suffices to exhibit for each d sufficiently many polynomials of degree at most d which define linearly independent functions on the set V.

The proof technique leads to an explicit basis for R_μ in the hook case, from which it is possible to analyze the restriction of the character to S_{n-1}. We obtain the following result.

PROPOSITION. *Let $H'_{r|s} = H'_{r|s}(X; t, q)$ be the Frobenius series of R_μ for μ the hook $(s+1, 1^r)$, and let Γ be the symmetric function operation corresponding under the Frobenius map to restriction of an S_n character to S_{n-1}. Then*

$$\begin{aligned} \Gamma H'_{r|s} &= \frac{1-t^r}{1-t} H'_{r-1|s} + t^r H'_{r-1|0} H'_{0|s-1} + q \frac{1-q^s}{1-q} H'_{r|s-1} \\ &= t \frac{1-t^r}{1-t} H'_{r-1|s} + q^s H'_{r-1|0} H'_{0|s-1} + \frac{1-q^s}{1-q} H'_{r|s-1}. \end{aligned} \tag{6.1}$$

The second equality in (6.1) gives a relation between $H'_{r-1|s}$, $H'_{r|s-1}$, and $H'_{r-1|0} H'_{0|s-1}$ which determines H' for any hook in terms of products of H' for a column by H' for a row. One readily shows that the Macdonald polynomials $H_\mu(X; t, q)$ satisfy the same relations and agree with H' for μ a column or a row. It follows that $H_\mu(X; t, q) = H'_{r|s}(X; t, q)$ when μ is a hook, proving *MPK*.

Our approach for two-rowed $\mu = (s, r)$ $(s \geq r)$ is somewhat indirect. We begin with the ring $R = \mathbb{Q}[u_1, \ldots, u_n]/E$, where E is the ideal generated by all symmetric polynomials without constant term. R is well-known to have dimension $n!$ as a vector space. In R we consider the ideal $J = (u_1^s, \ldots, u_n^s)$ and form the algebra

$$S = \mathrm{gr}_J R = R/J \oplus J/J^2 \oplus J^2/J^3 \oplus \cdots, \tag{6.2}$$

which can be given a ring structure in a standard way. In S we let y_i stand for the element u_i of R/J, and let x_i stand for the element u_i^s of J/J^2. Then we prove:

PROPOSITION. *The ring S is a graded Gorenstein ring, generated by its elements x_i and y_i.*

It follows that $S = \mathbb{Q}[X, Y]/I$, where I is the ideal of all polynomials which annihilate a certain polynomial Δ when applied as differential operators to Δ. One then easily shows that in this case $\Delta = \Delta_\mu$, hence the $n!$ conjecture holds, since S has dimension $n!$.

Identification of the character in the case $\mu = (r, s)$ takes place in two stages. First, using Theorem 2 on Tor modules, we show that the Frobenius series of R_μ is a linear combination of Macdonald polynomials H_ν for two-rowed shapes ν. This is achieved by showing that the restrictions of the Tor modules (5.1) to S_3 contain no alternants. Combining this with a recurrence for the restriction to S_{n-1} we obtain relations which determine the Frobenius series.

Computational experiments suggest that the ring S of (6.2) is isomorphic to R_μ not only for two-rowed shapes but for any 'near rectangle' (s^k, r), $s \geq r$.

A crucial step in our proof for two-rowed shapes involves induction on removal of corners, so a different method will be required for near-rectangles. For general μ it is not the case that R_μ is isomorphic to an algebra of the form (6.2).

Both authors received support for this work from National Science Foundation mathematical sciences research grants.

References.

1. Macdonald, I. G. (1988) *Actes 20^e Séminaire Lotharingien*, Publ. I.R.M.A. Strasbourg 372/S–20, 131–171.

2. Macdonald, I. G. (Unpublished manuscript) *Symmetric Functions and Hall Polynomials, 2nd Edition*, Chapter 6.

3. Macdonald, I. G. (1982) *SIAM J. Math. Anal.* **13**, no. 6, 988–1007.

4. Stembridge, J. R. (In press) *Proc. Amer. Math. Soc.*

5. de Concini, C. & Procesi, C. (1981) *Invent. Math.* **64**, 203–230.

6. Kraft, H. (1981) *Proc. 1980 Torun Conf., Poland*, Astérisque **87–88**, 195–205.

7. Garsia, A. M. & Procesi, C. (1992) *Advances in Math.* **94**, 82–138.

8. Bergeron, N. & Garsia, A. M. (1992) *Contemporary Mathematics* **138**, 51–86.