COMBINATORICS, SYMMETRIC FUNCTIONS, AND HILBERT SCHEMES

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Abstract. We survey the proof of a series of conjectures in combinatorics using new results on the geometry of Hilbert schemes. The combinatorial results include the positivity conjecture for Macdonald's symmetric functions, and the "$n!$" and "$(n+1)^{n-1}$" conjectures relating Macdonald polynomials to the characters of doubly-graded S_n modules. To make the treatment self-contained, we include background material from combinatorics, symmetric function theory, representation theory and geometry. At the end we discuss future directions, new conjectures and related work of Ginzburg, Kumar and Thomsen, Gordon, and Haglund and Loehr.

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1. Introduction

About ten years ago, Garsia and I [22] began to investigate certain doubly-graded S_n-modules whose characters appeared to be related to the new class of symmetric functions then recently introduced by Macdonald. Our modules are "doubled" analogs of well-known singly-graded modules arising in geometry and representation theory connected with the flag variety and Springer correspondence for GL_n. From the outset, we hoped to use them to prove the *positivity conjecture* for certain coefficients $K_{\lambda\mu}(q,t)$ occurring in Macdonald's theory, but by the time this was achieved [39], it had become clear that the solution of the positivity problem was just one chapter in a longer story. As the investigation advanced, we and others noticed that the characters of our doubled modules seemed to be related not only to Macdonald's symmetric functions but also to a number of classical "q-analogs" familiar to combinatorialists. Among these are q-enumerations of rooted forests and parking functions, two q-analogs of the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, and a q-analog of Lagrange inversion discovered independently by Garsia and Gessel. Eventually these various observations were formulated as a series of conjectures which have come to be known by the nicknames $n!$ and $(n+1)^{n-1}$ conjectures.

It develops that the doubled modules are associated naturally with the Hilbert scheme H_n of points in the plane. Understanding this geometric context has led to proofs of the $n!$ and $(n+1)^{n-1}$ conjectures, along lines first suggested by Procesi. As it turns out, the full explanation depends on properties of the Hilbert scheme that were not known before, and had to be established from scratch in order to complete the picture. One might say, then, that the main results are not the $n!$ and $(n+1)^{n-1}$ theorems, but new theorems in algebraic geometry. In a nutshell, the new theorems are, first, that the Hilbert scheme H_n is isomorphic to another kind of Hilbert scheme, which parametrizes orbits of the symmetric group S_n on $\mathbb{C}^n \oplus \mathbb{C}^n$, and second, a cohomology vanishing theorem for vector bundles on H_n provided by the first theorem.

I think it is fair to say that such results were unexpected, and became plausible to conjecture only because of evidence accumulated from the combinatorial study. Certainly it was this study that provided the incentive to prove them. One thing I wish to emphasize in these notes and in the accompanying lectures is how important the combinatorial origins of the problem were for all the subsequent developments.

It is perfectly possible to state the $n!$ and $(n+1)^{n-1}$ conjectures without reference to anything but elementary algebra, but to properly appreciate their content and the context in which they were discovered, it is helpful to understand some concepts which are familiar to combinatorialists and experts on special functions, but not so well known to a larger public. To this end I give in §2 and §3 a review of background material from combinatorics and the theory of Hall-Littlewood and Macdonald polynomials. To the same end, I will finish this introduction with a capsule history of some earlier developments which inspired the current results.

In the theory of Hall-Littlewood symmetric functions one meets q-analogs $K_{\lambda\mu}(q)$ of the *Kostka numbers* $K_{\lambda\mu}$, which count semi-standard Young tableaux of shape λ and content μ, or equivalently the weight multiplicity of μ in the irreducible representation of GL_n with highest weight λ. The $K_{\lambda\mu}(q)$ are called *Kostka-Foulkes polynomials*, and it is an important theorem that their coefficients

are positive integers. A major development in the combinatorial theory of symmetric functions was Lascoux and Schützenberger's interpretation of $K_{\lambda\mu}(q)$ as q-enumerating Young tableaux according to a numerical statistic called *charge*. This gives one of two proofs of the positivity theorem for Kostka-Foulkes polynomials; the other being that of Hotta and Springer, who interpreted $K_{\lambda\mu}(q)$ in terms of characters of cohomology rings of Springer fibers for GL_n.

In a 1987 preprint [62], Macdonald unified the theory of Hall-Littlewood symmetric functions with that of spherical functions on symmetric spaces, introducing the symmetric functions now known as Macdonald polynomials. For root systems of type A, they are symmetric functions in the classical sense, but with coefficients that depend on two parameters q and t. In this new context there are bivariate analogs $K_{\lambda\mu}(q,t)$ of the Kostka-Foulkes polynomials, specializing at $q=0$ to $K_{\lambda\mu}(t)$. Macdonald conjectured in [60] that these more general *Kostka-Macdonald polynomials* should also have positive integer coefficients. (Actually, the $K_{\lambda\mu}(q,t)$ were defined as rational functions and their being polynomials was part of the conjecture.) In light of Lascoux and Schützenberger's work, it was natural to try to prove Macdonald's positivity conjecture by suitably generalizing the definition of charge. Indeed, the proof of the positivity conjecture notwithstanding, it remains an open problem to give an explicit combinatorial proof along such lines.

As a first step toward the positivity conjecture, Garsia and Procesi [26] revisited and simplified the Hotta-Springer proof of the positivity theorem for Kostka-Foulkes polynomials. Beginning with an elementary description due to Tanisaki of the cohomology ring R_μ of a Springer fiber, they derived the formula relating $K_{\lambda\mu}(q)$ to the character of R_μ directly, without invoking geometric machinery. Their work led Garsia and me to the $n!$ conjecture, which similarly relates $K_{\lambda\mu}(q,t)$ to the character of a doubled analog of R_μ, and so implies Macdonald's positivity conjecture.

The spaces figuring in the $n!$ conjecture are quotients of the ring of coinvariants for the diagonal action of S_n on $\mathbb{C}^n \oplus \mathbb{C}^n$, and so we decided also to investigate the characters of the full coinvariant ring. That precipitated the key event leading to the discoveries recounted here: the recognition by us and others, especially Gessel and Stanley, of striking combinatorial patterns among characters of diagonal coinvariants. The space of coinvariants apparently had dimension equal to $(n+1)^{n-1}$, a combinatorially interesting number. Paying closer attention to the grading and the S_n action revealed known q-analogs of the number $(n+1)^{n-1}$ and the Catalan numbers C_n in the data. A menagerie of things studied earlier by combinatorialists for their own sake thus turned up unexpectedly in this new context.

It was Procesi who suggested, upon learning of their remarkable behavior, that the diagonal coinvariants might be interpreted as sections of a vector bundle on the Hilbert scheme H_n. Then it might be possible to compute their dimension and character using the Lefschetz formula of Atiyah and Bott, and so explain the phenomena. It soon became clear that the existence of Procesi's alleged vector bundle was in fact equivalent to the $n!$ conjecture. By 1994, we had managed to fully develop Procesi's idea, explaining the observations on the diagonal coinvariants by combining known combinatorial properties of Macdonald polynomials with a character formula predicated on the assumption that the geometric theorems alluded to earlier on the Hilbert scheme would hold. This explanation was successful enough to persuade us that the theorems surely must be true, although it took quite some time after that before the proof was finally complete.

2. Background from combinatorics

This section is a review of combinatorial facts needed later on. In part, the purpose here is to fix notation and terminology, but we will also introduce several fundamental concepts.

The *jeu-de-taquin* operations on Young tableaux have become central to the combinatorial theory of symmetric functions, and lead to the definition of *charge*, the basis of Lascoux and Schützenberger's combinatorial interpretation of the Kostka-Foulkes polynomials, discussed in more detail in §3.4.5. The positivity theorem for Kostka-Foulkes polynomials was the progenitor of Macdonald's positivity conjecture. The beautiful and subtle combinatorics associated with the earlier theorem was a major reason why Macdonald's conjecture seized the attention of combinatorialists.

Next we will touch on a few aspects of q-enumeration, involving Catalan numbers, trees and parking functions. These particular q-enumerations have especially appealing and well studied theories of their own, apart from their surprising connection with the diagonal coinvariants, which is our main concern here.

Finally, we outline Garsia and Gessel's theory of q-Lagrange inversion, which, as it happens, nicely ties together precisely the q-enumerations we have just discussed. This fact is hardly accidental, and indeed q-Lagrange inversion turns out to be indispensable to the full development of the theory relating the combinatorial phenomena to the Hilbert scheme. The reason for this is that to get specific predictions from the geometry, such as the dimension $(n+1)^{n-1}$ for the diagonal coinvariants, is not a trivial exercise. What makes it possible is a reformulation of q-Lagrange inversion in terms of symmetric function operators arising in the theory of Macdonald polynomials, as will be explained in §3.5.7.

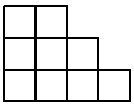
2.1. **Partitions, tableaux and jeu-de-taquin.** We write the parts of an integer partition as usual in decreasing order, as

$$\lambda = (\lambda_1, \dots, \lambda_l), \quad \text{where} \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l.$$

Its *length*, denoted $l(\lambda)$, is the number of parts, and its *size*, denoted $|\lambda|$, is the sum of the parts. The *diagram* of λ, sometimes called its *Young diagram* or its *Ferrers diagram*, is the array of lattice points

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : i+1 \leq \lambda_{j+1}\}.$$

It is drawn as an array of square cells, as for example



for the partition $(4,3,2)$. Note that our conventions follow the French style, and that the lower-left box is $(0,0)$, not $(1,1)$. We often abuse notation by identifying λ with its diagram, writing for instance $x \in \lambda$ to mean that $x = (i,j)$ is a cell in the diagram of λ. The *conjugate partition*, denoted λ', is the partition obtained by transposing the diagram of λ.

The *dominance order* is the partial order on partitions of a given size defined by

$$\lambda \geq \mu \quad \text{if} \quad \lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k \quad \text{for all } k. \tag{1}$$

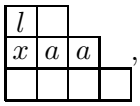
It is a slightly non-trivial exercise to prove that

$$\lambda \leq \mu \quad \Leftrightarrow \quad \mu' \leq \lambda'.$$

The *arm* of a cell $x \in \lambda$ is the set of cells to the right of x in its row; the *leg* is the set of cells above x in its column. The *hook* of x consists of x together with its arm and leg. We denote by

$$a(x), \quad l(x), \quad h(x) = 1 + a(x) + l(x)$$

the sizes of the arm, leg and hook of x. Thus in the example



we have $a(x) = 2$, $l(x) = 1$ and $h(x) = 4$.

A *standard Young tableau* of shape λ, where $|\lambda| = n$, is a bijective map $T\colon \lambda \to \{1, \ldots, n\}$ such that T is increasing along each row and column of λ (here already we are identifying λ with its diagram). A *semi-standard Young tableau* of shape λ is a map $T\colon \lambda \to \mathbb{N}$ which is weakly increasing along each row of λ and strictly increasing along each column. The number of standard Young tableaux of shape λ is given by the *hook formula* of Frame, Robinson and Thrall [19]:

$$|SYT(\lambda)| = \frac{n!}{\prod_{x\in\lambda} h(x)}. \tag{2}$$

An important numerical statistic associated with λ is

$$n(\lambda) \underset{\text{def}}{=} \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i'}{2}.$$

This number has the following representation-theoretical significance. Put $|\lambda| = n$, and let T be any standard Young tableau of shape λ. Let $S_T \cong S_{\lambda_1'} \times \cdots \times S_{\lambda_l'}$ be the subgroup of S_n consisting of elements that only permute the numbers within columns of T. The *Garnir polynomial* $g_T(x_1, \ldots, x_n)$ is defined to be the product of the Vandermonde determinants $\Delta(X_{C_0}) \cdots \Delta(X_{C_k})$ in the subsets X_{C_i} of the variables indexed by the entries C_i in column i of T, as for example

$$T = \begin{array}{|c|c|c|c|}\hline 3 & 6 & & \\ \hline 2 & 5 & 8 & \\ \hline 1 & 4 & 7 & 9 \\ \hline \end{array} \quad : \quad g_T = \Delta(x_1, x_2x_3)\Delta(x_4, x_5, x_6)\Delta(x_7, x_8).$$

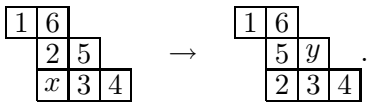
Thus g_T is the essentially unique homogeneous polynomial of minimal degree that is antisymmetric with respect to S_T. Its degree is $n(\lambda)$. As T ranges over standard tableaux of shape λ, the Garnir polynomials g_T form a basis of an S_n-invariant subspace of $\mathbb{C}[x_1, \ldots, x_n]$, which affords the irreducible representation V^λ of S_n whose character is denoted χ^λ in the standard indexing. In particular, its degree $\chi^\lambda(1)$ is the number of standard Young tableaux of shape λ.

A *skew shape* λ/ν, where $\nu \subseteq \lambda$, is the array of cells in the difference between the diagrams, as for example

$$(4,3,2)/(2,1) = \begin{array}{|c|c|}\hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \cline{3-4} \end{array}.$$

Standard and semistandard Young tableaux of skew shape are defined by the same rules as for straight shapes. Suppose given a semistandard tableau T of shape λ/ν and a cell x outside λ/ν but on its lower boundary, so that $\{x\} \cup (\lambda/\nu)$ is again a (skew) shape. There is a unique process, called a *forward slide*, by which an entry of T is moved from an adjacent cell above or to the right of x, then another entry into the cell thus vacated, and so on, until finally a cell is left vacant along

the upper boundary, in such a way that the result is again a tableau. This is best illustrated by an example:



Here y indicates the final vacated cell, and the tableau resulting from the slide consists of the other cells in the diagram on the right. The opposite process, in which tableau entries move up and to the right while the vacant cell moves down and to the left, is a *reverse slide*. It is easy to see that following a slide with an opposite slide into the cell just vacated undoes the the first slide. In particular, the figure above with the arrow reversed gives an example of a reverse slide. The general name for manipulation of tableaux by slides is *jeu-de-taquin*. The basic properties of jeu-de-taquin were established by Lascoux and Schützenberger [56, 77].

Proposition 2.1.1. *(1) When a skew tableau T is brought to a straight shape by jeu-de-taquin, the result depends only on T and not on the choice of slides used.*

(2) The number of tableaux T of a given skew shape λ/ν that go by jeu-de-taquin to a given tableau S of straight shape μ is the Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$, independent of S.

A semistandard tableau T has *content* μ if its entries are μ_1 1's, μ_2 2's, and so on. If $\mu_1 \geq \mu_2 \geq \cdots$, we say that T has *partition content*.

Definition 2.1.2. The *Kostka number* $K_{\lambda\mu}$ is the number of semistandard Young tableaux of shape λ and content μ. In particular, $K_{\lambda\mu} = 0$ unless $\lambda \geq \mu$, and $K_{\lambda\lambda} = 1$.

Lascoux and Schützenberger defined a numerical invariant called *charge* on tableaux with partition content and arbitrary skew shape. It is easiest to define it in terms of a complementary invariant, called *cocharge*.

Definition 2.1.3. The *cocharge* of a tableaux T with partition content μ is the integer $cc(T)$ uniquely characterized by the following properties:

(i) Cocharge is invariant under jeu-de-taquin slides.
(ii) Suppose the shape of T is disconnected, say $T = X \cup Y$, with X above and left of Y, and no entry of X is equal to 1. Let $S = Y \cup X$ be a tableau obtained by swapping X and Y. Then $cc(T) = cc(S) + |X|$.
(iii) If T is a single row, then $cc(T) = 0$.

The *charge* of T is defined as $c(T) = n(\mu) - cc(T)$.

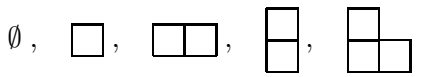
The existence of an invariant $cc(T)$ with properties (i)-(iii) is of course a theorem. To compute $cc(T)$ for a tableau T of straight shape λ with more than one row, one may first put it in the form $X \cup Y$ in (ii) using jeu-de-taquin, by sliding the bottom row to the right until it detaches from the rest of the shape. Then swapping the detached row to the top and normalizing again to straight shape by jeu-de-taquin, we obtain a new tableau S with $cc(S) = cc(T) - |\lambda| + \lambda_1$. This process is called *catabolism*. Since it diminishes the cocharge, repeated catabolism eventually produces a tableau with one row and cocharge zero.

2.2. **Catalan numbers and q-analogs.** The *Catalan numbers* are given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

They enumerate a wide range of interesting combinatorial objects. For example, C_n is the number of binary trees with n vertices, the number of ordered rooted trees with $n+1$ vertices, the number of standard Young tableaux of shape (n, n), and the number of partitions λ whose diagram is contained inside that of the partition $\delta_n = (n-1, n-2, \ldots, 1)$. For literally dozens of other combinatorial interpretations of C_n, one may consult the book of Stanley [82].

The formulation most relevant here is that C_n is the number of partitions $\lambda \subseteq \delta_n$, listed below for $n = 3$.



The list exhibits that $C_3 = 5$. The Carlitz-Riordan q-analog of the Catalan number [12] is the polynomial defined by

$$C_n(q) = \sum_{\lambda \subseteq \delta_n} q^{\binom{n}{2} - |\lambda|}. \tag{3}$$

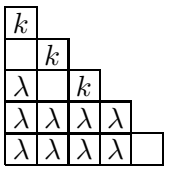
From the list of partitions for $n = 3$, we see that

$$C_3(q) = q^3 + q^2 + 2q + 1.$$

The Carlitz-Riordan q-Catalan numbers satisfy the recurrence

$$C_n(q) = \sum_{k=0}^{n-1} q^k C_k(q) C_{n-1-k}(q), \tag{4}$$

which is the q-analog of a classical recurrence for C_n. The recurrence (4) is easily proven by taking k to be the number of consecutive cells along the outermost diagonal of δ_n, beginning at the top left, that do not belong to λ, as illustrated here with $n = 6$ and $\lambda = (4, 4, 1)$, so $k = 3$.



For fixed k, the top k rows of δ_n/λ may be chosen independently of the bottom $n-1-k$, with the choice in the top k rows contributing a factor $q^k C_k(q)$ and the choice in the remaining rows contributing $C_{n-1-k}(q)$.

The ordinary generating function for Catalan numbers is

$$C(x) \underset{\text{def}}{=} \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}. \tag{5}$$

Thus $C(x)$ is the solution of a quadratic equation, which we will find it convenient to write in the form

$$xC(x)(1 - xC(x)) = x. \tag{6}$$

In other words, $xC(x)$ is the compositional inverse of the function $F(x) = x(1-x)$.

2.3. **Tree enumeration and q-analogs.** Among the things enumerated by Catalan numbers are various types of unlabeled trees. It is also classical in combinatorics (going back to Cayley) to enumerate trees and forests with vertices labeled by the integers $\{1, \dots, n\}$. A *tree* is a connected graph without cycles. A *rooted tree* is a tree with one vertex distinguished as the root. A *forest* is a possibly non-connected graph without cycles. Every forest is the disjoint union of its connected components, which are trees. A *rooted forest* is a forest in which each component is assigned a distinguished root.

Labeled rooted forests on the vertex set $\{1, \dots, n\}$ are in natural bijective correspondence with trees on the vertex set $\{0, 1, \dots, n\}$, the trees being thought of either as unrooted or as always rooted at vertex 0. The tree corresponding to a forest F is constructed by adding the vertex 0 and attaching it by an edge to the root vertex of each component of F. It is of little consequence whether we enumerate trees or forests; for the present discussion I prefer forests.

Proposition 2.3.1. *The number of rooted forests on n labeled vertices is $(n+1)^{n-1}$.*

There are many ways to prove this classical theorem, none of them entirely trivial. Perhaps the most direct is to consider the exponential generating function for trees

$$T(x) = \sum_{n=1}^{\infty} t_n \frac{x^n}{n!},$$

where t_n is the number of rooted trees on n labeled vertices. On the one hand, it follows from general principles that the exponential generating function $F(x)$ for rooted forests is given by $F(x) = e^{T(x)}$, because a forest is a disjoint union of trees. On the other hand, viewing a rooted tree as composed of a root vertex and an attached forest on the other vertices, one obtains $T(x) = xF(x)$. Hence we have $T(x) = xe^{T(x)}$, or better,

$$T(x)e^{-T(x)} = x, \tag{7}$$

so $T(x)$ is the compositional inverse of xe^{-x}. Now we apply the Lagrange inversion formula.

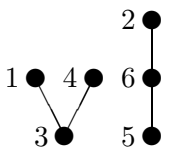
Proposition 2.3.2. *Let $xK(x)$ be the compositional inverse of the formal power series $x/E(x)$. Then the coefficients of $K(x)$ are given by*

$$k_n \underset{\text{def}}{=} [x^n]K(x) = [x^n]\frac{E(x)^{n+1}}{n+1}. \tag{8}$$

In our case, we take $E(x) = e^x$, so $xK(x) = T(x)$ and $K(x) = F(x)$. Then $k_n = f_n/n!$, where f_n is our desired number of forests. Proposition 2.3.2 yields $k_n = (n+1)^{n-1}/n!$ and hence $f_n = (n+1)^{n-1}$.

Definition 2.3.3. An *inversion* in a rooted forest F on vertices labeled $\{1, \dots, n\}$ is a pair of vertices $i < j$ such that vertex j is on the unique path from vertex i to the root of its component in F.

As an example, the forest



(drawn with the roots at the bottom) has 3 inversions: $(1,3)$, $(2,6)$ and $(2,5)$. This definition provides a combinatorial q-analog of the number $(n+1)^{n-1}$, the *inversion enumerator for forests*

$$J_n(q) \underset{\text{def}}{=} \sum_F q^{i(F)}, \tag{9}$$

where the sum is over rooted forests with vertices labeled $\{1,\dots,n\}$, and $i(F)$ denotes the number of inversions in the forest F. By listing the forests, one computes for example

$$J_3(q) = q^3 + 3q^2 + 6q + 6.$$

Mallows and Riordan [63] determined $J_n(q)$ by a generating function identity.

Proposition 2.3.4 ([63])**.** *We have*

$$\sum_{n=1}^{\infty} q^{n-1} J_{n-1}(q+1)\frac{x^n}{n!} = \log \sum_{n=0}^{\infty} (q+1)^{\binom{n}{2}} \frac{x^n}{n!}. \tag{10}$$

A pleasant proof of this result was given by Gessel and Wang [32]. The right-hand side of (10) is the exponential generating function for connected graphs with labeled vertices, enumerated with weight $q^{e(G)}$ for a graph G with $e(G)$ edges. Gessel and Wang described a search algorithm that selects a distinguished subtree in each graph. All graphs with a given distinguished tree T are gotten by adding to T some subset of certain optional edges. The number of optional edges is the number of inversions in T, regarded as rooted at vertex 1. Since vertex 1 never participates in an inversion of T, this is the same as the number of inversions in the rooted forest on the other vertices. Taking account of the $n-1$ fixed edges in T, together with the additional optional edges, and summing over all trees T on n vertices, gives the coefficient $q^{n-1}J_{n-1}(q+1)$ of $x^n/n!$ on the left-hand side.

2.4. ***q*-Lagrange inversion.** A q-analog of Lagrange inversion was defined independently by Garsia [30] and Gessel [31]. I will follow the approach of Garsia, who defined a q-analog of functional composition of formal power series by the identity

$$F \circ_q G(x) \underset{\text{def}}{=} \sum_n f_n G(x)G(qx)\cdots G(q^{n-1}x), \quad \text{where } F(x) = \sum_n f_n x^n.$$

He proved the following basic result, which shows that it is a good definition.

Proposition 2.4.1 ([30])**.** *We have*

$$F \circ_q G(x) = x \quad \textit{if and only if} \quad G \circ_{1/q} F(x) = x \tag{11}$$

More generally, when they hold, we have for any $\phi(x)$ and $\psi(x)$

$$\psi(x) = \phi \circ_q G(x) \quad \textit{if and only if} \quad \phi(x) = \psi \circ_{1/q} F(x). \tag{12}$$

Garsia also obtained a q-analog of the Lagrange inversion formula (8) for the q-compositional inverse $G(x)$ of $F(x) = x/E(x)$, as defined by (11). His formula involves operations called *roofing* and *starring*, which I will not go into here. Garsia and I later reformulated it more combinatorially as follows.

Proposition 2.4.2 ([24, 30])**.** *Let $E(x) = \sum_{n=0}^{\infty} e_n x^n$, with $e_0 = 1$, and let $G(x)$ be the q-compositional inverse of $F(x) = x/E(x)$ in the sense of* (11)*. Then $G(x) = xK(qx)$, where $K(x) = \sum_{n=0}^{\infty} k_n(q)x^n$, with $k_n(q)$ given by*

$$k_n(q) = \sum_{\lambda \subseteq \delta_n} q^{\binom{n}{2}-|\lambda|} e_{\alpha_0(\lambda)} e_{\alpha_1(\lambda)} \cdots e_{\alpha_{n-1}(\lambda)}.$$

Here $\alpha_i(\lambda)$ is the number of parts equal to i in λ, with $\alpha_0(\lambda)$ defined to make $\sum_{i=0}^{n-1} \alpha_i(\lambda) = n$.

One sees immediately that when $E(x) = 1/(1-x)$, so $e_n = 1$ for all n, then $k_n(q) = C_n(q)$, the Carlitz-Riordan q-Catalan number defined by (3). In other words the generating function for q-Catalan numbers

$$C(x;q) = \sum_{n=0}^{\infty} C_n(q)x^n$$

has the property that $G(x) = xC(qx;q)$ is the q-compositional inverse of $F(x) = x/E(x) = x(1-x)$. This is the the q-analog of equation (6).

It turns out that equation (7) also has a q-analog, which involves the inversion enumerators for forests.

Proposition 2.4.3 (Gessel [31])**.** *Let*

$$J(x;q) = \sum_{n=0}^{\infty} J_n(q)\frac{x^n}{n!}$$

be the exponential generating function for the forest inversion enumerators $J_n(q)$ in (9)*. Then*

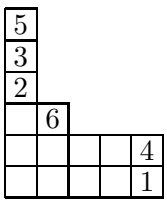
$$xe^{-x} \circ_q xJ(qx;q) = x. \tag{13}$$

This is proved using Proposition 2.3.4. Proposition 2.4.3 has an alternate interpretation that is worth mentioning. Combining (13) with Proposition 2.4.2, we see that $J_n(q)$ is equal to the specialization of $n!k_n(q)$ at $e_k = 1/k!$, or in symbols,

$$J_n(q) = \sum_{\lambda \subseteq \delta_n} q^{\binom{n}{2}-|\lambda|} \binom{n}{\alpha_0(\lambda), \alpha_1(\lambda), \ldots, \alpha_{n-1}(\lambda)}. \tag{14}$$

Consider the diagram of λ with its border extended by a line segment along the vertical axis ending at $(0, n)$. The numbers $\alpha_i(\lambda)$, including $\alpha_0(\lambda)$, are then the heights of the vertical segments along this extended border. The multinomial coefficient $\binom{n}{\alpha_0(\lambda),\alpha_1(\lambda),\ldots,\alpha_{n-1}(\lambda)}$ is the number of ways to place labels $\{1, \ldots, n\}$ to the right of the n vertical steps on the extended border of λ, so that the labels increase along each contiguous vertical segment. Here is an example with $n = 6$ and $\lambda = (4, 4, 1)$.

(15)



Such a labeled diagram is specified completely by giving the function $f\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ in which $f(i) - 1$ is the column occupied by label i. Not every function is admissible, because of the

condition $\lambda \subseteq \delta_n$. However, it is not hard to see that this condition on λ is precisely equivalent to the condition on f that

$$|f^{-1}(\{1,\dots,k\})| \geq k \quad \text{for all } k = 1,\dots,n. \tag{16}$$

Definition 2.4.4. A function $f : \{1,\dots,n\} \to \{1,\dots,n\}$ satisfying (16) is a *parking function.* The *weight* of f is the quantity $\binom{n+1}{2} - \sum_i f(i)$.

Note that the parts of λ are one less than the values of the corresponding parking function f, so the weight of f is simply $\binom{n}{2} - |\lambda|$. Hence Propositions 2.4.2 and 2.4.3 have the following corollary, which was also proved by Kreweras [51] using a combinatorial bijection.

Corollary 2.4.5. *The number of parking functions of weight d on $\{1,\dots,n\}$ is equal to the number of rooted forests on $\{1,\dots,n\}$ with d inversions.*

There is a natural action of S_n on parking functions defined by

$$w(f) = f \circ w^{-1}, \quad \text{for } w \in S_n.$$

This action is well-defined since it leaves condition (16) invariant; and it preserves the weight. We have the following nice result.

Proposition 2.4.6. *The permutation action of the symmetric group S_n on parking functions is isomorphic to its action on the finite Abelian group*

$$Q/(n+1)Q,$$

where $Q = \mathbb{Z}^n/\mathbb{Z}\cdot(1,1,\dots,1)$ and S_n acts on Q by permuting coordinates.

Proof. Fix the integers $\{0,\dots,n\}$ as representatives of the residue classes modulo $n+1$, and identify $(\mathbb{Z}/(n+1)\mathbb{Z})^n$ with the set of functions $f\colon \{1,..,n\} \to \{0,\dots,n\}$ in the obvious way. It is not hard to show that in every coset of $(\mathbb{Z}/(n+1)\mathbb{Z})\cdot(1,1,\dots,1)$ there is exactly one f that is a parking function. This gives an S_n-equivariant bijection between parking functions and the elements of $Q/(n+1)Q$. □

Note that the proposition gives in particular an easy proof that there are exactly $(n+1)^{n-1}$ parking functions. The Abelian group Q and its S_n action may be identified with the weight lattice for $\mathfrak{sl}_n$ and its Weyl group action.

3. Background from symmetric function theory

3.1. **Generalities.** We work with formal symmetric functions in infinitely many variables $z = z_1, z_2, \dots$, with coefficients implicitly assumed to be rational functions of any parameters q, t, etc. under discussion. I'll use the standard notation from Macdonald's book [61] for the classical families of symmetric functions: e_k for the k-th elementary symmetric function, h_k for the complete homogeneous symmetric function of degree k, p_k for the power-sum $z_1^k + z_2^k + \cdots$, and e_λ, h_λ, p_λ for products of these. The monomial symmetric functions are denoted m_λ, and the Schur functions s_λ.

The *Hall inner product* makes the Schur functions orthonormal, or equivalently, makes the monomial symmetric functions m_λ dual to the complete homogeneous symmetric functions h_λ:

$$\langle s_\lambda, s_\mu \rangle = \langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}. \tag{17}$$

Recall that the power-sum symmetric functions are orthogonal, but not orthonormal; their norms are given by

$$\langle p_\lambda, p_\lambda \rangle = z_\lambda \underset{\text{def}}{=} \prod_i \alpha_i! i^{\alpha_i}, \quad \text{where } \lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots). \tag{18}$$

The Kostka numbers $K_{\lambda\mu}$ of Definition 2.1.2 are related to symmetric functions by the identity

$$K_{\lambda\mu} = \langle s_\lambda, h_\mu \rangle, \tag{19}$$

which in light of (17) is equivalent to either of the expansions

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu \tag{20}$$

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda. \tag{21}$$

3.2. **The Frobenius map.** A classical theorem of Frobenius expresses the irreducible characters of the symmetric groups in terms of symmetric functions.

Proposition 3.2.1. *Let λ be a partition of n, and $w \in S_n$. The value of the irreducible character χ^λ of S_n at w is given by*

$$\chi^\lambda(w) = \langle s_\lambda, p_{\tau(w)} \rangle, \tag{22}$$

where $\tau(w)$ is the partition whose parts are the lengths of the disjoint cycles of the permutation w.

The number of permutations $w \in S_n$ with given cycle-type $\tau(w) = \lambda$ is equal to $n!/z_\lambda$, where $z_\lambda = \langle p_\lambda, p_\lambda \rangle$, as in (18). Hence Proposition 3.2.1 can also be written as the identity

$$s_\lambda = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w) p_{\tau(w)}.$$

This suggests the following definition.

Definition 3.2.2. The *Frobenius characteristic map* is the the linear map from class functions on S_n to symmetric functions homogeneous of degree n given by

$$F\chi \underset{\text{def}}{=} \frac{1}{n!} \sum_{w \in S_n} \chi(w) p_{\tau(w)},$$

or equivalently, the unique linear map sending the irreducible character χ^λ to the Schur function s_λ.

Note that F is an isometry of the usual inner product on characters onto the Hall inner product on symmetric functions.

We will often deal with graded, or doubly-graded, S_n-modules. In this context it is useful to extend the Frobenius map to a generating function for their graded characters.

Definition 3.2.3. Let $A = \bigoplus_r A_r$ be a graded S_n-module (with each A_r finite-dimensional). The *Frobenius series* of A is the generating function

$$F_A(z;t) = \sum_r t^r F \operatorname{char} A_r.$$

If $A = \bigoplus_{r,s} A_{r,s}$ is doubly graded, its Frobenius series is the bivariate generating function

$$F_A(z; q, t) = \sum_{r,s} t^r q^s F \operatorname{char} A_{r,s}.$$

Note that the degree of χ^λ is equal to $\chi^\lambda(1) = \langle s_\lambda, p_1^n \rangle$. It follows that the dimension of any finite-dimensional S_n-module V is given by $\langle p_1^n, F \operatorname{char} V \rangle$. Hence the Frobenius series of a graded or doubly graded module A determines its Hilbert series by the formula

$$H_A(t) = \langle p_1^n, F_A(z; t) \rangle \quad \text{or} \quad H_A(q, t) = \langle p_1^n, F_A(z; q, t) \rangle.$$

Of course the Frobenius series of any module is *a fortiori* Schur-positive, that is, the coefficients of its expansion in Schur functions are polynomials or formal series in t and/or q with positive integer coefficients. Our basic tool for establishing the Schur positivity of symmetric functions will be to interpret them as Frobenius series for suitable graded modules.

3.3. **Plethystic or λ-ring notation.** The algebra of symmetric functions $\Lambda_{\mathbb{F}}$ over a coefficient field $\mathbb{F}$ containing $\mathbb{Q}$ is freely generated by the power-sums p_k, that is,

$$\Lambda_{\mathbb{F}} \cong \mathbb{F}[p_1, p_2, \ldots].$$

Hence we may specify arbitrary values for the p_k's in any algebra $\mathcal{A}$ over $\mathbb{F}$ and extend uniquely to an $\mathbb{F}$-algebra homomorphism $\Lambda_{\mathbb{F}} \to \mathcal{A}$.

Now let A be a formal Laurent series with rational coefficients in indeterminates $a_1, a_2, \ldots$ (possibly including parameters q, t from the coefficient field). We define $p_k[A]$ to be the result of replacing each indeterminate a_i in A by a_i^k. Then for any $f \in \Lambda_{\mathbb{F}}$, the *plethystic substitution* of A into f, denoted $f[A]$, is the image of f under the homomorphism sending p_k to $p_k[A]$.

If A is a sum of indeterminates, $A = a_1 + \cdots + a_n$, then $p_k[A] = p_k(a_1, a_2, \ldots, a_n)$, and hence for every f we have $f[A] = f(a_1, a_2, \ldots, a_n)$. This is why we view this operation as a kind of substitution. In particular, when dealing with symmetric functions in an alphabet z, we always denote the sum of the variables by

$$Z = z_1 + z_2 + \cdots,$$

whence

$$f[Z] = f(z)$$

for all f. More generally, if A has a series expansion as a sum of monomials, then $f[A]$ is f evaluated on these monomials, for example

$$f[Z/(1-t)] = f(z_1, z_2, \ldots, tz_1, tz_2, \ldots, t^2 z_1, t^2 z_2, \ldots).$$

Among the virtues of this notation is that the substitution $Z \to Z/(1-t)$ as above has an explicit inverse, namely the substitution $Z \to Z(1-t)$.

One caution that must be observed with plethystic notation is that indeterminates must always be treated as formal symbols, never as variable numeric quantities. For instance, if f is homogeneous of degree d then it is true that

$$f[tZ] = t^d f[Z],$$

but it is *false* that $f[-Z] = (-1)^d f[Z]$, that is, we cannot set $t = -1$ in the equation above. Actually, $f[-Z]$ is an interesting quantity: it is equal to $(-1)^d \omega f(z)$, where ω is the classical involution on symmetric functions defined by $\omega p_k = (-1)^{k+1} p_k$, which interchanges the elementary

and complete symmetric functions e_k and h_k, and more generally exchanges the Schur function s_λ with $s_{\lambda'}$.

It is convenient when using plethystic notation to define

$$\Omega = \exp(\sum_{k=1}^{\infty} p_k/k). \tag{23}$$

Then since $p_k[A+B] = p_k[A] + p_k[B]$ and $p_k[-A] = -p_k[A]$ we have

$$\Omega[A+B] = \Omega[A]\Omega[B], \quad \Omega[-A] = 1/\Omega[A]. \tag{24}$$

From this and the single-variable evaluation $\Omega[x] = \exp(\sum_{k\geq 1} x^k/k) = 1/(1-x)$ we obtain

$$\Omega[Z] = \prod_i \frac{1}{1-z_i} = \sum_{n=0}^{\infty} h_n(z) \tag{25}$$

$$\Omega[-Z] = \prod_i (1-z_i) = \sum_{n=0}^{\infty} (-1)^n e_n(z). \tag{26}$$

The plethystic substitutions $Z \to Z(1-t)$ and $Z \to Z/(1-t)$ have an important representation-theoretical interpretation.

Proposition 3.3.1. *Let S^kV and $\wedge^k V$ denote the symmetric and exterior powers respectively of the defining representation $V = \mathbb{C}^n$ of S_n, and let $f(z) = F_A(z;t)$ be the Frobenius series of a graded S_n-module A. Then we have*

$$f[Z(1-t)] = \sum_k (-1)^k t^k F_{A\otimes \wedge^k V}(z;t), \tag{27}$$

$$f[\frac{Z}{1-t}] = \sum_k t^k F_{A\otimes S^k V}(z;t). \tag{28}$$

The proof is in two steps. First, a direct computation gives

$$h_n[Z(1-t)] = \sum_k (-1)^k t^k F_{\wedge^k V}(z;t),$$

$$h_n[\frac{Z}{1-t}] = \sum_k t^k F_{S^k V}(z;t).$$

Second, an easy exercise shows that if ϕ is a virtual character whose Frobenius image has the form $F\phi = h_n[ZQ]$ for some Q, then $F(\phi\otimes\chi) = (F\chi)[ZQ]$ for all χ, from which the result follows.

In §3.5 we will need the following classical generalization of the hook formula (2).

Proposition 3.3.2. *The Schur function specialization $s_\lambda(1,t,t^2,\ldots) = s_\lambda[1/(1-t)]$ is given by*

$$s_\lambda[1/(1-t)] = \frac{t^{n(\lambda)}}{\prod_{x\in\lambda}(1-t^{h(x)})}, \tag{29}$$

where $h(x)$ is the hook-length of the cell $x \in \lambda$.

3.4. **Hall-Littlewood polynomials.** Hall-Littlewood polynomials are symmetric functions with coefficients depending on a parameter q. They play an important role in the representation theory of $GL_n(\mathbb{F}_q)$ and in the geometry associated with it, involving character sheaves and the Springer correspondence (see §3.4.4). Together with Jack's symmetric functions, Hall-Littlewood polynomials were the precursors of Macdonald's symmetric functions.

The connecting coefficients between Hall-Littlewood polynomials and Schur functions are the *Kostka-Foulkes polynomials* $K_{\lambda\mu}(q)$. The positivity theorem for Kostka-Foulkes polynomials was the precursor to Macdonald's positivity conjecture, and the combinatorial proof of that theorem by Lascoux and Schützenberger was the inspiration for our work on Macdonald's conjecture.

3.4.1. *Definition; Kostka-Foulkes polynomials.* We begin with the classical definition of Hall-Littlewood polynomials, as in Macdonald [61]. We use the conventional q-notation (with t instead of q)

$$\begin{aligned} [k]_t &\underset{\text{def}}{=} \frac{1-t^k}{1-t} = t^{k-1}+t^{k-2}+\cdots+1; \\ [k]_t! &\underset{\text{def}}{=} [k]_t[k-1]_t\cdots[1]_t. \end{aligned} \tag{30}$$

Definition 3.4.1. The *Hall-Littlewood polynomial* $P_\lambda(z;t)$ is defined in $n \geq l(\lambda)$ variables $z = z_1,\ldots,z_n$ by the formula

$$P_\lambda(z;t) = \frac{1}{\prod_{i\geq 0}[\alpha_i]_t!} \sum_{w\in S_n} w\left(z^\lambda \frac{\prod_{i<j}(1-tz_j/z_i)}{\prod_{i<j}(1-z_j/z_i)}\right). \tag{31}$$

Here $\lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$, with α_0 defined so as to make $\sum_i \alpha_i = n$, and z^λ is shorthand for $z_1^{\lambda_1} z_2^{\lambda_2}\cdots z_l^{\lambda_l}$.

We will see below that the definition is stable with respect to changing the number of variables, so $P_\lambda(z;t)$ makes sense formally in infinitely many variables. At $t=0$, the denominator $\prod_{i\geq 0}[\alpha_i]_t!$ disappears and (31) reduces to the classical formula for Schur functions (equation (36) below), so we have

$$P_\lambda(z;0) = s_\lambda(z). \tag{32}$$

At $t=1$, the products inside the sum cancel, and $\prod_{i\geq 0}[\alpha_i]_t!$ becomes the number of permutations $w \in S_n$ that stabilize z^λ, so

$$P_\lambda(z;1) = m_\lambda(z). \tag{33}$$

Definition 3.4.2. The *Kostka-Foulkes polynomials* $K_{\lambda\mu}(t)$ are the coefficients in the expansion

$$s_\lambda(z) = \sum_\mu K_{\lambda\mu}(t) P_\mu(z;t).$$

In particular, by (32) and (33), we have

$$\begin{aligned} K_{\lambda\mu}(1) &= K_{\lambda\mu} \\ K_{\lambda\mu}(0) &= \delta_{\lambda\mu}. \end{aligned}$$

It is not yet obvious that $K_{\lambda\mu}(t)$ is in fact a polynomial, but this will become clear below.

3.4.2. *Jing's operators and transformed Hall-Littlewood polynomials.* There is another way of defining Hall-Littlewood polynomials which gives easier access to many of their properties. We begin with "vertex operators" due to Bernstein (see Macdonald [61] or Zelevinsky [86]) that have the effect of adding a part to a Schur function.

Definition 3.4.3. The *Bernstein operators* are the coefficients $S^0_m = [u^m]S^0(u)$ of the operator generating function $S^0(u)$ defined by

$$S^0(u)f = f[Z - u^{-1}]\Omega[uZ]. \tag{34}$$

Proposition 3.4.4. *The Bernstein operators add a part to the indexing partition of a Schur function, that is, for $m \geq \lambda_1$, we have $S^0_m s_\lambda(z) = s_{(m,\lambda)}(z)$.*

Proof. It suffices to work in finitely many variables $Z = z_1 + \cdots + z_n$. Then we have the partial fraction expansion

$$\Omega[uZ] = \prod_i \frac{1}{1-uz_i} = \sum_i \frac{1}{1-uz_i} \prod_{j\neq i} \frac{1}{1-z_j/z_i}. \tag{35}$$

The classical formula in n variables for a Schur function as a ratio of determinants, or equivalently the Weyl character formula for GL_n, can be written

$$s_\lambda(z) = \sum_{w\in S_n} w\left(\frac{z^\lambda}{\prod_{i<j}(1-z_j/z_i)}\right). \tag{36}$$

Now observe that for any polynomial f, we have

$$[u^m]f(u^{-1})\frac{1}{1-uz} = [u^0]u^{-m}f(u^{-1})\frac{1}{1-uz} = z^m f(z).$$

Combined with (34) and (35), this gives the formula for the Bernstein operator in n variables

$$S^0_m f(z) = \sum_i z_i^m \frac{f(Z-z_i)}{\prod_{j\neq i}(1-z_j/z_i)}.$$

The desired result now follows easily by induction using (36). □

It is customary to define s_λ by (36) when λ is any integer sequence, not necessarily a partition. Then, setting $\delta = (n-1, n-2, \ldots, 1, 0)$, we have $s_\lambda = 0$ if $\lambda + \delta$ does not have distinct parts, and otherwise $s_\lambda = \varepsilon(w)s_\nu$, where $w(\lambda+\delta) = \nu + \delta$ with ν a partition. With these conventions, the identity $S^0_m s_\lambda(z) = s_{(m,\lambda)}(z)$ holds for all $m \geq 0$.

We now introduce two t-deformations of Bernstein's operators that will allow us to construct Hall-Littlewood polynomials.

Definition 3.4.5. The *Jing operators* [45] are the coefficients $S^t_m = [u^m]S^t(u)$ of the operator generating function $S^t(u)$ defined by

$$S^t(u)f = f[Z + (t-1)u^{-1}]\Omega[uZ]. \tag{37}$$

Let $\Pi_{(1-t)}$ denote the plethystic substitution operator $\Pi_{(1-t)}f(z) = f[Z(1-t)]$. The *modified Jing operators* are

$$\tilde{S}^t_m = \Pi_{(1-t)} S^t_m \Pi^{-1}_{(1-t)},$$

or equivalently, the coefficients of the generating function

$$\tilde{S}^t(u)f = f[Z - u^{-1}]\Omega[(1-t)uZ]. \tag{38}$$

The *transformed Hall-Littlewood polynomials* are defined by

$$H_\mu(z;t) = S^t_{\mu_1} S^t_{\mu_1} \cdots S^t_{\mu_l}(1). \tag{39}$$

We also set $Q_\mu(z;t) \underset{\text{def}}{=} H_\mu[(1-t)Z;t]$, so

$$Q_\mu(z;t) = \tilde{S}^t_{\mu_1} \tilde{S}^t_{\mu_1} \cdots \tilde{S}^t_{\mu_l}(1). \tag{40}$$

The notation $Q_\mu(z;t)$ agrees with that used in Macdonald's book.

3.4.3. *Orthogonality and triangularity.* The basic orthogonality and triangularity properties of Hall-Littlewood polynomials are readily deduced using the Rodrigues-type formulas (39) and (40).

Lemma 3.4.6. *If $m \geq \mu_1$ and $\lambda \geq \mu$, then*

$$S^t_m s_\lambda \in \mathbb{Z}[t]\{s_\gamma : \gamma \geq (m,\mu)\}.$$

Moreover, $s_{(m,\mu)}$ occurs with coefficient 1 in $S^t_m s_\mu$.

Proof. Recall the Schur function identity (*dual Pieri rule*)

$$s_\lambda[Z+a] = \sum_k a^k \sum_{\lambda/\nu \in H_k} s_\nu,$$

where the notation $\lambda/\nu \in H_k$ means that the skew shape λ/ν is a *horizontal strip* of size k, that is, it has at most one cell in each column. Write the Jing operator in terms of the Bernstein operator: $S^t(u)f = S^0(u)f[Z + tu^{-1}]$. Taking the coefficient of u^m and applying the Pieri rule, we get

$$S^t_m s_\lambda = \sum_k t^k \sum_{\lambda/\nu \in H_k} s_{(m+k,\nu)}, \tag{41}$$

with the conventions discussed above if $(m+k,\nu)$ is not a partition. It is convenient to extend the dominance partial order to sequences that may not be partitions by maintaining the same definition (1). In particular, rewriting $s_{(m+k,\nu)}$, if nonzero, as $\pm s_\gamma$ where γ is a partition, we have

$$\gamma \geq (m+k,\nu) \geq (m,\lambda) \geq (m,\mu).$$

Equality can only occur for $k=0$, and does occur then if $\lambda = \mu$. □

Corollary 3.4.7. *We have*

$$H_\mu(z;t) = \sum_{\lambda \geq \mu} C_{\lambda\mu}(t) s_\lambda(z)$$

for suitable coefficients $C_{\lambda\mu}(t) \in \mathbb{Z}[t]$, with $C_{\mu\mu}(t) = 1$.

We will see presently that in fact $C_{\lambda\mu}(t) = K_{\lambda\mu}(t)$. Now let us use our second Rodrigues formula (40) to establish an opposite triangularity for $Q_\mu(z;t)$.

Lemma 3.4.8. *If $m \geq \lambda_1$, then*

$$\tilde{S}^t_m s_\lambda \in \mathbb{Z}[t]\{s_\gamma : \gamma \leq (m,\lambda)\},$$

and the coefficient of $s_{(m,\lambda)}$ in $\tilde{S}^t_m s_\lambda$ is equal to $1-t^\alpha$, where α is the multiplicity of m as a part of (m,λ).

Proof. In n variables, we can get an explicit formula for $\tilde{S}^t_m$, much as we did for S^0_m in the proof of Proposition 3.4.4, by using the partial fraction expansion

$$\Omega[(1-t)uZ] = \prod_i \frac{1-tuz_i}{1-uz_i} = t^n + (1-t)\sum_i \frac{1}{1-uz_i}\prod_{j\neq i}\frac{1-tz_j/z_i}{1-z_j/z_i}. \tag{42}$$

The resulting formula is

$$\tilde{S}^t_m f = \delta_{m,0}t^n f + (1-t)\sum_i z_i^m f[Z-z_i]\frac{\prod_{j\neq i}(1-tz_j/z_i)}{\prod_{j\neq i}(1-z_j/z_i)} \tag{43}$$

$$= \delta_{m,0}t^n f + (1-t)\sum_i\sum_k (-t)^k z_i^{m-k}\frac{e_k[Z-z_i]f[Z-z_i]}{\prod_{j\neq i}(1-z_j/z_i)}. \tag{44}$$

Taking $f = s_\lambda$ and applying the usual Pieri rule for multiplication by e_k, we obtain for $m > 0$

$$\tilde{S}^t_m s_\lambda = (1-t)\sum_k (-t)^k \sum_{\gamma/\lambda\in V_k} s_{(m-k,\gamma)}.$$

The worst case γ for each k, that is, the maximal one in the dominance order, is $\lambda + (1^k)$. For this case, $s_{(m-k,\gamma)}$, if non-zero, is $\pm s_\nu$ with ν having one of the forms

$$\nu = (\lambda_1,\dots,\lambda_i, m-k+i, \lambda_{i+1}+1,\dots,\lambda_k+1,\lambda_{k+1},\dots) \quad \text{and } i\leq k, \text{ or}$$
$$\nu = (\lambda_1,\dots,\lambda_k,\lambda_{k+1}-1,\dots,\lambda_i-1, m-k+i,\lambda_i,\dots) \quad \text{and } i > k.$$

In either case, $\nu \leq (m,\lambda)$ provided $m \geq \lambda_1$. Equality occurs when $m = \lambda_1 = \cdots = \lambda_i$ and $k = i$. Hence the coefficient of $s_{(m,\lambda)}$ in $\tilde{S}^t_m s_\lambda$ is equal to $(1-t)[\alpha]_t = (1-t^\alpha)$. □

Corollary 3.4.9. *We have*

$$Q_\mu(z;t) = \sum_{\lambda\leq\mu} B_{\lambda\mu}(t)s_\lambda$$

for suitable coefficients $B_{\lambda\mu}(t)\in\mathbb{Z}[t]$, and $B_{\mu\mu} = (1-t)^{l(\mu)}\prod_i [\alpha_i]_t!$, where $\mu = (1^{\alpha_1}, 2^{\alpha_2},\dots,)$.

The operator $\Pi_{(1-t)}$ is self-adjoint for the Hall inner product, that is, $\langle f, g[(1-t)Z]\rangle = \langle f[(1-t)Z], g\rangle$. More generally, for any A there holds the identity

$$\langle f, g[AZ]\rangle = \langle f[AZ], g\rangle.$$

This is easily seen using the Cauchy formula: homogeneous bases $\{u_\lambda\}$ and $\{v_\lambda\}$ are dual with respect to $\langle -,-\rangle$ if and only if

$$\sum_\lambda u_\lambda[Y]v_\lambda[Z] = \Omega[YZ]. \tag{45}$$

Hence $\{u_\lambda\}$ and $\{v_\lambda[AZ]\}$ are dual bases if and only if $\sum_\lambda u_\lambda[Y]v_\lambda[Z] = \Omega[YZ/A]$, and this condition is symmetric between $\{u_\lambda\}$ and $\{v_\lambda\}$.

By Corollaries 3.4.7 and 3.4.9, if $\langle H_\mu, H_\nu[(1-t)Z]\rangle \neq 0$, we must have $\mu\leq\nu$. By symmetry, we must also have $\nu\leq\mu$, so $\mu = \nu$. Together with the leading terms determined in Corollaries 3.4.7 and 3.4.9, this gives the following result.

Corollary 3.4.10. *The transformed Hall-Littlewood polynomials are orthogonal with respect to the inner product $\langle f, g[(1-t)Z]\rangle$, and their self-inner-products are given by*

$$\langle H_\mu, H_\mu[(1-t)Z]\rangle = (1-t)^{l(\mu)} \prod_i [\alpha_i]_t!, \quad \mu = (1^{\alpha_1}, 2^{\alpha_2}, \ldots). \tag{46}$$

The polynomials H_μ are uniquely characterized by any two of the three properties expressed by Corollaries 3.4.7, 3.4.9, and 3.4.10.

Equation (43) yields by induction an explicit formula for $Q_\lambda(z;t)$ in n variables, analogous to the classical formula (36) for Schur functions. In this way we can recover the classical formula (31) for Hall-Littlewood polynomials.

Proposition 3.4.11. *The symmetric functions $Q_\lambda(z;t) = H_\lambda[(1-t)Z;t]$ are given in n variables $z_1, \ldots, z_n$ by*

$$Q_\lambda(z;t) = (1-t)^{l(\lambda)}[n-l(\lambda)]_t! \sum_{w\in S_n} w\left(z^\lambda \frac{\prod_{i<j}(1-tz_j/z_i)}{\prod_{i<j}(1-z_j/z_i)}\right).$$

Hence the classical Hall-Littlewood polynomials $P_\lambda(z;t)$ are equal to

$$P_\lambda(z;t) = \frac{Q_\lambda(z;t)}{(1-t)^{l(\lambda)}\prod_i [\alpha_i]_t!}, \quad \lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots). \tag{47}$$

In the classical theory of Hall-Littlewood polynomials, one uses the t-inner product given in our language by

$$\langle f, g\rangle_t = \langle f, g[Z/(1-t)]\rangle.$$

Corollary 3.4.10 and equation (47) imply that $\langle P_\lambda, Q_\mu\rangle_t = \delta_{\lambda\mu}$, that is, $\{P_\lambda\}$ and $\{Q_\lambda\}$ are orthogonal (but not orthonormal) bases for $\langle -, -\rangle_t$, dual to each other. Hence

$$K_{\lambda\mu}(t) = \langle s_\lambda, Q_\mu\rangle_t = \langle s_\lambda, H_\mu\rangle.$$

This shows, as claimed earlier, that the coefficients $C_{\lambda\mu}(t)$ in Corollary 3.4.7 are equal to the $K_{\lambda\mu}(t)$.

To keep things organized, let me summarize in one place our conclusions so far.

Corollary 3.4.12. *The transformed Hall-Littlewood polynomials H_μ are related to the classical Hall-Littlewood polynomials by*

$$H_\mu[(1-t)Z;t] = Q_\mu(z;t) = (1-t)^{l(\mu)} \prod_{i=1}^{\mu_1} [\alpha_i(\mu)]_t! P_\mu(z;t). \tag{48}$$

They are uniquely characterized by the following properties.

(i) $H_\mu(z;t) \in \mathbb{Z}[t]\cdot\{s_\lambda : \lambda \geq \mu\}$,
(ii) $H_\mu[(1-t)Z;t] \in \mathbb{Z}[t]\cdot\{s_\lambda : \lambda \leq \mu\}$
(iii) $\langle s_\mu, H_\mu\rangle = 1$.

The Kostka-Foulkes polynomials $K_{\lambda\mu}(t)$ can be defined through the Schur function expansion

$$H_\mu(z;t) = \sum_{\lambda\mu} K_{\lambda\mu}(t) s_\lambda(z),$$

and enjoy the following properties.

(iv) $K_{\lambda\mu}(t) \in \mathbb{Z}[t]$,
(v) $K_{\lambda\mu}(t) = 0$ *unless* $\lambda \geq \mu$, *and* $K_{\mu\mu}(t) = 1$,
(vi) $K_{\lambda\mu}(0) = \delta_{\lambda\mu}$, *that is,* $H_\mu(z;0) = s_\mu(z)$,

(vii) $K_{\lambda\mu}(1) = K_{\lambda\mu}$, *that is,* $H_\mu(z;1) = h_\mu(z)$.

3.4.4. *Interpretation in geometry and representation theory.* The geometric interpretation of the Kostka-Foulkes polynomials that I wish to discuss is summarized in the next proposition, which combines results of Hotta, Lusztig and Springer [42, 58, 81]. To state it we need a bit of terminology. Let $\mathcal{N}$ be the variety of $n \times n$ nilpotent matrices, that is, the *nilpotent variety* in the Lie algebra $\mathfrak{gl}_n$. Via the exponential map, we can identify $\mathcal{N}$ with the variety of unipotent elements in GL_n. The adjoint action of GL_n on $\mathcal{N}$ is by similarities $x \mapsto gxg^{-1}$, so the GL_n-orbit of an element $x \in \mathcal{N}$ is given by its Jordan canonical form. For each partition λ of n, let O_λ denote the orbit whose elements have Jordan block sizes $\lambda_1, \dots, \lambda_l$.

The *flag variety* $\mathcal{B} = GL_n/B$ may be identified concretely either with the variety of Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{gl}_n$ or with the variety of flags of subspaces

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n, \quad \dim F_i = i. \tag{49}$$

A flag F is *compatible* with $x \in \mathcal{N}$ if $xF_i \subseteq F_{i-1}$ for each i, or equivalently, F is fixed by the unipotent element $\exp(x) \in GL_n$, or yet again equivalently, we have $x \in \mathfrak{b}$ for the corresponding Borel $\mathfrak{b}$, the Lie algebra of the stabilizer of F. The variety

$$Z_0 = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} : x \in \mathfrak{b}\} \tag{50}$$

is a vector bundle over $\mathcal{B}$, hence nonsingular. The *Grothendieck resolution* is the proper and birational map

$$g\colon Z_0 \to \mathcal{N}$$

given by projection on the first factor. Its fiber over any element $x \in O_\lambda$ is called a *Springer fiber*, and denoted $\mathcal{B}_\lambda$. Thus $\mathcal{B}_\lambda \subseteq \mathcal{B}$ is just the variety of flags fixed by a unipotent matrix of Jordan type λ. The dimension of $\mathcal{B}_\lambda$, which is equal to one-half the codimension of O_λ in $\mathcal{N}$, is the statistic $n(\lambda)$ discussed in §2.1.

Finally, let $GL_n(q)$ be the linear group over the finite field $\mathbb{F}_q$ with q elements. Identifying O_λ with a unipotent orbit in GL_n, its $\mathbb{F}_q$-rational points form a conjugacy class $O_\lambda(q) \subseteq GL_n(q)$ (here life is simpler for GL_n than for other semisimple Lie groups G; in general a unipotent orbit may break up into several conjugacy classes of the corresponding finite Chevalley group). The finite group $GL_n(q)$ acts on the finite set $\mathcal{B}(q)$ of $\mathbb{F}_q$-rational points of the flag variety, which in our case is just the set of flags as in (49), but in the finite vector space $\mathbb{F}_q^n$ instead of $\mathbb{C}^n$.

The characters χ_λ of those irreducible representations V_λ of $GL_n(q)$ that occur in the permutation representation on $\mathbb{C} \cdot \mathcal{B}(q)$ are called *unipotent characters*. Their indexing by partitions λ comes about as follows. Fix the Borel subgroup $B \subseteq GL_n(q)$ of upper triangular matrices. The *Hecke algebra* $H_n(q)$ is the subalgebra $H_n(q) \subseteq \mathbb{C}\, GL_n(q)$ of elements $\sum_g a_g g$ with coefficients a_g constant on double cosets BwB. As is well-known, $H_n(q)$ is the specialization at the integer q of a generic Hecke algebra in which q is an indeterminate, and its specialization at $q = 1$ is the group algebra of the Weyl group, that is, $H_n(1) = \mathbb{C}S_n$ for $G = GL_n$. Both the generic Hecke algebra and these specializations are semi-simple, so the irreducible representations V^λ of $H_n(q)$ are naturally identified with those of $\mathbb{C}S_n$. By general principles for double coset algebras, $H_n(q) \times GL_n(q)$ acts on $\mathbb{C} \cdot B(q)$ and this representation decomposes into irreducibles as

$$\mathbb{C} \cdot B(q) \cong \bigoplus_\lambda V^\lambda \otimes V_\lambda.$$

Thus there is a natural correspondence between the unipotent characters χ_λ of $GL_n(q)$ and the irreducible characters χ^λ of S_n.

Definition 3.4.13. The "cocharge" Kostka-Foulkes polynomials are

$$\tilde{K}_{\lambda\mu}(q) \underset{\text{def}}{=} q^{n(\mu)} K_{\lambda\mu}(q^{-1}).$$

It is not hard to see—from the proof of Lemma 3.4.6, for instance—that the degree of $K_{\lambda\mu}(t)$ is at most $n(\mu)$, so $\tilde{K}_{\lambda\mu}(q)$ is in fact a polynomial.

Proposition 3.4.14. *The Kostka-Foulkes polynomials $K_{\lambda\mu}(q)$ and their cocharge variants $\tilde{K}_{\lambda\mu}(q)$ have the following interpretations.*

(i) *The value of a unipotent character of $GL_n(q)$ on a unipotent conjugacy class is given by $\chi_\lambda(u) = \tilde{K}_{\lambda\mu}(q)$, for $u \in O_\mu(q)$.*

(ii) *The Poincaré series of the local intersection homology of the closure $\overline{O}_\lambda$ at any point $x \in O_\mu$ is given by $\sum_k q^k \, IH_x^{2k}(\overline{O}_\lambda) = q^{n(\mu)-n(\lambda)} K_{\lambda\mu}(q)$.*

(iii) *For the Springer action* [81] *of the Weyl group S_n on the cohomology ring $H^*(\mathcal{B}_\mu)$ of a Springer fiber, we have $\sum_q q^k \langle \chi^\lambda, \operatorname{char} H^{2k}(\mathcal{B}_\mu)\rangle = \tilde{K}_{\lambda\mu}(q)$. In other words, the Frobenius series of $H^*(\mathcal{B}_\mu)$ is given by $F_{H^*(\mathcal{B}_\mu)}(z;q) = q^{n(\mu)} H_\mu(z;q^{-1})$.*

It is somewhat difficult to gain an understanding of these results from the original papers, because the theory of perverse sheaves, which simplifies and clarifies the proofs, was developed later. I will not discuss the proofs in any detail, but will only mention a few points to make the present discussion self-contained. For further information I recommend the excellent exposition by Shoji [78].

One point that deserves attention is the definition of the Springer action referred to in part (iii) of the Proposition. The variety Z_0 in (50) is part of a larger bundle over the flag variety,

$$Z = \{(x, \mathfrak{b}) \in \mathfrak{gl}_n \times \mathcal{B} : x \in \mathfrak{b}\},$$

and projection on the first factor again yields a proper map

$$f \colon Z \to \mathfrak{gl}_n.$$

This map f is not birational, but it is generically finite. Specifically, the preimage of the set $(\mathfrak{gl}_n)_{\text{rss}}$ of regular semi-simple elements has a natural structure of principal S_n-bundle. Let $\mathbb{C}_Z$ be the trivial constant sheaf on Z. The fundamental *decomposition theorem* of Beilinson-Bernstein-Deligne-Gabber [4], together with an easy dimension argument, implies that the object $Rf_*\mathbb{C}_Z$ in the derived category of constructible sheaves on $\mathfrak{gl}_n$ is a perverse sheaf, and furthermore, it is the perverse extension of its restriction to $(\mathfrak{gl}_n)_{\text{rss}}$. Since S_n acts naturally on this restriction, it acts on $Rf_*\mathbb{C}_Z$. But the cohomology ring $H^*(\mathcal{B}_\mu)$ is just the stalk of $Rf_*\mathbb{C}_Z$ at $x \in O_\mu$. The action of S_n on the stalk is the Springer action.

The equivalence of interpretations (ii) and (iii) in Proposition 3.4.14 follows from similar considerations involving $Rg_*\mathbb{C}_{Z_0}$. Their further equivalence with interpretation (i) is part of Lusztig's theory of *character sheaves*.

As for the identification with $\tilde{K}_{\lambda\mu}(q)$ of the quantity described by all three interpretations, Shoji gave a procedure for computing the characters of the cohomology rings of Springer fibers (referred to in this context as *Green polynomials*) for the classical groups G, and this was extended by Lusztig to all G in [59]. Shoji and Lusztig characterize the Green polynomials by triangularity conditions,

which in the case of GL_n, when translated into symmetric function language using the Frobenius series and Proposition 3.3.1, amount to conditions (i)-(iii) in Corollary 3.4.12.

Finally, a word is in order here about the situation for a general semisimple Lie group G. The things described by the three parts of Proposition 3.4.14 again coincide for general G, with certain adjustments.

First of all, the cohomology rings $H^*(\mathcal{B}_u)$ depend, strictly speaking, on the choice of u in a nilpotent orbit O. On a fixed orbit they are all isomorphic, but the locally constant sheaf with stalk $H^*(\mathcal{B}_u)$ at u is usually not trivial. Various pairs (O, L) consisting of a nilpotent orbit O and an irreducible local system L on it arise in this way. These pairs, rather than the orbits themselves, are what correspond to unipotent conjugacy classes in the finite Chevalley group $G(q)$. To formulate the Proposition correctly, in (iii) one should consider separately the summands of $H^*(\mathcal{B}_u)$ corresponding to different local systems L on O. Also (ii) has to be rephrased to take account of the local systems.

Secondly, part (i) is only "almost" correct for general G. The unipotent characters χ_λ have to be replaced with certain linear combinations, called *almost characters*. This phenomenon occurs because Lusztig's character sheaves are imperfect geometric analogs of characters. The linear transformations required involve only a few characters at a time, and are given by certain small matrices which have been explicitly determined by Lusztig.

3.4.5. *Combinatorial interpretations.* The following theorem was discovered by Lascoux and Schützenberger [57] and its proof completed by Butler [11].

Theorem 3.4.15. *The Kostka-Foulkes polynomial $K_{\lambda\mu}(t)$ is given in terms of the charge statistic defined in §2.1, evaluated on semistandard tableaux T of shape λ and content μ, by*

$$K_{\lambda\mu}(t) = \sum_T t^{c(T)}.$$

Equivalently, the cocharge Kostka-Foulkes polynomials are given in terms of cocharge by

$$\tilde{K}_{\lambda\mu}(t) = \sum_T t^{cc(T)}.$$

An alternative combinatorial interpretation of $K_{\lambda\mu}(t)$ was given by Kirillov and Reshetikhin [47, 48]. For any partition ν, let

$$s_k(\lambda) = \lambda_1 + \cdots + \lambda_k$$

denote the sum of the first k parts of ν. A (λ, μ) *configuration* is a sequence of partitions

$$\nu = (\nu^0 = \mu', \nu^1, \ldots, \nu^{l(\lambda)})$$

of sizes

$$|\nu^k| = |\lambda| - s_k(\lambda).$$

Note that $\nu^{l(\lambda)}$ is empty by definition. The configuration ν is said to be *admissible* if the numbers

$$p_j^k(\nu) \underset{\text{def}}{=} s_j(\nu^{k-1}) - 2s_j(\nu^k) + s_j(\nu^{k+1}), \quad \text{for } i = 1, \ldots, l(\lambda) - 1 \text{ and all } j$$

are non-negative. Note that for j sufficiently large, $p_j^k(\nu) = \lambda_k - \lambda_{k+1}$, which is necessarily non-negative.

Theorem 3.4.16 (Kirillov-Reshetikhin [48])**.** *The Kostka-Foulkes polynomial $K_{\lambda\mu}(t)$ is given by the sum*

$$K_{\lambda\mu}(t) = \sum_{\nu} t^{n(\mu)-\sum_{j\geq 1}\mu'_j\nu^1_j+\sum_{k,j\geq 1}\nu^k_j(\nu^k_j-\nu^{k+1}_j)} \prod_{k,j\geq 1} \begin{bmatrix} p^k_j+\nu^k_j-\nu^k_{j+1} \\ \nu^k_j-\nu^k_{j+1} \end{bmatrix}_t \tag{51}$$

over all (λ,μ)-admissible configurations ν, where

$$\begin{bmatrix} n \\ k \end{bmatrix}_t \underset{\text{def}}{=} \frac{[n]_t!}{[k]_t![n-k]_t!}$$

are the Gauss binomial coefficients.

The theorem is proved by attaching to each configuration ν some additional data, called a *rigging*, so that the term corresponding to ν in (51) is a weighted enumeration of the possible riggings. Then Kirillov and Reshetikhin give a bijection between rigged configurations and semistandard tableaux in which the weight of the configuration corresponds to the charge of the tableau. This reduces Theorem 3.4.16 to Theorem 3.4.15.

Theorem 3.4.16 has an interesting origin. A technique from mathematical physics known as the *Bethe ansatz* enables one to produce highest weight vectors (called *Bethe vectors* in this context) for the irreducible constituents in tensor products of GL_n modules $V_{\mu_1}\otimes\cdots\otimes V_{\mu_r}$, where the μ_i are rectangular partitions. In this particular application of the Bethe ansatz it turns out that the resulting system of Bethe vectors is complete, and that they are naturally indexed by rigged configurations. The weight of the rigged configuration has a physical interpretation as a quantum number of the state described by the corresponding Bethe vector. When the rectangles μ_i are the rows of μ, the relevant configurations are the admissible configurations in Theorem 3.4.16. Thus the theorem says that $K_{\lambda\mu}(t)$ enumerates Bethe vectors in $V_{(\mu_1)}\otimes\cdots\otimes V_{(\mu_r)}$ by quantum number.

3.4.6. *The method of Garsia and Procesi.* The proof of Theorem 3.4.15 is complicated and not particularly illuminating, while the proof of Proposition 3.4.14 requires heavy intersection cohomology machinery. A simpler route than either of these to the positivity theorem for the Kostka-Foulkes polynomials was found by Garsia and Procesi [26], with some improvements by N. Bergergon and Garsia [8]. I will outline their approach.

The basic idea is to describe the cohomology ring $R_\mu = H^*(\mathcal{B}_\mu)$ in elementary terms, without reference to its geometric origin. In order to motivate the description it is helpful first to recall some geometrical facts about R_μ, although they are not actually needed for the construction. For any semisimple Lie group G, the cohomology ring $H^*(\mathcal{B})$ of the whole flag variety is isomorphic to the ring of *coinvariants* for the Weyl group W acting on a Cartan subalgebra $\mathfrak{h}$. The coinvariant ring is by definition $\mathbb{C}[\mathfrak{h}]/I$, where $I = (\mathbb{C}[\mathfrak{h}]^W_+)$ is generated by the W-invariant polynomials without constant term. The isomorphism $H^*(\mathcal{B}) \cong \mathbb{C}[\mathfrak{h}]/I$ is W-equivariant for the Springer action on $H^*(\mathcal{B})$ and the tautological W action on $\mathbb{C}[\mathfrak{h}]/I$.

For $G = GL_n$, the ring $\mathbb{C}[\mathfrak{h}]$ is a polynomial ring $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1,\ldots,x_n]$ in n variables, the Weyl group W is S_n, and $I = (e_1,\ldots,e_n)$ is generated by the elementary symmetric functions. The homomorphism $H^*(\mathcal{B}) \to H^*(\mathcal{B}_\mu)$ induced by the inclusion $\mathcal{B}_\mu \subseteq \mathcal{B}$ is always W-equivariant. For $G = GL_n$ these homomorphisms are surjective, so R_μ is $\mathbb{C}[\mathbf{x}]/I_\mu$ for a homogeneous S_n-invariant ideal $I_\mu \supseteq I$. The highest degree in which R_μ is nonzero is $\dim(\mathcal{B}_\mu) = n(\mu)$, and by the Springer correspondence, this top degree component of R_μ affords the irreducible representation V_μ of S_n.

Recall from §2.1 that $\mathbb{C}[\mathbf{x}]$ contains a unique copy of V_μ in degree $n(\mu)$, spanned by the Garnir polynomials $g_T(\mathbf{x})$ for standard tableaux T of shape μ. It develops that

> *I_μ is the unique largest homogeneous S_n-invariant ideal having zero intersection with the unique copy of V^μ in degree $n(\mu)$.*

So, following the Garsia-Procesi approach, let us forget about $H^*(\mathcal{B}_\mu)$, define I_μ and $R_\mu = \mathbb{C}[\mathbf{x}]/I_\mu$ by the above characterization, and proceed to study the graded character of R_μ. The first lemma is an easy general consequence of the definition of I_μ.

Lemma 3.4.17 (Bergeron-Garsia [8])**.** *Let $\mathbb{C}[\partial] = \mathbb{C}[\partial/\partial x_1, \dots, \partial/\partial x_n]$ act on $\mathbb{C}[\mathbf{x}]$ by differentiation, and let $G_\mu \subseteq \mathbb{C}[\mathbf{x}]$ be the $\mathbb{C}[\partial]$-module generated by the Garnir polynomials g_T for T of shape μ.*

(i) *A polynomial $f \in \mathbb{C}[\mathbf{x}]$ belongs to I_μ if and only if $f(\partial)g_T = 0$ for all T.*
(ii) *The canonical projection of $G_\mu \subseteq \mathbb{C}[\mathbf{x}]$ on $R_\mu = \mathbb{C}[\mathbf{x}]/I_\mu$ is an isomorphism.*

Let $\mathbf{x}_S$ denote the subset of the variables $x_1, \dots, x_n$ indexed by a subset $S \subseteq \{1, \dots, n\}$. Using Lemma 3.4.17 one can prove that the elementary symmetric function $e_k(\mathbf{x}_S)$ belongs to I_μ whenever

$$k > |S| - n + \mu'_1 + \mu'_2 + \cdots \mu'_{n-|S|}. \tag{52}$$

By considering the leading terms of derivatives of the Garnir polynomials, Bergeron and Garsia showed that G_μ and R_μ have dimension at least $\binom{n}{\mu_1,\dots,\mu_l}$. On the other hand Garsia and Procesi constructed a set of $\binom{n}{\mu_1,\dots,\mu_l}$ monomials which span $\mathbb{C}[\mathbf{x}]$ modulo the ideal generated by the functions $e_k(\mathbf{x}_S)$ with k and S satisfying (52). This proves the following result.

Proposition 3.4.18 (Garsia-Procesi [26])**.** *The* Tanisaki generators *$e_k(\mathbf{x}_S)$ for k and S satisfying* (52) *generate the ideal I_μ.*

With this established, Garsia and Procesi showed that the Frobenius series of R_μ satisfies a simple recurrence that also characterizes the "cocharge" version of the transformed Hall-Littlewood polynomials, thereby recovering the result in Proposition 3.4.14 (iii) by direct, elementary means.

Theorem 3.4.19 (Garsia-Procesi [26])**.** *The Frobenius series of R_μ is given by*

$$F_{R_\mu}(z;t) = t^{n(\mu)} H_\mu(z;t^{-1}) = \sum_\lambda \tilde{K}_{\lambda\mu}(t) s_\lambda.$$

In particular, $\tilde{K}_{\lambda\mu}(t)$ is a polynomial with non-negative integer coefficients.

Another aspect of Garsia and Procesi's approach is worth mentioning here. Fix distinct complex numbers $\alpha_1, \dots, \alpha_l$, and let $a \in \mathbb{C}^n$ be a point with μ_1 coordinates equal to α_1, μ_2 equal to α_2, and so on. Let $I_a \subseteq \mathbb{C}[\mathbf{x}]$ be the ideal of polynomials vanishing on the orbit $S_n \cdot a$, and let $\operatorname{gr} I_a$ denote its ideal of leading forms. Clearly $\dim \mathbb{C}[\mathbf{x}]/\operatorname{gr} I_a = \dim \mathbb{C}[\mathbf{x}]/I_a = |S_n \cdot a| = \binom{n}{\mu_1,\dots,\mu_l}$. It is not hard to show, as Garsia and Procesi do in [26], that the Tanisaki generators $e_k(\mathbf{x}_S)$ belong to $\operatorname{gr} I_a$, and hence the dimension count implies that $\operatorname{gr} I_a = I_\mu$. In particular, R_μ affords the induced representation $1 \uparrow_{S_\mu}^{S_n}$, where $S_\mu \cong S_{\mu_1} \times \cdots \times S_{\mu_l}$ is the stabilizer of a. Written in terms of the Frobenius characteristic, this says

$$F_{R_\mu}(z;1) = h_\mu(z).$$

Since the top degree part of R_μ is V^μ, it follows that $t^{n(\mu)} F_{R_\mu}(z;t^{-1})$ has properties (i) and (iii) in the characterization of $H_\mu(z;t)$ in Corollary 3.4.12. The remaining property (ii) is a consequence of Proposition 3.3.1 and the following proposition.

Proposition 3.4.20. *The S_n-modules $\mathrm{Tor}_i^{\mathbb{C}[\mathbf{x}]}(R_\mu, \mathbb{C})$ contain only irreducible representations V_λ with $\lambda \geq \mu'$. In particular, R_μ has an S_n-equivariant graded free resolution over $\mathbb{C}[\mathbf{x}]$ whose terms are generated by S_n-modules containing only those irreducibles.*

This proposition follows from the geometric theorems around the $n!$ conjecture, but I think an elementary proof should be possible. This would improve further on the results of Bergeron, Garsia and Procesi by providing an even more direct route to the identification of the character of R_μ.

3.4.7. *Characterization of the cocharge Kostka-Foulkes polynomials.* For comparison with the Macdonald polynomials discussed in the next section it will be useful to reformulate the conditions characterizing $H_\mu(z;t)$ in Corollary 3.4.12 in terms of their cocharge variant

$$\tilde{H}_\mu(z;t) \underset{\text{def}}{=} t^{n(\mu)} H_\mu(z;t^{-1}) = \sum_\lambda \tilde{K}_{\lambda\mu}(t) s_\lambda(t).$$

To do this, we note that $\tilde{K}_{(n)\mu}(t) = 1$ for all μ. This is clear from the geometric interpretation, since $\tilde{K}_{(n)\mu}(t)$ is the Hilbert series of the S_n invariants in R_μ. Alternatively, it is not difficult to deduce from (41) that $K_{(n)\mu}(t) = t^{n(\mu)}$. We also note that $\tilde{H}_\mu[(1-t)Z;t] = t^{n(\mu)}(-t)^{|\mu|}\omega H_\mu[(1-t^{-1})z;t^{-1}]$, which contains only Schur functions s_λ with $\lambda' \leq \mu$, or equivalently $\lambda \geq \mu'$. Hence the desired characterization is as follows.

Corollary 3.4.21. *The cocharge variant transformed Hall-Littlewood polynomials $\tilde{H}_\mu(z;t)$ are uniquely characterized by the properties*

(i) $\tilde{H}_\mu(z;t) \in \mathbb{Z}[t]\{s_\lambda : \lambda \geq \mu\}$;
(ii) $\tilde{H}_\mu[(1-t)Z;t] \in \mathbb{Z}[t]\{s_\lambda : \lambda \geq \mu'\}$;
(iii) $\tilde{H}_\mu[1;t] = \langle s_{(n)}, \tilde{H}_\mu(z;t)\rangle = 1$.

3.5. **Macdonald polynomials.**

3.5.1. *Definition and transformed version; Kostka-Macdonald coefficients.* We begin with Macdonald's original definition [60, 61] of his polynomials as deformations of the Hall-Littlewood polynomials $P_\lambda(z;t)$ with an extra parameter q. Macdonald first defines a q,t-deformation of the Hall inner product which in our notation is

$$\langle f, g\rangle_{q,t} \underset{\text{def}}{=} \langle f(z),\ g[\tfrac{1-q}{1-t}Z]\rangle.$$

Definition 3.5.1. The *Macdonald symmetric functions* $P_\mu(z;q,t)$ are uniquely characterized by the orthogonality and triangularity conditions

(i) $P_\mu(z;q,t) = s_\mu + \sum_{\lambda<\mu} a_{\lambda\mu}(q,t)s_\lambda$, for suitable coefficients $a_{\lambda\mu} \in \mathbb{Q}(q,t)$;
(ii) $\langle P_\lambda, P_\mu\rangle = 0$ if $\lambda \neq \mu$.

For all purposes that we will be concerned with here it is better to work with transformed versions of the Macdonald polynomials, which are q-deformations of the cocharge variant transformed Hall-Littlewood polynomials $\tilde{H}_\mu(z;t)$.

Definition 3.5.2. The *transformed Macdonald symmetric functions* $\tilde{H}_\mu(z;q,t)$ are uniquely characterized by the conditions

(i) $\tilde{H}_\mu[(1-q)Z;q,t] \in \mathbb{Q}(q,t)\{s_\lambda : \lambda \geq \mu\}$;
(ii) $\tilde{H}_\mu[(1-t)Z;q,t] \in \mathbb{Q}(q,t)\{s_\lambda : \lambda \geq \mu'\}$;
(iii) $\tilde{H}_\mu[1;q,t] = 1$.

Observe that from Corollary 3.4.21 it immediately follows that $\tilde{H}_\mu(z;0,t) = \tilde{H}_\mu(z;t)$. Of course the existence of polynomials meeting the conditions in either of the above definitions requires proof. We will prove in §3.5.2 that the $\tilde{H}_\mu(z;q,t)$ exist, so let us now see that this implies that the $P_\mu(z;q,t)$ exist. To this end, we define another q,t-inner product

$$\langle f, g\rangle_* \underset{\text{def}}{=} \langle f,\ \omega g[(1-q)(1-t)Z]\rangle = (-t)^d\langle f,\ g[(1-q)(1-t^{-1})Z]\rangle, \tag{53}$$

where the last equality is for f and g homogeneous of degree d in z. We have $\langle \tilde{H}_\mu, \tilde{H}_\nu\rangle_* = \langle \tilde{H}_\mu[(1-q)Z;q,t],\ \omega\tilde{H}_\nu[(1-t)Z;q,t]\rangle$. If this is non-zero then by (i) and (ii), we must have $\mu \le \nu$. By symmetry, we must also have $\nu \le \mu$, so we have

(iv) $\langle \tilde{H}_\mu, \tilde{H}_\nu\rangle_* = 0$ if $\mu \ne \nu$.

It follows that the symmetric functions

$$J_\mu(z;q,t) \underset{\text{def}}{=} t^{n(\mu)}\tilde{H}_\mu[(1-t^{-1})Z;q,t^{-1}] \tag{54}$$

have the orthogonality property (ii) of Definition 3.5.1, and they also have the triangularity property (i), except that the leading coefficient need not be 1. But we can of course divide by the leading coefficient to obtain the required polynomials $P_\mu(z;q,t)$.

Definition 3.5.3. The *Kostka-Macdonald polynomials* $\tilde{K}_{\lambda\mu}(q,t)$ are defined through the Schur function expansion

$$\tilde{H}_\mu(z;q,t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t)s_\lambda(z).$$

From the usual proof of the existence theorem for Macdonald symmetric functions (see §3.5.2, below) it is by no means obvious that the "polynomials" $\tilde{K}_{\lambda\mu}(q,t)$ are anything more than rational functions of q and t. Their *integrality* property,

$$\tilde{K}_{\lambda\mu}(q,t) \in \mathbb{Z}[q,t],$$

remained unproven until circa 1995, when several proofs were independently discovered by a number of people using a variety of methods [27, 28, 29, 49, 50, 55, 76]. I will mention one way of proving integrality later.

Definition 3.5.3 is related to the original one of Macdonald in the following way. The symmetric functions $J_\mu(z;q,t)$ defined in (54) are the *integral forms* of Macdonald, who defined coefficients $K_{\lambda\mu}(q,t)$ through the expansion

$$J_\mu(z;q,t) = \sum_\lambda K_{\lambda\mu}(q,t)s_\lambda[Z/(1-t)].$$

By (54), this is equivalent to $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,t^{-1})$. Macdonald defined the integral forms $J_\mu(z;q,t)$ to be scalar multiples of $P_\mu(z;q,t)$ by an explicit normalizing factor. To see that our J_μ is the same scalar multiple of P_μ as Macdonald's, one may compare the identity $K_{(n)\mu}(q,t) = t^{n(\mu)}$ obtained by Macdonald [61] with our $\tilde{K}_{(n)\mu}(q,t) = 1$, which is another way of stating part (iii) of Definition 3.5.2.

3.5.2. *Existence.* We will prove that the polynomials $\tilde{H}_\mu(z;q,t)$ meeting the conditions in Definition 3.5.2 exist by exhibiting them as eigenfunctions of the operator

$$D \underset{\text{def}}{=} \frac{1}{(1-q)(1-t)}(1-D_0), \tag{55}$$

where

$$D_0 f = [u^0] f[Z + (1-q)(1-t)u^{-1}]\Omega[-uZ]. \tag{56}$$

We shall see that the existence theorem also follows as a consequence of the geometric results to be discussed later. This is the case for various other aspects of the elementary theory of Macdonald polynomials as well, such as integrality results. I have included the elementary existence proof here for clarity and to keep this part of the discussion self-contained.

Lemma 3.5.4. *Set $\hat{D} = \Pi_{(1-t^{-1})} D_0 \Pi^{-1}_{(1-t^{-1})}$, or explicitly,*

$$\hat{D} f = [u^0] f[Z - (1-q)u^{-1}]\Omega[(1-t)uZ].$$

Then $\hat{D}$ is lower-triangular with respect to the Schur basis. More precisely,

$$\hat{D} s_\mu = (1 - (1-q)(1-t)B_\mu(q,t))s_\mu + \sum_{\lambda<\mu} b_{\lambda\mu}(q,t)s_\lambda \tag{57}$$

for suitable coefficients $b_{\lambda\mu}(q,t) \in \mathbb{Z}[q,t]$, where

$$B_\mu(q,t) \underset{\text{def}}{=} \sum_{(i,j)\in\mu} q^i t^j. \tag{58}$$

Proof. Write $\hat{D}$ in terms of the modified Jing operators as $\hat{D} f = [u^0]\tilde{S}^t(u) f[Z + qu^{-1}]$. Then using the dual Pieri rule as in the proof of Lemma 3.4.6, we have

$$\hat{D} s_\mu = \sum_m q^m \sum_{\mu/\lambda \in H_m} \tilde{S}^t_m s_\lambda.$$

As in the proof of Lemma 3.4.8, this is equal in n variables $z_1, \dots, z_n$ to

$$t^n s_\mu + (1-t)\sum_{m,k} q^m (-t)^k \sum_{\mu/\lambda\in H_m} \sum_{\gamma/\lambda \in V_k} s_{(m-k,\gamma)}.$$

Also as in the proof of Lemma 3.4.8, the worst case is $\gamma = \lambda + (1^k)$, and then $s_{(m-k,\gamma)} = 0$ or $\pm s_\nu$ with ν of the form indicated there. The condition $\mu/\lambda \in H_m$ implies that $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots$, and it follows with little difficulty that $\nu \leq \mu$.

By carefully examining how equality can be attained, one sees that the contribution to the coefficient of s_μ from the sum over m and k is equal to $\sum_{j=1}^n t^{j-1} q^{\mu_j}$, with μ_j interpreted as zero for $j > l(\mu)$. But this is equal to $[n]_t - (1-q)B_\mu(q,t)$, completing the proof. □

Lemma 3.5.4 implies that $\hat{D}$ has distinct eigenvalues and that its eigenfunction $F(z;q,t)$ with eigenvalue $1-(1-q)(1-t)B_\mu(q,t)$ belongs to the space $\mathbb{Q}(q,t)\{s_\lambda : \lambda \leq \mu\}$. Then $H(z;q,t) = F[Z/(1-t^{-1});q,t]$ is an eigenfunction of D with eigenvalue $B_\mu(q,t)$, and $H[(1-t)Z;q,t]$ belongs to $\mathbb{Q}(q,t)\{s_\lambda : \lambda \geq \mu'\}$. But D is symmetric between q and t, so we have the following corollary.

Corollary 3.5.5. *The operator D has eigenfunctions $\tilde{H}_\mu(z;q,t)$ with eigenvalue $B_\mu(q,t)$ satisfying conditions (i) and (ii) in Definition 3.5.2.*

To see that condition (iii) can also be satisfied we only have to verify that the eigenfunctions have $\langle s_{(n)}, \tilde{H}_\mu(z;q,t)\rangle \neq 0$. But their specializations at $q=0$ are non-zero scalar multiples of the Hall-Littlewood polynomials $\tilde{H}_\mu(z;t)$, so this is clear from Corollary 3.4.21.

Corollary 3.5.6. *Polynomials $\tilde{H}_\mu(z;q,t)$ satisfying conditions (i)-(iii) in Definition 3.5.2 exist.*

Note that the solution of (i)-(iii) is necessarily unique, since the matrix giving any other solution in terms of the basis $\{\tilde{H}_\mu\}$ would have to be upper triangular by (i), lower triangular by (ii), and 1 on the diagonal by (iii).

3.5.3. *Specializations.* For special values of the parameters, there are simpler expressions for the Macdonald symmetric functions, as follows.

Proposition 3.5.7. *The Macdonald symmetric function $\tilde{H}_\mu(z;0,t)$ at $q=0$ is equal to the Hall-Littlewood symmetric function $\tilde{H}_\mu(z;t)$. Equivalently, the Kostka-Macdonald polynomials at $q=0$ reduce to the Kostka-Foulkes polynomials:*

$$\tilde{K}_{\lambda\mu}(0,t) = \tilde{K}_{\lambda\mu}(t). \tag{59}$$

Proof. Compare Corollary 3.4.21 with Definition 3.5.2. □

Proposition 3.5.8. *The Macdonald symmetric function at $q=1$ is given by*

$$\tilde{H}_\mu(z;1,t) = (1-t)^{|\mu|}\big(\prod_i [\mu'_i]_t!\big)\; h_{\mu'}[Z/(1-t)]. \tag{60}$$

Corollary 3.5.9. *The Macdonald symmetric function at $q=t=1$ is given by*

$$\tilde{H}_\mu(z;1,1) = e_1^n,$$

for every partition μ. In other words, the Kostka-Macdonald polynomials satisfy

$$\tilde{K}_{\lambda\mu}(1,1) = \chi^\lambda(1),$$

the number of standard Young tableaux of shape λ, independent of μ.

Proof. The operator D in (55) is well-defined in the limit as $q \to 1$, and a calculation shows that $D_{q=1}$ is a derivation: $D_{q=1}(fg) = (D_{q=1}f)g + f(D_{q=1}g)$. Since $B_\mu(1,t) = \sum_i [\mu'_i]_t$, it follows that

$$\tilde{H}_\mu(z;1,t) = \prod_i \tilde{H}_{(1^{\mu'_i})}(z;1,t).$$

Therefore we only need to show that $\tilde{H}_{(1^n)}(z;1,t) = (1-t)^n[n]_t! h_n[Z/(1-t)]$. This latter fact holds even for $q \neq 1$, since condition (ii) in Definition 3.5.2 implies that $\tilde{H}_{(1^n)}(z;q,t)$ is a scalar multiple of $h_n[Z/(1-t)]$, and the identity $h_n[1/(1-t)] = 1/([n]_t!(1-t)^n)$, which is a special case of (29), fixes the scalar factor.

For the corollary, the Cauchy formula and (29) imply that

$$\begin{aligned}(1-t)^n[n]_t! h_n[Z/(1-t)] &= (1-t)^n[n]_t! \sum_{|\lambda|=n} s_\lambda(z) s_\lambda[\frac{1}{1-t}] \\ &= \sum_{|\lambda|=n} s_\lambda(z) \frac{[n]_t!}{\prod_{x\in\lambda}[h(x)]_t!}.\end{aligned}$$

Setting $t = 1$, this becomes

$$\sum_{|\lambda|} s_\lambda(z)\chi^\lambda(1),$$

by the hook formula (2), and this is finally equal to e_1^n, the Frobenius characteristic of the regular representation of S_n. □

Proposition 3.5.10. *The Macdonald symmetric function $\tilde{H}_\mu(z; q, q^{-1})$ at $t = 1/q$ is given by*

$$\tilde{H}_\mu(z; q, q^{-1}) = q^{-n(\mu)} \prod_{x\in\mu} (1 - q^{h(x)}) s_\mu[Z/(1-q)], \tag{61}$$

where $h(x) = 1 + a(x) + l(x)$ denotes the hook-length of the cell x in the diagram of μ.

Proof. One verifies immediately that $s_\mu[Z/(1-q)]$ satisfies conditions (i) and (ii) in Definition 3.5.2 when $t = 1/q$, and hence $\tilde{H}_\mu(z; q, q^{-1})$ is a scalar multiple of $s_\mu[Z/(1-q)]$. Formula (29) fixes the scalar factor. □

In addition to the specializations the transformed Macdonald polynomials obey two fundamental symmetries. The first one is obvious from the definition.

Proposition 3.5.11. *For every μ we have $\tilde{H}_{\mu'}(z; q, t) = \tilde{H}_\mu(z; t, q)$, or equivalently,*

$$\tilde{K}_{\lambda\mu'}(q, t) = \tilde{K}_{\lambda\mu}(t, q).$$

Proposition 3.5.12. *For every μ we have $\omega\tilde{H}_\mu(z; q, t) = t^{n(\mu)}q^{n(\mu')}\tilde{H}_\mu(z; q^{-1}, t^{-1})$, or equivalently,*

$$\tilde{K}_{\lambda'\mu}(q, t) = t^{n(\mu)}q^{n(\mu')}\tilde{K}_{\lambda\mu}(q^{-1}, t^{-1}).$$

Proof. It is easy to see that $\omega\tilde{H}_\mu(z; q^{-1}, t^{-1})$ satisfies conditions (i) and (ii) in Definition 3.5.2, and hence is a scalar multiple of $\tilde{H}_\mu(z; q, t)$. To fix the scalar, we need the identity

$$\tilde{K}_{(1^n),\mu} = t^{n(\mu)}q^{n(\mu')}.$$

Somewhat surprisingly, this is one of the more subtle results in the elementary theory of Macdonald polynomials. We will prove something a little more general in the next section, in Corollary 3.5.20. □

3.5.4. *The positivity problem.* From the observations in the preceding section, we see that the specializations $\tilde{K}_{\lambda\mu}(0, t)$, $\tilde{K}_{\lambda\mu}(q, 0)$, $\tilde{K}_{\lambda\mu}(1, t)$ and $\tilde{K}_{\lambda\mu}(q, 1)$ of the Kostka-Macdonald polynomials $\tilde{K}_{\lambda\mu}(q, t)$ have non-negative integer coefficients. We shall also see explicitly in Corollary 3.5.20 below that $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$ when λ is a hook shape partition. These facts and tables which he had computed for $n \leq 6$ led Macdonald to conjecture the following theorem, already in his 1988 paper [60].

Theorem 3.5.13 (Macdonald positivity conjecture)**.** *The Kostka-Macdonald polynomials $\tilde{K}_{\lambda\mu}(q, t)$ are polynomials with non-negative integer coefficients.*

Part of the conjecture was the *integrality* property, $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{Z}[q, t]$, which as we have mentioned has since been proved by elementary methods. The additional *positivity* property, $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$, lies deeper, as might be expected from the fact that it is a serious theorem even for the Kostka-Foulkes polynomials $\tilde{K}_{\lambda\mu}(0, t)$. The only proof known at present is the one based on the geometry of Hilbert schemes that I will describe in §5.4.

In the course of working on the positivity conjecture, Garsia and I and others were led to integrality and positivity conjectures for other quantities related to the Kostka-Macdonald polynomials. Some of these conjectures have now also been proven using geometric methods. In the rest of this section I will explain some additional aspects of the theory that lead to these further conjectures.

3.5.5. *Operators Δ and ∇ and the plethystic formula.* For any symmetric function f we define a homogeneous linear operator Δ_f on symmetric functions with coefficients in $\mathbb{Q}(q,t)$ by the formula

$$\Delta_f \tilde{H}_\mu(z;q,t) = f[B_\mu]\tilde{H}_\mu(z;q,t), \tag{62}$$

where $B_\mu = B_\mu(q,t)$ is given by (58). In particular, the operator D of §3.5.2 is $D = \Delta_{e_1}$ in this notation. In [7] one can find formulas similar to (56), but more complicated, for the operators Δ_f. We also define

$$\nabla \tilde{H}_\mu(z;q,t) = t^{n(\mu)}q^{n(\mu')}\tilde{H}_\mu(z;q,t), \tag{63}$$

which is equivalent to setting $\nabla f(z) = \Delta_{e_n} f(z)$ for f homogeneous of degree n. The operator ∇ plays an especially important role in the theory.

F. Bergeron, Garsia, Tesler and I studied these operators in [7, 25]. We proved integrality properties and we conjectured positivity properties for them, and we showed that various results previously discovered by others could be deduced easily with their aid. The next proposition, which was proved in [7] by elementary means, also follows (in part) from a natural geometric interpretation of the operators, as we shall see in §5.4.4.

Proposition 3.5.14 ([7])**.** *The operators Δ_f and hence also ∇ are integral, in the sense that if f and F have coefficients in $\mathbb{Z}[q,t]$, then so does $\Delta_f F$. The operator ∇^{-1} is Laurent-integral, that is, $\nabla^{-1}F$ has coefficients in $\mathbb{Z}[q,t,q^{-1},t^{-1}]$*

Conjecture 3.5.15 ([7])**.** *When expanded on the Schur basis $\{s_\lambda\}$, the following quantities all have coefficients in $\mathbb{N}[q,t]$:*

(I)
$$(-1)^{i(\mu)}\nabla^m s_\mu(z)$$
for all μ and all $m \geq 1$, where $i(\mu) = \binom{\mu_1}{2} + \sum_{\mu'_i<i-1}(i-1-\mu'_i)$;

(II)
$$(-1)^{|\mu|-l(\mu)}\nabla\tilde{H}_\mu(z;0,t)$$
for all μ;

(III)
$$\nabla\omega\tilde{H}_\mu(z;0,t^{-1})$$
for all μ, and also
$$\nabla\omega\tilde{H}_\mu(z;0,t^{-1}) - \nabla\omega\tilde{H}_\nu(z;0,t^{-1})$$
whenever $\mu \geq \nu$;

(IV)
$$(-1)^{|\mu|-l(\mu)}\nabla m_\mu(z)$$
for all μ, and moreover the coefficients are doubly unimodal in q and t;

(V)
$$\Delta_{s_\nu} e_n(z)$$
for all ν and n.

All except (II) above have as special cases the quantity

$$\nabla e_n(z),$$

whose special significance will be explained below. Each of (I)–(V) has a geometric interpretation in the Hilbert scheme setting, and in principle it should eventually be possible to prove all five parts of the conjecture using geometric methods. To date, this has only been carried out for the following weakened form of (V).

Theorem 3.5.16 ([40]). *When expanded on the Schur basis, the quantity*

$$\Delta_{s_\nu}\nabla e_n(z)$$

has coefficients in $\mathbb{N}[q,t]$ *for all* ν *and* n.

In [7] we proved Proposition 3.5.14 by considering the operator D_1 defined by taking the coefficient of u^1 instead of u^0 in (56), and showing that there is an integral basis of the algebra of symmetric functions consisting of elements of the form

$$u_\lambda(z;q,t) = e_1^a D_1 e_1^{\lambda_1-1} D_1 e_1^{\lambda_2-1}\cdots D_1 e^{\lambda_k-1},$$

where $\lambda = (\lambda_1,\ldots,\lambda_k,1^a)$ with $\lambda_k > 1$. The proposition then follows from various commutation relations between the operators Δ_f, D_1 and multiplication by e_1. I will not go into further detail on these relations here. The following fundamental result is proved in a similar way.

Proposition 3.5.17 ([25]). *For any symmetric function* f, *we have*

$$\langle f, \tilde{H}_\mu[Z+1;q,t]\rangle_* = K_f[(1-q)(1-t)B_\mu - 1;\ -1],$$

where

$$K_f(z;u) \underset{\text{def}}{=} \nabla^{-1}(f[Z-u])$$

and $\langle -,-\rangle_*$ *is defined in* (53).

Let me mention two important consequences that follow by straightforward calculations from this proposition. The first of these is the *plethystic formula* for Kostka-Macdonald polynomials.

Corollary 3.5.18. *Fix a partition* γ *of size* k. *Then the Kostka-Macdonald coefficients* $\tilde{K}_{(n-k,\gamma)\mu}(q,t)$, *where* $n = |\mu| \geq k+\gamma_1$, *are given for all* μ *simultaneously by*

$$\tilde{K}_{(n-k,\gamma)\mu}(q,t) = k_\gamma[B_\mu;q,t],$$

where

$$k_\gamma[\tfrac{Z+1}{(1-q)(1-t)};q,t] = \nabla^{-1}\omega\left(s_\lambda[\tfrac{Z+1}{(1-q)(1-t)}-1]\right).$$

This corollary was first proved prior to Proposition 3.5.17, by Garsia and Tesler in [28], but without the simple formula for k_γ. Another consequence of Proposition 3.5.17 is the reciprocity formula of Koornwinder and Macdonald, which takes the following pleasant form when written in terms of the transformed Macdonald symmetric functions.

Proposition 3.5.19 ([61]). *For all pairs of partitions* λ, μ, *we have*

$$\tilde{H}_\mu[1-uA_\lambda;q,t]\Omega[uB_\mu] = \tilde{H}_\lambda[1-uA_\mu;q,t]\Omega[uB_\lambda],$$

where $B_\mu = B_\mu(q,t)$ *is given by* (58) *and* $A_\mu = 1-(1-q)(1-t)B_\mu$.

As a particular consequence, by taking $\lambda = \emptyset$ in the reciprocity formula, we obtain the specialization theorem of Macdonald [61].

Corollary 3.5.20. *Under the plethystic specialization $Z \mapsto 1-u$, we have*

$$\tilde{H}_\mu[1-u;q,t] = \Omega[-uB_\mu].$$

Equivalently, for $\lambda = (n-r, 1^r)$ a hook shaped partition and all partitions μ of n, we have

$$\tilde{K}_{\lambda\mu} = e_r[B_\mu - 1]. \tag{64}$$

3.5.6. *Raising operators.* Operators on Macdonald symmetric functions analogous to the Jing operators S^t_m in Definition 3.4.5 were first found by Lapointe and Vinet [55] and Kirillov and Noumi [49]. Their work provides several different families of operators B_m which add a part to the indexing partition of a Macdonald polynomial, that is,

$$B_m \tilde{H}_\mu(z;q,t) = \tilde{H}_{(m,\mu)}(z;q,t) \quad \text{for } m \geq \mu_1. \tag{65}$$

Equivalently, for every partition μ, there holds the Rodrigues formula

$$\tilde{H}_\mu(z;q,t) = B_{\mu_1} B_{\mu_2} \cdots B_{\mu_l}(1). \tag{66}$$

Note that in light of the symmetry given by Proposition 3.5.11 it is the same thing to give operators B_{1^m} with the property

$$B_{1^m} \tilde{H}_\mu(z;q,t) = \tilde{H}_{\mu+(1^m)}(z;q,t) \quad \text{for } m \geq l(\mu), \tag{67}$$

so that

$$\tilde{H}_{\mu'}(z;q,t) = B_{1^{\mu'_1}} B_{1^{\mu'_2}} \cdots B_{1^{\mu'_r}}(1). \tag{68}$$

Zabrocki subsequently discovered that such raising operators can be manufactured at will from *any* operators which have the property in (67) for the Hall-Littlewood polynomials.

Theorem 3.5.21 ([29])**.** *For any operator V on symmetric functions, define its q-deformation V^q by the formula*

$$V^q f = V_Y(\, f[qZ + (1-q)Y]\,)\,|_{Y \mapsto Z},$$

where V_Y denotes V acting on symmetric functions in the variables y and treating the variables z as scalars (so in particular, $V^0 = V$ and $V^1 f = V(1)\cdot f$). Let T_{1^m} be any linear operators whatsoever that satisfy

$$T_{1^m} \tilde{H}_\mu(z;t) = \tilde{H}_{\mu+(1^m)}(z;t) \quad \textit{for } m \geq l(\mu). \tag{69}$$

Then

$$T^q_{1^m} \tilde{H}_\mu(z;q,t) = \tilde{H}_{\mu+(1^m)}(z;q,t) \quad \textit{for } m \geq l(\mu). \tag{70}$$

It turns out rather amazingly that this theorem is almost trivial to prove, once it is given that there exists *some* family of operators satisfying both (69) and (70). Conveniently, the operators of Kirillov and Noumi have the correct form, *i.e.*, they are the Zabrocki q-deformations of their $q=0$ specializations, and the result follows. One nice application of Zabrocki's theorem is to give the simplest of many proofs of the integrality theorem.

Corollary 3.5.22 ([29])**.** *The Kostka-Macdonald polynomials are polynomials, $\tilde{K}_{\lambda\mu}(q,t) \in \mathbb{Z}[q,t]$.*

Proof. It is easy to see that Theorem 3.5.21 is equivalent to the corresponding result for $H_\mu(z;t)$ and $H_\mu(z;q,t) \underset{\text{def}}{=} t^{n(\mu)}\tilde{H}(z;q,t^{-1})$. In this variant of the theorem, take for T_{1^m} the *trivial* operator defined by

$$T_{1^m}H_\mu(z;t) = \begin{cases} H_{\mu+(1^m)}(z;t) & \text{if } m \geq l(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.4.12, part (v) shows that the transformed Hall-Littlewood symmetric functions $H_\mu(z;t)$ form an integral basis for the algebra of symmetric functions with coefficients in $\mathbb{Z}[t]$. In particular, the operators T_{1^m} act on the Schur basis with coefficients in $\mathbb{Z}[t]$, and their q-deformations $T_{1^m}^q$ act with coefficients in $\mathbb{Z}[q,t]$. Then the Rodrigues formula implies that $H_\mu(z;q,t) \in \mathbb{Z}[q,t]\{s_\lambda\}$, that is, $K_{\lambda\mu}(q,t) \in \mathbb{Z}[q,t]$ for all λ and μ. By the symmetry in Proposition 3.5.12, this is equivalent to $\tilde{K}_{\lambda\mu} \in \mathbb{Z}[q,t]$ for all λ and μ. □

In is worthy of remark that the Jing operators S_m^t are just the Zabrocki t-deformations of the Bernstein operators S_m^0.

3.5.7. *Operator ∇ and q-Lagrange inversion.* The q-Catalan numbers discussed in §2.2, the inversion enumerator for forests in §2.3, and more generally the q-Lagrange inversion coefficients $k_n(q)$ in Proposition 2.4.2 turn out to have remarkable expressions involving the operator ∇ specialized at $t = 1$.

Proposition 3.5.23. *The quantity*

$$\nabla_{t=1}e_n$$

is given by the formula for $k_n(q)$ in Proposition 2.4.2, when the indeterminates e_k in the formula are interpreted as elementary symmetric functions.

Corollary 3.5.24. *The Carlitz-Riordan q-Catalan numbers are given in terms of ∇ by*

$$C_n(q) = \langle e_n, \nabla_{t=1}e_n \rangle, \tag{71}$$

while the inversion enumerator for forests, or weight enumerator for parking functions, is given by

$$J_n(q) = \langle e_1^n, \nabla_{t=1}e_n \rangle. \tag{72}$$

Proof of the corollary. The inner product $\langle e_n, e_\nu \rangle$ is equal to 1 for all ν, so the right-hand side of (71) is the specialization of $k_n(q)$ at $e_k = 1$ for all k. We have seen in §2.4 that this is the same as $C_n(q)$.

The inner product $\langle e_1^n, e_\nu \rangle$ is equal to the multinomial coefficient $\binom{n}{\nu_1,\ldots,\nu_l}$. Using (14), we obtain (72). □

Proposition 3.5.23 was proved in [23]. Here I will sketch a slick version of the proof. There are two key points. The first is that Garsia's original solution of the q-Lagrange inversion problem in [30] can be recast in symmetric function language when the coefficients e_k are understood as elementary symmetric functions, to give another formula for $k_n(q)$, different from the one in Proposition 2.4.2, namely

$$k_n(q) = \sum_{|\mu|=n} q^{n(\mu')} h_\mu[Z/(1-q)] f_\mu[1-q],$$

where the $f_\mu(z) = \omega m_\mu(z)$ are the so-called *forgotten* symmetric functions. The second is that the specialization in Proposition 3.5.8, taken with $t = 1$ instead of $q = 1$, implies that

$$\nabla_{t=1} h_\mu[Z/(1-q)] = q^{n(\mu')} h_\mu[Z/(1-q)].$$

But $\{h_\mu[Z/(1-q)]\}$ and $\{m_\mu[(1-q)Z]\}$ are dual bases with respect to the Hall-inner product, so the Cauchy formula (45) yields

$$h_n(YZ) = \sum_{|\mu|=n} h_\mu[Z/(1-q)] m_\mu[(1-q)Y].$$

Now applying ω in the Y variables and then setting $Y = 1$ yields

$$e_n(z) = \sum_{|\mu|=n} h_\mu[Z/(1-q)] f_\mu[1-q],$$

and hence

$$k_n(q) = \nabla_{t=1} e_n.$$

The $t = 1/q$ specialization of ∇e_n can be derived by similar devices.

Proposition 3.5.25. *We have*

$$\nabla_{t=q^{-1}} e_n = q^{-\binom{n}{2}} \frac{1}{[n+1]_q} e_n\big[[n+1]_q Z\big].$$

Note that the above formula is a kind of naive q-analog of the classical Lagrange inversion formula (8).

The proof of Proposition 3.5.23 given above and the similar (unstated) proof of Proposition 3.5.25 evade the issue of determining ∇e_n explicitly. We did, however, find an explicit formula in [23]. Since we will need it later, I will review its derivation.

In order to apply ∇, we need to write e_n in terms of the transformed Macdonald polynomials. To do this, we use the fact that the $\tilde{H}_\mu(z; q, t)$ are orthogonal with respect to the inner product $\langle -, - \rangle_*$ in (53), together with the identity

$$\langle \tilde{H}_\mu, \tilde{H}_\mu \rangle_* = t^{n(\mu)} q^{n(\mu')} \prod_{x \in \mu} (1 - t^{1+l(x)} q^{-a(x)})(1 - t^{-l(x)} q^{1+a(x)}). \tag{73}$$

Here $a(x)$ and $l(x)$ denote the arm and leg of x, as defined in §2.1. Identity (73) follows from Macdonald's formula for the inner product $\langle P_\mu, P_\mu \rangle$. However, rather than repeat its derivation here, I prefer to appeal to the geometric proof that will be given in §5.4.3. The latter proof is more illuminating than the elementary one because it explains the meaning of the factors in (73).

The $\langle -, - \rangle_*$ orthogonality and (73) yield as an instance of the Cauchy formula

$$\omega\Omega\left[\frac{YZ}{(1-q)(1-t)}\right] = \sum_\mu \frac{t^{-n(\mu)} q^{-n(\mu')} \tilde{H}_\mu(y; q, t) \tilde{H}_\mu(z; q, t)}{\prod_{x \in \mu} (1 - t^{1+l(x)} q^{-a(x)})(1 - t^{-l(x)} q^{1+a(x)})}.$$

Taking the homogeneous component of degree n, setting $Y = 1 - u$, and using Corollary 3.5.20, we arrive at

$$e_n\left[\frac{(1-u)Z}{(1-q)(1-t)}\right] = \sum_{|\mu|=n} \frac{t^{-n(\mu)} q^{-n(\mu')} \Omega[-uB_\mu] \tilde{H}_\mu(z; q, t)}{\prod_{x \in \mu} (1 - t^{1+l(x)} q^{-a(x)})(1 - t^{-l(x)} q^{1+a(x)})}.$$

Dividing both sides by $1-u$ and setting $u=1$ in what remains produces

$$(-1)^{n-1}p_n\left[\frac{Z}{(1-q)(1-t)}\right]=\sum_{|\mu|=n}\frac{t^{-n(\mu)}q^{-n(\mu')}\Pi_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in\mu}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})},$$

where

$$\Pi_\mu \underset{\text{def}}{=} \Omega[B_\mu-1]=\prod_{\substack{(i,j)\in\mu\\(i,j)\neq(0,0)}}(1-q^it^j). \tag{74}$$

Finally, applying the operator $D=\Delta_{e_1}$ to both sides yields, by a calculation on the left-hand side and using Corollary 3.5.5 on the right,

$$e_n(z)=\sum_{|\mu|=n}\frac{t^{-n(\mu)}q^{-n(\mu')}(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in\mu}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}.$$

This proves the following proposition.

Proposition 3.5.26. *We have explicitly*

$$\nabla e_n(z)=\sum_{|\mu|=n}\frac{(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in\mu}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}, \tag{75}$$

with $B_\mu(q,t)$ as in (58) *and $\Pi_\mu(q,t)$ as in* (74).

4. The $n!$ and $(n+1)^{n-1}$ conjectures

4.1. **The $n!$ conjecture.** The proof of the Macdonald positivity conjecture rests on the interpretation of $\tilde{H}_\mu(z;q,t)$ as the Frobenius series of a suitable doubly graded S_n module. This interpretation, which Garsia and I proposed in 1991, is what has come to be known as the *n! conjecture*. I will present it here in a way that I hope gives a flavor of how we discovered it. The results below and some of our other early results on the $n!$ conjecture were announced in [22] and given a fuller treatment in [24].

In §3.4.6, we saw that the Kostka-Foulkes polynomials $\tilde{K}_{\lambda\mu}(t)$ describe multiplicities of S_n characters χ^λ in the graded character of the Garsia-Procesi ring R_μ, which is the same thing as the cohomology ring $H^*(\mathcal{B}_\mu)$ of a springer fiber. We also saw that these rings are quotients of the ring

$$R_{1^n}(\mathbf{x})=\mathbb{C}[\mathbf{x}]/I$$

of coinvariants for the natural action of S_n on the polynomial ring $\mathbb{C}[\mathbf{x}]=\mathbb{C}[x_1,\dots,x_n]$.

From Propositions 3.5.7 and 3.5.11 we know that the Kostka-Macdonald polynomials $\tilde{K}_{\lambda\mu}(q,t)$ specialize to

$$\tilde{K}_{\lambda\mu}(0,t)=\tilde{K}_{\lambda\mu}(t);\quad \tilde{K}_{\lambda\mu}(q,0)=\tilde{K}_{\lambda\mu'}(q).$$

This suggests that $\tilde{H}_\mu(z;q,t)$ might be the Frobenius series of a doubly graded ring $R_\mu(\mathbf{x},\mathbf{y})$, a quotient of the polynomial ring $\mathbb{C}[\mathbf{x},\mathbf{y}]=\mathbb{C}[x_1,y_1,\dots,x_n,y_n]$ in two sets of variables, whose components of degree zero in the $\mathbf{y}$ and $\mathbf{x}$ variables are $R_\mu(\mathbf{x})$ and $R_{\mu'}(\mathbf{y})$, respectively. In other words, we may expect $R_\mu(\mathbf{x},\mathbf{y})$ to be a suitable quotient of

$$R_\mu(\mathbf{x})\otimes R_\mu(\mathbf{y})\cong H^*(\mathcal{B}_\mu)\otimes H^*(\mathcal{B}_{\mu'}).$$

Other information about the $\tilde{K}_{\lambda\mu}(q,t)$ gives more clues. In particular, by Corollary 3.5.9, we want $R_\mu(\mathbf{x},\mathbf{y})$ to afford the regular representation of S_n. To this end, recall that the defining ideal of $R_\mu(\mathbf{x})$ is generated by the leading forms of polynomials $p(\mathbf{x})$ that vanish on the S_n-orbit of a point $a = (a_1,\dots,a_n) \in \mathbb{C}^n$ with μ_i coordinates a_j equal to α_i, for distinct complex numbers α_i. Suppose now that we fix a point

$$b = (a_1, b_1, \dots, a_n, b_n) \in \mathbb{C}^{2n} \tag{76}$$

by assigning distinct numbers α_j to each row j in the diagram of μ and β_i to each column i, and setting $a_k = \alpha_{j_k}$, $b_k = \alpha_{i_k}$, where $(i_1,j_1),\dots,(i_n,j_n)$ is a list of the cells $(i,j) \in \mu$ in some arbitrary order. The n pairs (a_k,b_k) are all distinct, so the S_n orbit of b in $\mathbb{C}^{2n} = (\mathbb{C}^2)^n$ is a *regular* orbit, with $n!$ distinct points. Let J_b be the ideal of leading forms of polynomials $p(\mathbf{x},\mathbf{y})$ that vanish on the orbit $S_n \cdot b$, and put $R_b = \mathbb{C}[\mathbf{x},\mathbf{y}]/J_b$. Then we have automatically that R_b affords the regular representation of S_n. Projecting b on the $\mathbf{x}$ coordinates gives a point with coordinate multiplicities μ, so the subring of R_b generated by $\mathbf{x}$ is isomorphic to R_μ. The subring generated by $\mathbf{y}$ is isomorphic to $R_{\mu'}$ by symmetry. In short, the ring R_b has the properties we desire, except that it is apparently only singly, not doubly, graded.

In practice, it always turns out that the ideal J_b is doubly homogeneous. Another way to say this is that if we assign the $\mathbf{x}$ variables different weights from the $\mathbf{y}$ variables for the purpose of defining "leading forms," the resulting ideal does not depend on the weights. It also turns out that the ring R_b is always *Gorenstein*, and this property is the key to reformulating the problem independently of the choice of b. A special case of the definition will suffice for the moment.

Definition 4.1.1. The *socle* of a finite-dimensional graded $\mathbb{C}$-algebra R is the set $\mathrm{soc}(R)$ of elements $x \in R$ annihilated by the maximal homogeneous ideal R_+. The algebra R is *Gorenstein* if $\dim \mathrm{soc}(R) = 1$.

Note that the highest degree homogeneous component $R_{d_{\max}}$ is always contained in $\mathrm{soc}(R)$, so if R is Gorenstein, then $R_{d_{\max}} = \mathrm{soc}(R)$. Recall also that the only one-dimensional representations of S_n are the trivial representation and the sign representation. Hence if R has an action of S_n by algebra automorphisms, and R is Gorenstein, then the socle of R must afford either the trivial or the sign representation.

Now suppose that R is a graded Gorenstein quotient of $R_\mu(\mathbf{x}) \otimes R_{\mu'}(\mathbf{y})$. The ring R_μ contains only S_n modules V_λ with $\lambda \geq \mu$, while $R_{\mu'}$ contains only V_λ with $\lambda \geq \mu'$ (see Corollary 3.4.21). However, the sign representation $\varepsilon = V_{(1^n)}$ can only occur in tensor products of the form $V_\lambda \otimes V_{\lambda'}$. Hence it occurs only once in $R_\mu \otimes R_{\mu'}$, in the top degree component $(R_\mu)_{n(\mu)} \otimes (R_{\mu'})_{n(\mu')}$. If we assume that this unique copy of the sign representation is not in the kernel of the canonical projection $R_\mu \otimes R_{\mu'} \to R$, then its image in R must be equal to $\mathrm{soc}(R)$. These considerations lead to the following result.

Proposition 4.1.2. *There is a unique S_n-invariant Gorenstein (doubly) graded ideal $J_\mu \subseteq R_\mu(\mathbf{x}) \otimes R_{\mu'}(\mathbf{y})$ such that $R_\mu(\mathbf{x},\mathbf{y}) = \mathbb{C}[\mathbf{x},\mathbf{y}]/J_\mu$ contains a copy of the sign representation. A polynomial $f \in \mathbb{C}[\mathbf{x},\mathbf{y}]$ belongs to J_μ if and only if the principal ideal (f) generated by its image in $R_\mu(\mathbf{x}) \otimes R_{\mu'}(\mathbf{y})$ has zero intersection with the unique copy of the sign representation.*

Proof. Define J_μ to be the set of polynomials f satisfying the stated criterion. It is easy to see that J_μ is a doubly homogeneous ideal and that $R_\mu(\mathbf{x},\mathbf{y}) = \mathbb{C}[\mathbf{x},\mathbf{y}]/J_\mu$ is a quotient of $R_\mu(\mathbf{x}) \otimes R_{\mu'}(\mathbf{y})$ in which the socle is the sign representation. Hence it is Gorenstein. For uniqueness, note that any ideal in a finite-dimensional graded algebra has non-zero intersection with the socle. On one hand,

if R is any Gorenstein quotient R of $R_\mu(\mathbf{x}) \otimes R_{\mu'}(\mathbf{y})$ in which the sign representation survives, this implies that every $f \in J_\mu$ must vanish in R. Hence R is a quotient of $R_\mu(\mathbf{x}, \mathbf{y})$. On the other hand, any proper ideal in $R_\mu(\mathbf{x}, \mathbf{y})$ contains the socle, so $R = R_\mu(\mathbf{x}, \mathbf{y})$. □

Henceforth we will simply write

$$R_\mu = \mathbb{C}[\mathbf{x}, \mathbf{y}]/J_\mu$$

for the ring in Proposition 4.1.2, and use the notation $R_\mu(\mathbf{x})$, $R_{\mu'}(\mathbf{y})$ to distinguish the Garsia-Procesi rings in one set of variables. Before continuing further, let me give also a more elementary description of R_μ. As before let $(i_1, j_1), \ldots, (i_n, j_n)$ be a list of the cells $(i, j) \in \mu$ in some order, and set

$$\Delta_\mu(\mathbf{x}, \mathbf{y}) = \det \begin{bmatrix} x_1^{j_1} y_1^{i_1} & x_2^{j_1} y_2^{i_1} & \ldots & x_n^{j_1} y_n^{i_1} \\ \vdots & & \ddots & \vdots \\ x_1^{j_n} y_1^{i_n} & x_2^{j_n} y_2^{i_n} & \ldots & x_n^{j_n} y_n^{i_n} \end{bmatrix}. \tag{77}$$

It is not hard to prove the following characterization of the defining ideal J_μ of R_μ.

Proposition 4.1.3. *The ideal J_μ in Proposition 4.1.2 coincides with the ideal of polynomial differential operators that annihilate Δ_μ, that is, we have $f \in J_\mu$ if and only if*

$$f(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_n})\Delta_\mu = 0.$$

Consider once again the ideal of leading forms J_b of the vanishing ideal of an orbit $S_n \cdot b$, with b as in (76). We will prove that every $f \in J_b$ satisfies the condition in Proposition 4.1.2, so $J_b \subseteq J_\mu$. If $f \in J_b$, then by definition f coincides on $S_n \cdot b$ with some polynomial g of lower degree. There is an essentially unique function ϵ on $S_n \cdot b$ that affords the sign representation. It cannot be represented by a polynomial of degree less than $n(\mu) + n(\mu')$, because $R_\mu(\mathbf{x}) \otimes R_{\mu'}(\mathbf{y})$ has no copy of the sign representation in lower degree. Suppose now that hf is a homogeneous multiple of f of degree $n(\mu) + n(\mu')$ which coincides on $S_n \cdot b$ with ϵ. Then hg also represents ϵ and has smaller degree, a contradiction. This shows that f satisfies the condition in Proposition 4.1.2. The containment $J_b \subseteq J_\mu$, has the following consequences.

Proposition 4.1.4. *We have $\dim R_\mu \leq n!$. If equality holds, then $J_b = J_\mu$. In particular, J_b is then doubly homogeneous and does not depend on the choice of b, and R_μ affords the regular representation of S_n.*

At this point the interpretation we have been seeking for the Macdonald symmetric function $\tilde{H}_\mu(z; q, t)$ becomes natural and plausible to conjecture.

Theorem 4.1.5 ($n!$ conjecture)**.** *The Frobenius series of R_μ is the transformed Macdonald symmetric function*

$$F_{R_\mu}(z; q, t) = \tilde{H}_\mu(z; q, t).$$

In particular R_μ affords the regular representation of S_n, and hence $\dim R_\mu = n!$.

The "in particular" follows by Corollary 3.5.9. The Macdonald positivity conjecture, Theorem 3.5.13, is a corollary to this theorem.

4.1.1. *Examples.* Let us briefly consider the simplest case of the $n!$ conjecture, when $\mu = (1^n)$. In this case Δ_μ is the Vandermonde determinant $\Delta(\mathbf{x})$ in the $\mathbf{x}$ variables, and $R_{(1^n)} = R_{(1^n)}(\mathbf{x})$ is the coinvariant ring of S_n acting on $\mathbb{C}^n$, or the cohomology ring $H(\mathcal{B})$ of the flag variety. Here we are in a classical situation, and the $n!$ conjecture is an instance of the following more general result.

Proposition 4.1.6. *Let W be a group generated by complex reflections on a vector space $\mathfrak{h}$. Then the coinvariant ring $R = \mathbb{C}[\mathfrak{h}]/W$ affords the regular representation of W. It is Gorenstein, and the representation of W on its socle is the determinant of the representation on $\mathfrak{h}$. When W is the Weyl group of a semisimple Lie algebra $\mathfrak{g}$, then R is isomorphic to the cohomology ring of the flag variety for the corresponding Lie group G.*

It is also instructive to relate the known symmetries and specializations of the Kostka-Macdonald polynomials to Theorem 4.1.5. We have already seen that the specializations $q = 0$ and $t = 0$, and also $q = t = 1$, are in agreement with the theorem. These led us to the construction of R_μ in the first place.

The specialization $q = 1$ (and by symmetry $t = 1$) can be obtained with a bit of effort from the identification of J_μ as the ideal of leading forms J_b for an orbit $S_n \cdot b$. To do this one uses the fact noted above that J_b is independent of the choice of weights for leading forms, which allows one to take leading forms in the $\mathbf{x}$ variables first and the $\mathbf{y}$ variables afterwards.

The symmetry $\tilde{H}_{\mu'}(z; q, t) = \tilde{H}_\mu(z; q, t)$ is in obvious agreement with the theorem.

More interesting is the symmetry $\tilde{K}_{\lambda'\mu}(q, t) = t^{n(\mu)} q^{n(\mu')} \tilde{K}_{\lambda\mu}(q^{-1}, t^{-1})$. This symmetry reflects the Gorenstein property of R_μ, which implies that multiplication gives a perfect pairing of complementary degrees

$$(R_\mu)_{r,s} \otimes (R_\mu)_{n(\mu)-r,n(\mu)-s} \to (R_\mu)_{n(\mu),n(\mu')} = \mathrm{soc}(R_\mu).$$

The pairing is S_n equivariant, and it pairs complementary irreducible representations V_λ and $V_{\lambda'}$ because the socle affords the sign representation. Hence V_λ and $V_{\lambda'}$ have the same multiplicities in complementary degrees.

4.2. The $(n+1)^{n-1}$ conjecture.

4.2.1. *Diagonal harmonics and coinvariants.* The rings R_μ involved in the $n!$ conjecture are graded quotients of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ that afford the regular representation of S_n. In particular, their only copy of the trivial representation is given by the constants, and so every R_μ is a quotient of the *diagonal coinvariant ring*

$$R_n = \mathbb{C}[\mathbf{x}, \mathbf{y}]/I_n, \tag{78}$$

where I_n is the ideal generated by the space $\mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{S_n}$ of invariant polynomials without constant term. Because of this connection, Garsia and I were led during our early investigation of the rings R_μ to study also the ring R_n. The result was the discovery of its remarkable combinatorial aspects, as related in the introduction. Below I will review some of these aspects in more detail.

First we should take note of some basic facts about R_n. The first is that the ideal I_n has a well-known set of generators.

Proposition 4.2.1 (Weyl [85])**.** *The ideal I_n is generated by the bivariate power-sums*

$$p_{r,s}(\mathbf{x}, \mathbf{y}) \underset{\text{def}}{=} \sum_{i=1}^{n} x^r y^s, \quad 1 \le r + s \le n.$$

The corresponding result also holds for more than two sets of variables $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}$.

The second fact is that as far as the study of its Frobenius series is concerned, it doesn't matter whether we work with the coinvariant ring R_n or the space of harmonics.

Definition 4.2.2. The space DH_n of *diagonal harmonics* for S_n is the set of polynomials $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ annihilated by all S_n-invariant constant coefficient differential operators without constant term, that is, by $\mathbb{C}[\partial\mathbf{x}, \partial\mathbf{y}]^{S_n}_+$.

Since R_μ is a quotient of R_n, it follows from Proposition 4.1.3 that the polynomial Δ_μ is a diagonal harmonic. It at first seemed more natural in the early days to work with the space of derivatives of Δ_μ rather than with R_μ, and with DH_n rather than R_n, but in the end it is immaterial, by the following easy proposition.

Proposition 4.2.3. *The canonical projection of DH_n on R_n is an isomorphism. Similarly, if $D_\mu = \mathbb{C}[\partial\mathbf{x}, \partial\mathbf{y}]\Delta_\mu$, then the canonical projection of D_μ on R_μ is an isomorphism.*

4.2.2. *The initial conjectures.* In §2 we have described q-enumerations of Catalan numbers, forests and parking functions. They turn out to be connected with the Frobenius and Hilbert series of R_n.

Theorem 4.2.4 ($(n+1)^{n-1}$ conjecture)**.** *Let R_n denote the ring of diagonal coinvariants and R_n^ε its subspace of antisymmetric elements. Their characters, dimensions, and Hilbert series enjoy the following properties.*

(i)
$$\dim R_n = (n+1)^{n-1}.$$

(ii)
$$\dim R_n^\varepsilon = C_n = \frac{1}{n+1}\binom{2n}{n},$$
the n-th Catalan number.

(iii)
$$H_{R_n}(q, q^{-1}) = q^{-\binom{n}{2}}[n+1]_q^{n-1}.$$

(iv)
$$H_{R_n^\varepsilon}(q, q^{-1}) = q^{-\binom{n}{2}}\frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q.$$

(v)
$$H_{R_n}(q, 1) = J_n(q) = \sum_F q^{i(F)},$$
the enumerator of forests on n vertices by number of inversions, or parking functions by weight.

(vi)
$$H_{R_n^\varepsilon}(q, 1) = C_n(q),$$
the Carlitz-Riordan q-Catalan number.

(vii) *As an S_n-module, R_n is isomorphic to $\varepsilon \otimes PF$, where PF is the permutation representation on parking functions.*

(viii) *Ignoring the y-degrees and grading R_n only by x-degrees, the isomorphism $\varepsilon \otimes R_n \cong PF$ is homogeneous, where PF is graded by weights.*

We have listed items (i)-(viii) cumulatively for clarity, but of course some of them are redundant. Specifically, (i)-(ii) are special cases of both (iii)-(iv) and (v)-(vi), (viii) is of course stronger than (vii), and both (v) and (vi) follow from (viii).

4.2.3. *The master conjecture.* To appreciate the effect that the empirical discovery of the facts in Theorem 4.2.4 had at the time, it must be borne in mind that the connection between q-Lagrange inversion and the operator ∇ discussed in §3.5.7 was not then known. In hindsight, however, armed with Propositions 3.5.23 and 3.5.25, we can readily recognize that everything in Theorem 4.2.4 is implied by the following master formula.

Theorem 4.2.5. *The Frobenius series of the diagonal coinvariant ring is given by*

$$F_{R_n}(z;q,t) = \nabla e_n(z).$$

Indeed, part (viii) is equivalent to Proposition 3.5.23, and parts (v)-(vi) are its Corollary 3.5.24. Parts (iii) and (iv) follow from Proposition 3.5.25 using the symmetric function identities

$$\langle e_1^n, e_n\big[[n+1]_q Z\big]\rangle = [n+1]_q^n$$
$$\langle e_n, e_n\big[[n+1]_q Z\big]\rangle = \begin{bmatrix}2n\\ n\end{bmatrix}_q.$$

5. Hilbert scheme interpretation

Theorems 4.1.5 and 4.2.5 and the various entities associated with them—the Macdonald symmetric functions $\tilde{H}_\mu(z;q,t)$ and the operators Δ_f and ∇—can be understood geometrically in the context of Hilbert schemes of points in the plane. I will explain in some detail how this comes about in §5.4, below. First we need some background material on Hilbert schemes, and a review of classically known results and the new theorems that have made possible the solution of the $n!$ and $(n+1)^{n-1}$ conjectures.

5.1. The Hilbert scheme and isospectral Hilbert scheme.

Definition 5.1.1. Let $R = \mathbb{C}[x,y] = \mathcal{O}(\mathbb{C}^2)$ denote the coordinate ring of the affine plane. The *Hilbert scheme* $H_n = \mathrm{Hilb}^n(\mathbb{C}^2)$ *of* n *points in the plane* is the algebraic variety parametrizing ideals $I \subseteq R$ such that $\dim_{\mathbb{C}} R/I = n$.

The definition of Hilbert schemes and other facts about them stated below without specific attribution are due to Grothendieck [35]. The term *parametrizing* in the definition has the following technical meaning. By definition, the ideals parametrized by H_n correspond one-to-one with zero-dimensional subschemes $S \subseteq \mathbb{C}^2$ of length n, so we may also think of H_n as parametrizing these subschemes. Now there is a flat family, the *universal scheme*

$$\begin{array}{c} F_n \ \subseteq H_n \times \mathbb{C}^2 \\ \big\downarrow \pi \\ H_n, \end{array} \tag{79}$$

over H_n, whose fiber over a point of H_n is the corresponding subscheme S. The structure of H_n as an algebraic variety is characterized by the universal property that any family $Y \subseteq T \times \mathbb{C}^2$, flat and finite of degree n over a scheme T, is the pullback of the universal family F via a unique morphism $T \to H_n$.

Let us pause to see what various points of H_n might look like. In the "generic" case, the subscheme $S \subseteq \mathbb{C}^2$ is a set of n distinct points and $I = I(S)$ its vanishing ideal. The least generic case, but most important for us, is when I is a monomial ideal. In that case, there are n monomials

$x^j y^i$ not in I, which form a basis of R/I, and their exponents (i, j) are the cells in the diagram of a partition μ. We index the monomial ideals by their corresponding partitions, writing

$$I = I_\mu.$$

For example, $I_{(1^n)} = (x^n, y)$ and $I_{(n)} = (x, y^n)$. The subscheme $S = V(I_\mu)$ defined by a monomial ideal is non-reduced and concentrated at the origin. It may be usefully and correctly pictured as an infinitesimal copy of the diagram of μ.

The first result on H_n is Fogarty's theorem.[1]

Theorem 5.1.2 ([18])**.** *The Hilbert scheme of points in the plane (or any smooth surface) is irreducible and nonsingular, of dimension* $2n$*.*

The irreducibility part of the theorem means that the "generic" subschemes $S \subseteq \mathbb{C}^2$ consisting of n distinct points really are generic, in the sense that they are dense in H_n.

Next we need an auxiliary variety, called the *isospectral Hilbert scheme*. To define it, we first need to introduce the *Hilbert-Chow morphism*. For each ideal $I \in H_n$, the ring R/I is isomorphic to the direct product of its local rings $(R/I)_P$ at each point $P \in V(I)$. Hence the multiplicities $m_P(I) = \dim_{\mathbb{C}}(R/I)_P$ sum to n, so that to I corresponds a zero-dimensional algebraic cycle

$$\sigma(I) = \sum_P m_P(I) \cdot P \tag{80}$$

of weight n. These cycles may be identified with the points of the the orbit variety $(\mathbb{C}^2)^n/S_n$, where S_n acts on $(\mathbb{C}^2)^n$ by permuting coordinates.

Proposition 5.1.3. *The map*

$$\sigma\colon H_n \to \mathbb{C}^{2n}/S_n$$

described by (80) *is a projective and birational morphism of algebraic varieties. In particular it is a desingularization of the quotient singularity* $\mathbb{C}^{2n}/S_n$*.*

Definition 5.1.4. The *isospectral Hilbert scheme* is the reduced fiber product

$$\begin{array}{ccc} X_n & \xrightarrow{f} & \mathbb{C}^{2n} \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow \\ H_n & \xrightarrow{\sigma} & \mathbb{C}^{2n}/S_n. \end{array} \tag{81}$$

Specifically, X_n is the underlying reduced subscheme of the scheme-theoretic fiber product (which is not reduced).

Let me explain the motivation for the term *isospectral*. For each point I of H_n, the operators of multiplication by x and y are commuting endomorphisms of the n-dimensional vector space R/I. Their joint spectrum consists of n pairs of eigenvalues $(\alpha_i, \beta_i) \in \mathbb{C}^2$, which are just the points of $\sigma(I)$, with repetitions given by the multiplicities. A point of X_n is then a tuple $(I, P_1, \ldots, P_n)$ in $H_n \times (\mathbb{C}^2)^n$ where $(P_1, \ldots, P_n)$ is some ordering of the joint spectrum.

The projection ρ of X_n on H_n is finite. Its degree is $n!$, since for a generic $I = I(S)$ there are $n!$ possible orderings $(P_1, \ldots, P_n)$ of the n distinct points of S.

[1] I am told that this theorem was actually proved earlier but not published by Hartshorne.

5.1.1. *Torus action.* The algebraic torus group

$$\mathbb{T}^2 = (\mathbb{C}^*)^2 \tag{82}$$

acts linearly on $\mathbb{C}^2$ as the group of 2×2 diagonal matrices

$$\tau_{t,q} = \begin{bmatrix} t^{-1} & 0 \\ 0 & q^{-1} \end{bmatrix}. \tag{83}$$

The inverse signs in (83) make $\mathbb{T}^2$ act on the coordinate ring $\mathbb{C}[x, y]$ of $\mathbb{C}^2$ by

$$\tau_{t,q} x = tx; \quad \tau_{t,q} y = qy.$$

The action of $\mathbb{T}^2$ on $\mathbb{C}^2$ induces an action on H_n, X_n, the universal family F_n, etc., so that all relevant morphisms are equivariant. There are induced $\mathbb{T}^2$ actions on various vector spaces such as the coordinate ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ of $\mathbb{C}^{2n}$, the space of global sections of any $\mathbb{T}^2$-equivariant vector bundle, or the fiber of such a bundle at a torus-fixed point in H_n. The $\mathbb{T}^2$ action on such a space is equivalently described by a $\mathbb{Z}^2$-grading: namely, an element f is homogeneous of degree (r, s) if and only if $\tau_{t,q} f = t^r q^s f$. In most cases, the grading induced by the torus action coincides with an obvious "natural" double grading, as for example in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$.

Proposition 5.1.5. *The $\mathbb{T}^2$-fixed points of H_n are the monomial ideals I_μ, and every $I \in H_n$ has a fixed point in the closure of its orbit.*

Proof. An ideal $I \subseteq R = \mathbb{C}[x, y]$ is fixed if and only if it is doubly homogeneous, *i.e.*, if and only if it is a monomial ideal. The initial ideal of I with respect to any term order on R is a monomial ideal in the closure of the $\mathbb{T}^2$-orbit of I. □

5.1.2. *Zero-fiber.* We will need some basic results on the zero-fiber of the Hilbert-Chow morphism $\sigma\colon H_n \to \mathbb{C}^{2n}/S_n$ over the origin in $\mathbb{C}^{2n}/S_n$, which we denote by

$$Z_n = \sigma^{-1}(\{0\}).$$

The definition may be understood either in the set-theoretic or scheme-theoretic sense, since we will see shortly that the scheme-theoretic zero-fiber is reduced. In a naive sense, Z_n parametrizes subschemes $S \subseteq \mathbb{C}^2$ supported at the origin. One has to be a bit careful about this, however: by Skjelnes [80], the functor of families supported at the origin is not representable. What Z_n does represent is the restriction of this functor to families with reduced base scheme.

The first property of Z_n that we need is classical.

Proposition 5.1.6 (Briançon [9])**.** *The zero-fiber Z_n is irreducible, of dimension $n - 1$.*

We also need a further property of Z_n that had not been known before.

Proposition 5.1.7 ([38])**.** *The zero-fiber Z_n is scheme-theoretically reduced and Cohen-Macaulay.*

This is proved by considering the zero-fiber $\tilde{Z}_n$ in the universal family F_n over H_n, that is, the fiber over the origin of the composite map

$$\sigma \circ \pi\colon F_n \to H_n \to \mathbb{C}^{2n}/S_n.$$

The universal family F_n is Cohen-Macaulay, since it is flat and finite over H_n (see the discussion following Theorem 5.2.1 below for more on this point). An explicit calculation in local coordinates on F_n, using the generators of the ideal of $\{0\}$ in $\mathbb{C}^{2n}/S_n$ given by Proposition 4.2.1, shows that $\tilde{Z}_n$ is a generically reduced local complete intersection in F_n, hence reduced and Cohen-Macaulay.

It is easy to show that the projection $\pi\colon F_n \to H_n$ maps $\tilde{Z}_n$ isomorphically onto Z_n, completing the proof.

5.2. **Main theorem on the isospectral Hilbert scheme.** The next theorem is the key result from which all else follows.

Theorem 5.2.1 ([39])**.** *The isospectral Hilbert scheme X_n is Cohen-Macaulay and Gorenstein. Equivalently, the projection*

$$\rho\colon X_n \to H_n$$

is flat and the coordinate ring of its scheme-theoretic fiber over each point I_μ is a finite dimensional (doubly) graded Gorenstein algebra, in the sense of Definition 4.1.1.

A few clarifying remarks may be in order here. By definition, a scheme is Cohen-Macaulay (resp. Gorenstein) if its local ring at every point is so. A local ring is *Cohen-Macaulay* if some, or equivalently every, system of parameters is a regular sequence. Phrased geometrically, this means that a scheme X is Cohen-Macaulay if some, or equivalently every, finite morphism $X \to H$ with H nonsingular, is flat. In particular this applies to the morphism ρ.

A local ring A is Gorenstein if it is Cohen-Macaulay first of all, and in addition, for some, or equivalently every, ideal $\mathcal{J}$ generated by a system of parameters, the Artin local ring $A/\mathcal{J}$ has one-dimensional socle. In the graded case this reduces to Definition 4.1.1. Since I_μ is a $\mathbb{T}^2$-fixed point, the coordinate ring $\mathcal{O}(\rho^{-1}(I_\mu))$ of its fiber is doubly graded, and it is precisely of the form $A/\mathcal{J}$, where $A = \mathcal{O}_{X_n,Q_\mu}$ is the local ring of X_n at the unique point Q_μ lying over I_μ, and $\mathcal{J}$ is generated by local coordinates on H_n at I_μ. So the rings $\mathcal{O}(\rho^{-1}(I_\mu))$ are Gorenstein if and only if X_n is Gorenstein at each point I_μ. However, the Gorenstein locus in X_n is open and $\mathbb{T}^2$-invariant, so this is equivalent to X_n being Gorenstein everywhere, by Proposition 5.1.5.

There is an important way to reformulate Theorem 5.2.1. For any finite group G of order g acting faithfully on $V = \mathbb{C}^m$, one defines the *G-Hilbert scheme*

$$V/\!/G \tag{84}$$

as in Ito and Nakamura [43] to be the closure in $\mathrm{Hilb}^g(V)$ of the locus parametrizing regular G-orbits in V. The fibers of the universal family X over $V/\!/G$ are G-invariant subschemes $S \subseteq V$ whose coordinate rings afford the regular representation of G. In particular, G acts on X, and $X/G = V/\!/G$. The projection $X \to V$ then induces a Hilbert-Chow morphism $V/\!/G \to V/G$.

The case of interest here is $V = \mathbb{C}^{2n}$, $G = S_n$. Because the subgroup S_{n-1} has index n in S_n, the quotient X/S_{n-1} of the universal family over $\mathbb{C}^{2n}/\!/S_n$ is flat and finite of degree n. A simple argument based on Proposition 4.2.1 shows that the coordinates x_n and y_n on $\mathbb{C}^{2n}$ generate the S_{n-1}-invariants in the coordinate ring of each fiber of X. Hence X/S_{n-1} can be identified with a subscheme of $(\mathbb{C}^{2n}/\!/S_n)\times\mathbb{C}^2$. The universal property of the Hilbert scheme then yields a morphism

$$\phi\colon \mathbb{C}^{2n}/\!/S_n \to H_n,$$

and it is easy to see that it commutes with the Hilbert-Chow morphisms.

Conversely, the flatness of the projection $\rho\colon X_n \to H_n$, given by Theorem 5.2.1, means that we have a morphism $H_n \to \mathrm{Hilb}^{n!}(\mathbb{C}^{2n})$ mapping generic subschemes $S \subseteq \mathbb{C}^2$ to regular S_n-orbits in $\mathbb{C}^{2n}$. Since H_n is irreducible, this factors through a morphism

$$\eta\colon H_n \to \mathbb{C}^{2n}/\!/S_n.$$

Because both morphisms ϕ and η commute with the Hilbert-Chow morphism, they are mutually inverse generically and hence everywhere. We have proved the following corollary.

Corollary 5.2.2. *The Hilbert scheme of S_n-orbits $\mathbb{C}^{2n}//S_n$ is isomorphic to the Hilbert scheme H_n of points in the plane. The isomorphism identifies the isospectral Hilbert scheme X_n with the universal family on $\mathbb{C}^{2n}//S_n$.*

We remark that Theorem 5.2.1 and Corollary 5.2.2 hold with $\mathbb{C}^2$ replaced by any smooth quasiprojective surface, since the properties in question are local in the étale topology.

5.3. **Theorem of Bridgeland, King and Reid.** When G is a finite subgroup of $SL(V)$, the quotient singularity V/G is Gorenstein, and in some cases it possesses a particularly nice type of desingularization.

Definition 5.3.1. Let $V = \mathbb{C}^m$ and $G \subseteq SL(V)$. A resolution of singularities

$$Y \to V/G$$

is *crepant* if the sheaf ω_Y of differential m-forms is trivial, *i.e.*, $\omega \cong \mathcal{O}_Y$.

A crepant resolution is automatically minimal and has other interesting properties, among them the *generalized McKay correspondence*, conjectured by Reid [72] and proved by Batyrev [2], which says that the Betti numbers of Y enumerate the conjugacy classes of G according to certain weights. The classical McKay correspondence is its two-dimensional case.

In dimensions two and three, the G-Hilbert scheme $V//G$ turns out always to be a crepant resolution. Seeking to explain and generalize this fact, Bridgeland, King and Reid [10] proved the following theorem.

Theorem 5.3.2 ([10])**.** *Let G be a finite subgroup of $SL(V)$, and consider the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & V \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ V//G & \xrightarrow{\sigma} & V/G, \end{array} \tag{85}$$

Let $D(V//G)$ be the bounded derived category of coherent sheaves on $V//G$ and $D^G(V)$ the similar derived category of G-equivariant sheaves on V. Suppose that the Hilbert-Chow morphism $\sigma\colon V//G \to V/G$ satisfies the following smallness criterion: for every d, the locus of points x with $\dim \sigma^{-1}(x) \geq d$ has codimension at least $2d-1$ in V/G. Then

(i) $V//G$ is a crepant resolution V/G;

(ii) the functor

$$\Phi = Rf_* \circ \rho^*\colon D(V//G) \to D^G(V)$$

is an equivalence of categories.

Proposition 5.1.6 implies that the Chow morphism $\sigma\colon H_n \to \mathbb{C}^{2n}/S_n$ satisfies the smallness criterion in Theorem 5.3.2. Combining this theorem with Corollary 5.2.2 we therefore obtain a powerful tool for the study of the Hilbert scheme H_n.

Corollary 5.3.3. *In diagram* (81)*, the functor*

$$\Phi = Rf_* \circ \rho^*\colon D(H_n) \to D^{S_n}(\mathbb{C}^{2n})$$

is an equivalence of categories.

We remark that it is has long been known (see Beauville [3]) that H_n is a crepant resolution of $\mathbb{C}^{2n}/S_n$. Thus the interesting aspect of the Bridgeland-King-Reid theorem in this context is part (ii).

5.4. **How the conjectures follow.** Now let us see how Theorems 4.1.5 and 4.2.5 follow from our theorems on the Hilbert scheme. To begin, we need names for some vector bundles of geometric origin on H_n. We put

$$B = \pi_* \mathcal{O}_{F_n}, \tag{86}$$

the sheaf of regular functions on the universal family, pushed down to a sheaf on the Hilbert scheme, and

$$P = \rho_* \mathcal{O}_{X_n}, \tag{87}$$

the push-down of the sheaf of regular functions on the isospectral Hilbert scheme. Here we are identifying vector bundles with their (locally free) sheaves of sections. The sheaf P is locally free by virtue of Theorem 5.2.1.

The bundle B is the *tautological bundle* whose fiber at $I \subseteq R$ is R/I. In particular it is a bundle of algebras which are quotients of $R = \mathbb{C}[x, y]$. The bundle P is the tautological bundle on $\mathbb{C}^{2n}//S_n$, regarded as a sheaf on H_n via the isomorphism in Corollary 5.2.2. As such it is a bundle of algebras which are quotients of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ by S_n-invariant ideals, with S_n acting by the regular representation on each fiber. The Gorenstein property of X_n in Theorem 5.2.1 means that the fibers of P are Gorenstein algebras.

We also put

$$\mathcal{O}(1) = \wedge^n B, \tag{88}$$

the *tautological line bundle* on the Hilbert scheme. In fact $\mathcal{O}(1)$ is a very ample line bundle for H_n as a variety projective over $\mathbb{C}^{2n}/S_n$, as I will explain later (see Proposition 6.1.5). As usual we also write $\mathcal{O}(k)$ with $k \in \mathbb{Z}$ for the tensor powers of $\mathcal{O}(1)$ or its dual $\mathcal{O}(-1)$.

5.4.1. *The $n!$ conjecture.* The first result is the "$n!$" part of the $n!$ conjecture.

Proposition 5.4.1. *The ring R_μ in the $n!$ conjecture is the fiber $P(I_\mu)$ of the bundle P at the point I_μ in the Hilbert scheme. In particular,* $\dim R_\mu = n!$.

Proof. Consider an orbit $S_n \cdot b \subseteq \mathbb{C}^{2n}$ with b associated to μ as in (76). Its ideal J_b is a point of $\mathbb{C}^{2n}//S_n$. The corresponding point of H_n is the ideal $I = I(S)$, where S is the set of coordinate pairs (a_i, b_i) in b. In effect, the set S is the picture in $\mathbb{C}^2$ of the diagram of μ.[2]

The ideal J of leading forms of J_b is the limit as $u \to 0$ of J_{ub}. More precisely, consider the one-parameter torus $\mathbb{C}^* = \{\tau_{u^{-1},u^{-1}}\} \subseteq \mathbb{T}^2$. The $\mathbb{C}^*$ orbit of J_b consists of the ideals J_{ub} for $u \neq 0$. This orbit extends to an affine line $\mathbb{C}^1 \hookrightarrow \mathbb{C}^{2n}//S_n$, with the origin mapping to J.

The point of H_n corresponding to J is similarly the limit as $u \to 0$ of $I(uS)$, that is, the ideal of leading forms of $I(S)$. For (r, s) outside the diagram of μ, the polynomial

$$f(x, y) = \prod_{j<s} (x - \alpha_j) \prod_{i<r} (y - \beta_i)$$

[2]For this to come out right we should think of the x-axis as vertical and the y-axis as horizontal. Unfortunately neither the English nor the French convention for Young diagrams matches the Cartesian convention for axes in the plane.

belongs to $I(S)$. The leading form of f is $x^s y^r$, so the ideal of leading forms of $I(S)$ is I_μ. Hence J is the defining ideal of the fiber $P(I_\mu) = \mathbb{C}[\mathbf{x},\mathbf{y}]/J$, and in particular $\mathbb{C}[\mathbf{x},\mathbf{y}]/J$ is Gorenstein.

Recall from §4.1 that R_μ is uniquely characterized as the Gorenstein quotient of $\mathbb{C}[\mathbf{x},\mathbf{y}]/J$ in which the sign representation is not killed. But $\mathbb{C}[\mathbf{x},\mathbf{y}]/J$ is already Gorenstein, so it is equal to R_μ. □

To identify the Frobenius series of R_μ we can use the functorial equivalence Φ in Corollary 5.3.3. The method I will outline is a bit different from the one in [39].

Proposition 5.4.2. *The S_n-modules $\mathrm{Tor}_i^{\mathbb{C}[\mathbf{x}]}(R_\mu, \mathbb{C})$ contain only irreducible representations V^λ of S_n with $\lambda \geq \mu'$.*

Before I indicate the proof of this proposition, let us take note of its consequences. For one thing it implies Proposition 3.4.20. Just as Proposition 3.4.20 provides a triangularity condition on the Frobenius series of the Garsia-Procesi ring, so does Proposition 5.4.2 for its doubled analog R_μ. Specifically, using Proposition 3.3.1, it implies

$$F_{R_\mu}[(1-t)Z; q, t] \in \mathbb{Z}[q,t] \cdot \{s_\lambda : \lambda \geq \mu'\},$$

which is condition (ii) in Definition 3.5.2. Taking $\mathbf{y}$ in place of $\mathbf{x}$ we see by symmetry that $F_{R_\mu}(z; q, t)$ also satisfies condition (i) in the definition of $\tilde{H}_\mu(z; q, t)$. Finally, the trivial representation of S_n occurs uniquely in degree $(0,0)$ in R_μ, so $\langle s_{(n)}, F_{R_\mu}(z; q, t)\rangle = 1$, which is another way of stating condition (iii) in the definition. Hence we have

$$F_{R_\mu}(z; q, t) = \tilde{H}_\mu(z; q, t).$$

Now let us turn to the proof of Proposition 5.4.2. We first need to say something about the locus $W_y \subseteq H_n$ parametrizing subschemes S with support on the y-axis $V(x) \subseteq \mathbb{C}^2$. The locus W_y has a well-known explicit description. Given a partition μ of n, consider the set of points $S \in W_y$ for which the multiplicities of the points of S are the parts of μ. For every μ this set has dimension n. First there are $l(\mu)$ degrees of freedom to choose the support of S on the y-axis. For each point of the support, the choice of a non-reduced scheme structure there is locally (in the analytic topology) the same as the choice of a point in the zero-fiber of the Hilbert-Chow morphism for H_{μ_i}. By Briançon's theorem (Proposition 5.1.6) this contributes $\mu_i - 1$ degrees of freedom for each i, for a total of

$$l(\mu) + \sum_i (\mu_i - 1) = n.$$

The locus W_y is the union of these n-dimensional locally closed subsets $W_{y,\mu}$, whose closures are its irreducible components.

The functor Φ in Corollary 5.3.3 has the explicit inverse

$$\Psi = \mathcal{O}(-1) \otimes (\rho_* \circ Lf^*(-))^\varepsilon, \tag{89}$$

where the notation $(\cdot)^\varepsilon$ stands for the subsheaf of S_n-antisymmetric elements. Note that $D^{S_n}(\mathbb{C}^{2n})$ is just the derived category of finitely generated S_n-equivariant $\mathbb{C}[\mathbf{x},\mathbf{y}]$-modules. The key point in proving Proposition 5.4.2 is to identify the objects

$$\Psi(V^\lambda \otimes \mathbb{C}[\mathbf{y}]) = \Psi(V^\lambda \otimes \mathbb{C}[\mathbf{x},\mathbf{y}]/(\mathbf{x}))$$

in the derived category $D(H_n)$. To do this we observe that the coordinates $\mathbf{x} = x_1, x_2, \ldots, x_n$ form a regular sequence in $\mathcal{O}_{X_n}$ at every point where they vanish. The reason for this is that their vanishing locus $V(\mathbf{x}) \subseteq X_n$ is finite over W_y, hence has dimension n, so it is a complete

intersection in the Cohen-Macaulay scheme X_n. Moreover, a simple calculation at a generic point in each component of W_y shows that $V(\mathbf{x})$ is scheme-theoretically reduced, generically and hence everywhere. These facts imply that $Lf^*(V^\lambda \otimes \mathbb{C}[\mathbf{y}]) = V^\lambda \otimes \mathcal{O}_{V(\mathbf{x})}$. By (89), we have

$$\Psi(V^\lambda \otimes \mathbb{C}[\mathbf{y}]) = \mathcal{O}(-1) \otimes \mathrm{Hom}_{S_n}(V^{\lambda'}, \rho_* \mathcal{O}_{V(\mathbf{x})}).$$

Now $\rho_* \mathcal{O}_{V(\mathbf{x})}$ is supported on W_y and its S_n action is induced from that on X_n. Over the subset $W_{y,\mu}$ of W_y, the fiber of $V(\mathbf{x})$ consists of $\binom{n}{\mu_1,\dots,\mu_l}$ reduced points, and S_n permutes them transitively, with the stabilizer of each one having the form $S_{\mu_1} \times \cdots \times S_{\mu_l}$. From this one sees that $\mathrm{Hom}_{S_n}(V^{\lambda'}, \rho_* \mathcal{O}_{V(\mathbf{x})})$ is supported on the components $\overline{W}_{y,\mu}$ with $\mu \le \lambda'$. In turn one deduces that $\mathrm{Hom}_{S_n}(V^{\lambda'}, \rho_* \mathcal{O}_{V(\mathbf{x})})$ is locally zero at I_μ unless $\mu' \le \lambda'$.

To complete the proof, Proposition 5.4.2 is equivalent to

$$\mathrm{Ext}^i(V_\lambda \otimes \mathbb{C}[\mathbf{y}], R_\mu) = 0 \quad \text{for all } i, \text{ unless } \lambda' \ge \mu' \tag{90}$$

in the derived category $D^{S_n}(\mathbb{C}^{2n})$. But Proposition 5.4.1, stated in the language of Corollary 5.3.3, says simply that $\Phi k_{I_\mu} = R_\mu$, where $k_{I_\mu} = \mathcal{O}_{H_n, I_\mu}/\mathfrak{m}_{I_\mu} \in D(H_n)$ is the "skyscraper sheaf" of the point I_μ. Hence by Corollary 5.3.3, the Ext groups in (90) are equal to the Ext groups

$$\mathrm{Ext}^i(\Psi(V^\lambda \otimes \mathbb{C}[\mathbf{y}]), k_{I_\mu}).$$

We have just seen that these are zero unless $\lambda' \ge \mu'$.

5.4.2. *The* $(n+1)^{n-1}$ *conjecture.* To prove Theorem 4.2.5, we need stronger consequences of Corollary 5.3.3 than the ones we used in the preceding section. The next result is the second of our two main theorems on H_n (the first being Theorem 5.2.1).

Theorem 5.4.3 ([40])**.** *With the notation* B *and* P *as in* (86), (87) *for the tautological vector bundles on the Hilbert scheme, we have*

$$\begin{gathered} H^i(H_n, P \otimes B^{\otimes l}) = 0 \quad \textit{for all } i > 0 \textit{ and all } l\textit{; and} \\ H^0(H_n, P \otimes B^{\otimes l}) = R(n,l), \end{gathered} \tag{91}$$

where $R(n,l)$ *is the coordinate ring of a certain union of linear subspaces* $Z(n,l) \subseteq \mathbb{C}^{2n+2l}$ *(the* polygraph*). Moreover, we have the similar statements on the zero-fiber*

$$\begin{gathered} H^i(Z_n, P \otimes B^{\otimes l}) = 0 \quad \textit{for all } i > 0 \textit{ and all } l\textit{; and} \\ H^0(Z_n, P \otimes B^{\otimes l}) = R(n,l)/\mathfrak{m}R(n,l), \end{gathered} \tag{92}$$

where $\mathfrak{m} = \mathbb{C}[\mathbf{x},\mathbf{y}]_+^{S_n}$ *is the homogeneous maximal ideal in the ring of invariants* $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$*, that is, the ideal of* $\{0\}$ *in* $\mathbb{C}^{2n}/S_n$*.*

The definition of the polygraph $Z(n,l)$ will be given in §6, and need not concern us here except when $l = 0$. In that case, $Z(n,0) = \mathbb{C}^{2n}$ and $R(n,0) = \mathbb{C}[\mathbf{x},\mathbf{y}]$. In particular, the ring $R(n,l)/\mathfrak{m}R(n,l)$ on the right-hand side in (92) reduces for $l = 0$ to the diagonal coinvariant ring R_n. We can use Theorem 5.4.3 to calculate the Frobenius series of R_n in terms of Macdonald symmetric functions, with the aid of the following auxiliary result, which is a corollary to the proof of Proposition 5.1.7.

Proposition 5.4.4. *On H_n, we have a $\mathbb{T}^2$-equivariant locally free resolution of the coordinate sheaf $\mathcal{O}_{Z_n}$ of the zero-fiber,*

$$\cdots \to B \otimes \wedge^k(B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \to \cdots \to B \otimes (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \to B \to \mathcal{O}_{Z_n} \to 0, \tag{93}$$

where B' is a canonical direct summand of the tautological bundle B,

$$B = \mathcal{O} \oplus B',$$

and $\mathcal{O}_t$, $\mathcal{O}_q$ denote the twistings $\mathcal{O} \otimes \mathbb{C}_t$ and $\mathcal{O} \otimes \mathbb{C}_q$ by one-dimensional representations of $\mathbb{T}^2$ in which $\tau_{t,q}$ acts as t or q, respectively.

This given, we can make the desired calculation using a suitable version [83] of the classical Lefschetz formula of Atiyah and Bott [1]. What the formula gives us is actually the Frobenius series Euler characteristic

$$\sum_i (-1)^i F_{H^i(Z_n,P)}(z;q,t). \tag{94}$$

But from (92) we see that the only nonzero term here is $i = 0$, and that term is just $F_{R_n}(z;q,t)$. The formula gives (94) as a sum of local data for each $\mathbb{T}^2$-fixed point I_μ on the Hilbert scheme. The required local datum at I_μ is the Frobenius series of the fiber of $\mathcal{O}_{Z_n} \otimes P$ there, divided by the determinant of the $\mathbb{T}^2$ action on the cotangent space at I_μ.

From Proposition 5.4.1 and Theorem 4.1.5, we know that the Frobenius series of $P(I_\mu)$ is $\tilde{H}_\mu(z;q,t)$. We have to multiply this by a Hilbert series Euler characteristic for the fiber at I_μ of the resolution of $\mathcal{O}_{Z_n}$ in (93). Note that the Hilbert series of $B(I_\mu) = R/I_\mu$ is none other than the expression $B_\mu(q,t)$ in (58). From this we calculate the factor attributable to (93) as

$$(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t),$$

with $\Pi_\mu(q,t)$ defined as in (74).

For the denominator factor we need the following lemma, which is a reformulation of a classical result of Ellingsrud and Strömme [17].

Lemma 5.4.5 ([38])**.** *The eigenvalues of $\tau_{t,q} \in \mathbb{T}^2$ on the cotangent space of the Hilbert scheme at the fixed point I_μ are given by the $2n$ monomials*

$$t^{1+l(x)}q^{-a(x)},\ t^{-l(x)}q^{1+a(x)} :\ x \in \mu,$$

where $a(x)$ and $l(x)$ are the arm and leg of the cell x in the diagram of μ.

Putting all this together yields the formula

$$F_{R_n}(z;q,t) = \sum_{|\mu|=n} \frac{(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in\mu}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}.$$

Comparing this with Proposition 3.5.26 we see that we have proved Theorem 4.2.5.

5.4.3. *Orthogonality of the Macdonald symmetric functions.* As promised earlier, there is a nice geometric proof of the identity (73), along with the orthogonality of the Macdonald symmetric functions $\tilde{H}_\mu(z;q,t)$ with respect to the inner product $\langle -,-\rangle_*$. The ideas that go into it will also enable us to deduce other results.

The equivalence of categories in Corollary 5.3.3 induces an isomorphism of Grothendieck groups

$$\Phi\colon K^0_{\mathbb{T}}(H_n) \cong K^0_{S_n\times\mathbb{T}}(\mathbb{C}^{2n}).$$

Here $K^0_{\mathbb{T}}(H_n)$ is the Grothendieck group of torus-equivariant coherent sheaves on the Hilbert scheme, and $K^0_{S_n\times\mathbb{T}}(\mathbb{C}^{2n})$ is the Grothendieck group of finitely generated S_n-equivariant doubly-graded $\mathbb{C}[\mathbf{x},\mathbf{y}]$-modules. These Grothendieck groups are modules over the representation ring

$$\mathbb{Z}[q,t,q^{-1},t^{-1}]$$

of the torus $\mathbb{T}^2$. More specifically, $K^0_{S_n\times\mathbb{T}}(\mathbb{C}^{2n})$ is freely generated as a $\mathbb{Z}[q,t,q^{-1},t^{-1}]$ module by the free $\mathbb{C}[\mathbf{x},\mathbf{y}]$-modules $V^\lambda\otimes\mathbb{C}[\mathbf{x},\mathbf{y}]$. By Proposition 3.3.1, the Frobenius series of $V^\lambda\otimes\mathbb{C}[\mathbf{x},\mathbf{y}]$ is

$$F_{V^\lambda\otimes\mathbb{C}[\mathbf{x},\mathbf{y}]}(z;q,t)=s_\lambda\left[\frac{Z}{(1-q)(1-t)}\right].$$

An object $A\in D(H_n)$ is supported on the zero-fiber if and only if ΦA is supported at 0, and a finitely-generated graded module is supported at 0 if and only if it is finite-dimensional. The Grothendieck group of finite-dimensional S_n-equivariant $\mathbb{C}[\mathbf{x},\mathbf{y}]$-modules is freely generated by the irreducible S_n-modules V^λ, regarded as $\mathbb{C}[\mathbf{x},\mathbf{y}]$-modules annihilated by $(\mathbf{x},\mathbf{y})$. We can summarize these observations as follows.

Proposition 5.4.6. *The Frobenius series composed with the functor Φ in Corollary 5.3.3 gives an isomorphism of the Grothendieck group $K^0_{\mathbb{T}^2}(H_n)$ onto the algebra of symmetric functions f with the property that $f[(1-q)(1-t)Z]$ has coefficients in $\mathbb{Z}[q,t,q^{-1},t^{-1}]$. Under this isomorphism, the subgroup of objects supported on the zero-fiber corresponds to the subalgebra of symmetric functions f which already have coefficients in $\mathbb{Z}[q,t,q^{-1},t^{-1}]$.*

To make this more explicit, if $[I_\mu]\in K^0_{\mathbb{T}^2}(H_n)$ denotes the class of the skyscraper sheaf k_{I_μ}, then Proposition 5.4.1 and Theorem 4.1.5 give

$$\Phi[I_\mu]=\tilde{H}_\mu(z;q,t). \tag{95}$$

Also, the formula for the inverse map in (89) implies that

$$\Psi(V^\lambda\otimes\mathbb{C}[\mathbf{x},\mathbf{y}])=\mathrm{Hom}_{S_n}(V^{\lambda'},\mathcal{O}(-1)\otimes P).$$

With $\lambda=(n)$, this shows in particular that the line bundle of antisymmetric elements P^ε is isomorphic to $\mathcal{O}(1)$. Recall that fibers of P are Gorenstein algebras, and the fibers of P^ε are their socles. The Gorenstein property implies that there is a perfect pairing

$$P\otimes P\to P^\varepsilon\cong\mathcal{O}(1), \tag{96}$$

so $\mathcal{O}(-1)\otimes\varepsilon\otimes P\cong P^*$. Now define *character bundles*

$$P_\lambda \underset{\text{def}}{=} \mathrm{Hom}_{S_n}(V^\lambda,P).$$

Then the above considerations yield

$$\Phi P^*_\lambda=s_\lambda[Z/(1-q)(1-t)]. \tag{97}$$

Corollary 5.4.7. *For $\mathbb{T}^2$ equivariant objects A, B in $D(H_n)$, the Hilbert series Euler characteristic of their Tor groups is given by*

$$\sum_i(-1)^i H_{\mathrm{Tor}_i(A,B)}(q,t)=\langle\Phi[A],\Phi[B]\rangle.,$$

where

$$\langle f,g\rangle. \underset{\text{def}}{=} \langle\nabla^{-1}f,g\rangle_*$$

Proof. Both sides depend bilinearly on the classes $[A]$ and $[B]$ in the Grothendieck group. Evaluating at $A = P_\lambda^*$ and $B = k_{I_\mu}$, we have $\tilde{K}_{\lambda\mu}(q^{-1}, t^{-1})$ on the left-hand side. On the right-hand side, using (95) and (97), we have $\langle \tilde{H}_\mu, s_\lambda[Z/(1-q)(1-t)]\rangle_.$, which is also equal to $\tilde{K}_{\lambda\mu}(q^{-1}, t^{-1})$. □

Now for the promised proof of (73).

Corollary 5.4.8. *We have $\langle \tilde{H}_\mu, \tilde{H}_\nu\rangle_* = 0$ if $\mu \neq \nu$, and*

$$\langle \tilde{H}_\mu, \tilde{H}_\mu\rangle_* = t^{n(\mu)} q^{n(\mu')} \prod_{x\in\mu} (1 - t^{1+l(x)} q^{-a(x)})(1 - t^{-l(x)} q^{1+a(x)}).$$

Proof. From the definition of ∇ we have $\langle \tilde{H}_\mu, \tilde{H}_\nu\rangle_* = t^{n(\mu)} q^{n(\mu')} \langle \tilde{H}_\mu, \tilde{H}_\nu\rangle_.$. The result now follows from Corollary 5.4.7 and Lemma 5.4.5, since $\mathrm{Tor}_i(k_{I_\mu}, k_{I_\mu})$ is the i-th exterior power of the cotangent space to H_n at I_μ. □

5.4.4. *Integrality and positivity for the operators Δ_f.* The operators Δ_f defined in (62) have a simple interpretation in terms of the isomorphism of Grothendieck groups in Proposition 5.4.6. To explain it we need to recall that the *Schur functor* S^λ corresponding to a partition λ of n is defined by

$$S^\lambda(W) = \mathrm{Hom}_{S_n}(V^\lambda, W^{\otimes n}).$$

If W is (doubly) graded, then so is $S^\lambda(W)$, and its Hilbert series is given by

$$H_{S^\lambda(W)}(q,t) = s_\lambda[H_W(q,t)].$$

In particular, the Hilbert series of the fiber at I_μ of the vector bundle $S^\lambda(B)$, where B is the tautological bundle on the Hilbert scheme, is given by $s_\lambda[B_\mu(q,t)]$. Together with (95), this implies the following result.

Proposition 5.4.9. *The operator Δ_{s_λ} on symmetric functions corresponds under the isomorphism in Proposition 5.4.6 to the operator on the Grothendieck group of the Hilbert scheme induced by the functor*

$$(S^\lambda(B) \otimes -).$$

In particular, the operators ∇ and ∇^{-1} correspond to the functors

$$(\mathcal{O}(1) \otimes -),\ (\mathcal{O}(-1) \otimes -).$$

This gives immediately a partial proof of Proposition 3.5.14.

Corollary 5.4.10. *The operators Δ_{s_λ}, ∇ and ∇^{-1} have the Laurent-integrality property in Property 3.5.14, that is, they map the algebra of symmetric functions with coefficients in $\mathbb{Z}[q,t,q^{-1},t^{-1}]$ into itself.*

Finally let us turn to the proof of Theorem 3.5.16. In the language of Proposition 5.4.6, Theorem 4.2.5 tells us that

$$\Phi\mathcal{O}_{Z_n} = \nabla e_n(z),$$

and therefore we also have

$$\Phi(S^\lambda(B) \otimes \mathcal{O}_{Z_n}) = \Delta_{s_\lambda} \nabla e_n(z).$$

To establish that this has coefficients in $\mathbb{N}[q,t]$, it suffices to show that

$$H^i(Z_n, P \otimes S^\lambda(B)) = 0, \quad \text{for } i > 0$$

and that the graded $\mathbb{C}[\mathbf{x},\mathbf{y}]$-module $H^0(Z_n, P \otimes S^\lambda(B))$ is zero in negative degrees. But $S^\lambda(B)$ is a direct summand of $B^{\otimes l}$, where $l = |\lambda|$, so both assertions follow from Theorem 5.4.3.

6. Discussion of proofs of the main theorems

In this section I will outline the proofs of Theorem 5.2.1 (the Cohen-Macaulay and Gorenstein properties of X_n) and Theorem 5.4.3 (the cohomology vanishing theorem). Procesi has also given a nice synopsis of the proof of Theorem 5.2.1 in his review article [71], and I have followed here his way of organizing the logic.

6.1. **Theorem on the isospectral Hilbert scheme.** We are to prove that X_n is Cohen-Macaulay and Gorenstein. We proceed by induction on n. Over a point I of the Hilbert scheme whose support $\sigma(I)$ is not of the form $n \cdot P$, we have locally a product structure

$$X_n \cong X_k \times X_l. \tag{98}$$

So we can assume by induction that X_n is locally Cohen-Macaulay and Gorenstein where the n points do not all coincide. In order to handle the most degenerate points, we make use of the *nested Hilbert scheme* $H_{n-1,n}$ and its corresponding isospectral scheme $X_{n-1,n}$. The nested Hilbert scheme parametrizes pairs of ideals

$$I_n \subseteq I_{n-1} \subseteq R$$

such that $\dim R/I_n = n$ and $\dim R/I_{n-1} = n-1$. Parallel to Fogarty's theorem we have the theorem of Tikhomirov and Cheah [13] that $H_{n-1,n}$ is irreducible and nonsingular. Now all these schemes for $n-1$ and n fit into a big commutative diagram

$$\begin{array}{ccc} X_{n-1,n} & \xrightarrow{g} & X_n \\ & \searrow & \downarrow \\ \downarrow & H_{n-1,n} & \longrightarrow H_n \\ & \downarrow & \\ X_{n-1} & \longrightarrow H_{n-1}. & \end{array} \tag{99}$$

By induction, the bottom horizontal arrow is flat of degree $(n-1)!$ with Gorenstein fibers. From this fact it is easy to deduce that the square involving the bottom arrow is a scheme-theoretic fiber square. Consequently $X_{n-1,n}$ is Cohen-Macaulay and Gorenstein, since $H_{n-1,n}$ is nonsingular and the diagonal arrow is flat and finite with Gorenstein fibers.

For the next step we need to calculate the canonical line bundles on some of the schemes in (99). On $H_{n-1,n}$, and also on $X_{n-1,n}$ via pullback, we define

$$\mathcal{O}(k,l) = p_{n-1}^* \mathcal{O}_{H_{n-1}}(k) \otimes p_n^* \mathcal{O}_{H_n}(l),$$

where p_{n-1}, p_n are the projections on H_{n-1}, H_n and $\mathcal{O}(1)$ denotes the highest exterior power of the tautological bundle, as in (88). The canonical bundle on H_{n-1} is trivial, and the pairing in (96), which holds on H_{n-1} by induction, implies that the canonical bundle on X_{n-1} is $\mathcal{O}(-1)$. It follows that the relative canonical bundle of $X_{n-1,n}$ over $H_{n-1,n}$ in our fiber square is $\mathcal{O}(-1,0)$ in the notation above.

Lemma 6.1.1 ([39])**.** *The canonical sheaf on $H_{n-1,n}$ is $\omega_{H_{n-1,n}} = \mathcal{O}(1,-1)$.*

This is proved by direct computation in local coordinates. We conclude that the canonical bundle on $X_{n-1,n}$ is

$$\omega_{X_{n-1,n}} = \mathcal{O}(0,-1).$$

The important point here is that $\omega_{X_{n-1,n}}$ is the pullback of a line bundle on X_n through the map g in diagram (99).

Claim 6.1.2. *For the map g in* (99) *we have $Rg_*\mathcal{O}_{X_{n-1,n}} = \mathcal{O}_{X_n}$.*

Suppose we prove the claim. Since $\omega_{X_{n-1,n}} = g^*\mathcal{O}_{X_n}(-1)$ we shall also have $Rg_*\omega_{X_{n-1,n}} = \mathcal{O}_{X_n}(-1)$. By duality theory it follows that $\mathcal{O}_{X_n}(-1)$ is the dualizing complex on X_n. But this means exactly that X_n is Cohen-Macaulay and Gorenstein, with canonical bundle $\mathcal{O}(-1)$, so our theorem will be proved.

As to the claim, the nested isospectral Hilbert scheme, like X_n, has a local product structure

$$X_{n-1,n} \cong X_{k,k-1} \times X_l$$

where the the n points do not all coincide. The map g in diagram (99) factors locally as

$$g_k \times 1_l \colon X_{k,k-1} \times X_l \to X_k \times X_l.$$

We can assume that Claim 6.1.2 holds for g_k as part of the induction. Then the claim holds locally away from the most degenerate points. In particular, it holds outside the locus defined by the equations

$$y_1 = y_2 = \ldots = y_n.$$

The following lemma is proved by a standard local cohomology argument.

Lemma 6.1.3. *Let $g \colon Y \to X$ be a proper morphism of algebraic varieties. Suppose given m global regular functions $z_1, \ldots, z_m$ on X, and let Z be the subvariety of X where they vanish; $U = X \setminus Z$ its complement. Assume the following conditions hold.*

(i) *The z_i form a regular sequence in every local ring $\mathcal{O}_{X,x}$, $x \in Z$.*
(ii) *The z_i form a regular sequence in every local ring $\mathcal{O}_{Y,y}$, $y \in g^{-1}(Z)$.*
(iii) *Every fiber of g has dimension $< m - 1$.*
(iv) *On the open set U, the canonical homomorphism $\mathcal{O}_X \to Rg_*\mathcal{O}_Y$ is an isomorphism.*

Then the canonical homomorphism $\mathcal{O}_X \to Rg_*\mathcal{O}_Y$ is an isomorphism everywhere.

To prove Claim 6.1.2 we apply this to the morphism $g \colon X_{n-1,n} \to X_n$, taking $z_i = y_i - y_{i+1}$, $i = 1, \ldots, n-1$. We need to verify assumptions (i)-(iv).

We have already seen that we can assume (iv) by induction.

For (iii) we have to analyze the fibers of the map $H_{n-1,n} \to H_n$. The fiber over a point I in H_n is just a projective space of dimension $\dim \operatorname{soc}(R/I) - 1$. This is always maximized at a fixed point $I = I_\mu$, and then a simple calculation shows that $\dim \operatorname{soc}(R/I) < n - 2$ when $n > 3$. For $n \leq 3$ the induction hypotheses have to be verified directly.

Both (i) and (ii) reduce to the corresponding problem for the sequence $y_1, \ldots, y_n$. For (ii), the local rings of $X_{n-1,n}$ are Cohen-Macaulay, so we only need to show that the locus $\mathbf{y} = 0$ in $X_{n-1,n}$ has dimension n. This follows from the cell decomposition of $H_{n-1,n}$ in Cheah [13].

The crucial part is (i), which amounts to the following statement.

Proposition 6.1.4. *The morphism $X_n \to \mathbb{C}^n$ given by projection on the y coordinates is flat.*

At this point we have pushed our geometric induction argument as far as it will go, and must approach the proof of Proposition 6.1.4 head-on. The first step is construct H_n and X_n as blowups.

Proposition 6.1.5 ([38]). *Let $A = \mathbb{C}[\mathbf{x},\mathbf{y}]^{\varepsilon}$ be the subspace of antisymmetric elements and let $J = \mathbb{C}[\mathbf{x},\mathbf{y}]A$ be the ideal it generates. Define graded Rees algebras*

$$S = \bigoplus_{d\geq 0} A^d; \quad T = \mathbb{C}[\mathbf{x},\mathbf{y}][tJ] = \bigoplus_{d\geq 0} J^d,$$

with the convention that $A^0 = \mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$. Then $H_n = \operatorname{Proj} S$ and $X_n = \operatorname{Proj} T$.

Proof. The result for X_n follows easily from the one for H_n. For H_n, observe that the fibers of the tautological bundle B are quotients of $R = \mathbb{C}[x,y]$, so the fibers of $\mathcal{O}(1) = \wedge^n B$ are quotients of $\wedge^n R = A$. So we get an algebra homomorphism $\phi\colon S \to \bigoplus_d H^0(H_n, \mathcal{O}(d))$, and the sections of $\mathcal{O}(1)$ coming from S do not vanish simultaneously anywhere on H_n. Hence ϕ is induced by a morphism $f : H_n \to \operatorname{Proj} S$, and $\mathcal{O}_{H_n}(1)$ is the pullback $f^*\mathcal{O}(1)$ of the twisting sheaf on $\operatorname{Proj} S$. Since f is a birational morphism of schemes projective over $\mathbb{C}^{2n}/S_n$, it is projective and hence surjective.

Consider the section α of $\mathcal{O}_{H_n}(1)$ given by the Vandermonde determinant $\Delta(\mathbf{x}) \in A$. The open set U_α where $\alpha \neq 0$ consists of the ideals I such that $\{1, x, \ldots, x^{n-1}\}$ is a basis of R/I. One can see with explicit coordinates that $U_\alpha \cong \mathbb{C}^{2n}$. In fact U_α is the open cell in the cell decomposition of H_n given by Ellingsrud and Strömme [17]. From the cell decomposition one also sees that the divisor $Z = H_n \setminus U$ is irreducible. These facts imply that the Picard group of H_n is isomorphic to $\mathbb{Z}$, and $\mathcal{O}_{H_n}(1) = f^*\mathcal{O}(1)$ is ample. It follows that f is injective. □

To prove Proposition 6.1.4 it suffices to prove that for every d the ideal J^d is a free $\mathbb{C}[\mathbf{y}]$-module. To do this we introduce an auxiliary object.

Definition 6.1.6. Let $[l]$ stand for the set $\{1,\ldots,l\}$. To every function $f\colon [l] \to [n]$ associate the map $\pi_f\colon (\mathbb{C}^2)^n \to (\mathbb{C}^2)^l$ sending $(P_1,\ldots,P_n)$ to $(P_{f(1)},\ldots,P_{f(l)})$ and let $W_f \subseteq \mathbb{C}^{2n+2l}$ be its graph. The *polygraph* $Z(n,l)$ is the union of the linear subspaces $W_f \subseteq \mathbb{C}^{2n+2l}$ over all functions $f\colon [l] \to [n]$.

We denote the coordinates on $\mathbb{C}^{2n}$ by x_i, y_i and those on $\mathbb{C}^{2l}$ by a_i, b_i, so the coordinate ring $R(n,l)$ of $Z(n,l)$ is the quotient

$$R(n,l) = \mathbb{C}[\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}]/(\bigcap_f I_f),$$

where

$$I_f = \sum_{i=1}^{l} (a_i - x_{f(i)}, b_i - y_{f(i)}).$$

The connection with the ideals J^d is as follows.

Proposition 6.1.7. *Put $l = dn$ and let S_n^d act on $R(n,l) = R(n,dn)$ by permuting the dn coordinate pairs a_i, b_i in blocks of size n. Then J^d is isomorphic as a $\mathbb{C}[\mathbf{x},\mathbf{y}]$-module to the space $R(n,dn)^{\varepsilon}$ of antisymmetric elements with respect to S_n^d.*

Proof. Fix the function $f_0\colon [dn] \to [n]$ defined by $f_0(kn+i) = i$ for $i \in [n]$ and $0 \leq k < d$. Let W_{f_0} be the corresponding component of $Z(n,l)$. Restriction of functions from $Z(n,l)$ to W_{f_0} is a homomorphism of $\mathbb{C}[\mathbf{x},\mathbf{y}]$-algebras $\phi\colon R(n,l) \to \mathbb{C}[\mathbf{x},\mathbf{y}]$ mapping a_{kn+i}, b_{kn+i} to x_i, y_i. It is easy to see that ϕ maps $R(n,l)^{\varepsilon}$ surjectively onto J^d.

We need to prove that ϕ is injective on $R(n,l)^\varepsilon$. Every antisymmetric function p vanishes on W_f if $f(kn+i) = f(kn+j)$ for some $0 \le k < d$ and some $i, j \in [n]$ with $i \ne j$. But if $f(kn+1), \dots, f(kn+n)$ are distinct for each k, then f is in the S_n^d-orbit of f_0 and so p is determined on W_f by its restriction to W_{f_0}. □

Hence our problem is reduced to a special case of the following theorem.

Theorem 6.1.8. *The coordinate ring $R(n,l)$ of the polygraph is a free module over the polynomial ring $\mathbb{C}[\mathbf{y}]$.*

This theorem is proved in [39] using induction and some commutative algebra to produce a basis of $R(n,l)$ as a free module. The proof is quite constructive and one can extract from it an algorithm to generate the basis elements in any given degree. More usefully, one can deduce that that the basis elements are indexed by some simple combinatorial data. Unfortunately, the proof is also extremely complicated, and I will be the first to admit that it is rather unsatisfactory from a conceptual point of view.

I should point out, however, that the polygraph is not merely an artifice to make the proof of Proposition 6.1.4 more combinatorial. It is also the object whose coordinate ring $R(n,l)$ appears in Theorem 5.4.3 as the space of global sections of the tensor product of tautological bundles $P \otimes B^{\otimes l}$. As such it is a geometrically natural entity. From this point of view we can see for example that the identification of J^d with $R(n,dn)^\varepsilon$ in Proposition 6.1.7 is secretly the representation of the line bundle $\mathcal{O}(d)$ as the summand $(\wedge^n B)^{\otimes d}$ of $B^{\otimes dn}$.

6.2. **Vanishing theorem.** Now let us see how the vanishing theorem, Theorem 5.4.3, comes about. The first point is to restate the theorem using the Bridgeland-King-Reid functor as the pair of identities

$$\begin{aligned} \Phi(B^{\otimes l}) &= R(n,l), \\ \Phi(\mathcal{O}_{Z_n} \otimes B^{\otimes l}) &= R(n,l)/\mathfrak{m}R(n,l). \end{aligned} \tag{100}$$

The next point is that the second identity can be deduced from the first in a straightforward way, using the resolution of $\mathcal{O}_{Z_n}$ in Proposition 5.4.4. So we only have to prove the first identity, which we may rewrite using the Bridgeland-King-Reid theorem as

$$\Psi R(n,l) = B^{\otimes l}.$$

To put it a bit more precisely, the fiber product over the Hilbert scheme $X_n \times F_n^l/H_n$ projects on $\mathbb{C}^{2n+2l}$ with image $Z(n,l)$. Functions on $Z(n,l)$ pull back to global functions on $X_n \times F_n^l/H_n$, that is, to global sections of $P \otimes B^{\otimes l}$, and this induces a map $R(n,l) \to \Phi(B^{\otimes l})$. Applying Ψ we have a canonical map $\Psi R(n,l) \to B^{\otimes l}$ and we are to prove it is an isomorphism.

I wish to point out here the full strength of this seemingly trivial substitution. The functor Φ involves sheaf cohomology, in the guise of the derived functor Rf_*. The inverse functor Ψ, however, involves only commutative algebra: to compute ΨA for any object A you only need an S_n-equivariant free resolution of A.

It develops that Theorem 6.1.8 is exactly what we need in order to calculate $\Psi R(n,l)$. The theorem implies immediately that $R(n,l)$ has a free resolution over $\mathbb{C}[\mathbf{x},\mathbf{y}]$ of length at most n, but we can do a little better using the translation invariance of the polygraph with respect to $\mathbb{C}^2$. This yields that $R(n,l)$ is actually a free $\mathbb{C}[x_1,\mathbf{y}]$-module and hence has a free resolution of length $n-1$.

Next we observe that when not all n points coincide, so we have the local product structure in (98), both $R(n,l)$ and $B^{\otimes l}$ decompose into smaller pieces, in a manner compatible with the map $\Psi R(n,l) \to B^{\otimes l}$. Therefore we can assume by induction that our map is locally an isomorphism away from the most degenerate points.

This much turns out to be almost enough to prove the theorem. Consider an exact triangle

$$C[-1] \to \Psi R(n,l) \to B^{\otimes l} \to C$$

in the derived category $D(H_n)$. We are to prove that $C = 0$. We have just shown that it has a locally free resolution of length n (from the resolution of $R(n,l)$ plus the extra term $B^{\otimes l}$), and that it vanishes on an open set whose complement has codimension $n-1$. If the codimension were $n+1$ instead, it would imply that $C = 0$ by the intersection theorem of Peskine, Szpiro and Roberts [70, 74, 75]. All we need to do is kill off C on a slightly bigger open set.

Let U_x be the open set, called U_α in the proof of Proposition 6.1.5, where the monomials $1, x, \ldots, x^{n-1}$ are independent modulo I, and let U_y be the same with y in place of x. The theorem follows from two more facts.

Lemma 6.2.1. *Let W be the locus where not all n points coincide. Then the complement of the open set $W \cup U_x \cup U_y$ in the Hilbert scheme has codimension $n+1$.*

Lemma 6.2.2. *On U_x, the canonical map $\Psi R(n,l) \to B^{\otimes l}$ is an isomorphism.*

Lemma 6.2.1 is an easy application of the Ellingsrud-Strömme cell decomposition.

Lemma 6.2.2 is a computation in explicit coordinates on U_x. Its proof contains two delicate points, which I will mention without giving details. First we have to check that the free resolution of $R(n,l)$ remains exact when pulled back to the open set in X_n lying over U_x. This is done using Theorem 6.1.8 again. Second, we have to check that on U_x the canonical map $\mathcal{O}(-1) \otimes (\rho_* f^* R(n,l))^\varepsilon \to B^{\otimes l}$ is an isomorphism (recall the formula for Ψ in (89)). Once written out in coordinates, it is easy to see that this map is surjective. For injectivity one has to show that certain elements that are not obviously zero do in fact vanish in $f^*R(n,l)$, which requires a bit of care.

7. Current developments

7.1. **Combinatorial advances.** It remains an open problem to find a combinatorial formula for the Kostka-Macdonald polynomials $\tilde{K}_{\lambda\mu}(q,t)$ along the lines of Theorem 3.4.15 or 3.4.16, but there has recently been encouraging progress on the corresponding problem for the coefficients of the Frobenius series of diagonal coinvariants, $\nabla e_n(z)$.

In view of (71) there exist t-analogs of the Carlitz-Riordan q-Catalan numbers

$$C_n(q,t) \underset{\text{def}}{=} \langle e_n, \nabla e_n \rangle.$$

By Theorem 4.2.5, this *q,t-Catalan polynomial* is the Hilbert series of the antisymmetric part R_n^ε of the diagonal coinvariant ring. Hence it is a polynomial with positive integer coefficients. Unlike most of the other quantities that can be shown to be positive by geometric methods, this one has a known combinatorial interpretation, discovered by Haglund and proved by him and Garsia. I will state a variant of their theorem that lends itself better to generalization.

Theorem 7.1.1 ([20, 21])**.** *Denote by $b(\lambda)$ the number of cells x in the diagram of λ for which the arm and leg of x are related by $l(x) \le a(x) \le l(x)+1$. Let δ_n be the staircase partition*

$(n-1, n-2, \ldots, 1)$. *Then the q,t-Catalan polynomial is the sum over partitions λ with diagram contained inside the diagram of δ_n*

$$C_n(q,t) = \sum_{\lambda \subseteq \delta_n} q^{\binom{n}{2}-|\lambda|} t^{b(\lambda)}. \tag{101}$$

To prove this, define more generally $S_{n,k}(q,t)$ to be the same sum taken only over those λ that contain precisely k cells on the outermost diagonal of δ_n. Garsia and Haglund find a recurrence for $S_{n,k}(q,t)$ and an expression in terms of symmetric functions that satisfies the same recurrence. Their expression reduces to $\langle e_n, \nabla e_n \rangle$ for $S_{n+1,0}(q,t)$, which is the same as the sum in (101).

Interestingly, there is a way of rewriting the combinatorial formula (101) as a sum of products of Gauss binomial coefficients, formally resembling the Kirillov-Reshetikhin formula (51) for the Kostka-Foulkes polynomials.

Corollary 7.1.2. *The q,t-Catalan polynomial can be written as*

$$C_n(q,t) = \sum_r \sum_{k_1+\cdots+k_r=n} q^{\sum_i (i-1)k_i}\, t^{\sum_i \binom{k_i}{2}} \prod_{i=1}^{r-1} \begin{bmatrix} k_i + k_{i+1} - 1 \\ k_i - 1 \end{bmatrix}_t, \tag{102}$$

the sum ranging over compositions of n into positive integers $k_1, \ldots, k_r$.

Proof. Encode each $\lambda \subseteq \delta_n$ by the sequence $(e_1, e_2, \ldots, e_n)$ with $e_i = n - i - \lambda_i$ and $e_n = 0$. The constraints on the possible sequences are $0 \le e_i \le e_{i+1} + 1$ for each i. Let k_m be the number of e_i's equal to $m-1$. If $r = 1 + \max(e_1, \ldots, e_n)$, the constraints imply that $k_1, \ldots, k_r$ are all non-zero.

The exponent $\binom{n}{2} - |\lambda|$ of q in (101) is $\sum_j e_j = \sum_i (i-1)k_i$. The exponent $b(\lambda)$ of t is the number of pairs $i < j$ with $e_j = e_i$, contributing $\sum_i \binom{k_i}{2}$, plus the number of pairs with $e_j = e_i + 1$. The constraints imply that for each m, the subsequence formed by the e_i's in $\{m-1, m\}$ ends with an $m-1$. The others may be interspersed in any order, contributing the factor $\begin{bmatrix} k_m + k_{m+1} - 1 \\ k_m - 1 \end{bmatrix}_t$ as we sum over all possibilities. One can show that the choices of interspersing order for each m may be made independently, and together they determine the sequence. □

Curiously, the symmetry $C_n(q,t) = C_n(t,q)$, which is obvious from the definition, appears as quite a surprising property of formula (101) or (102).

The Garsia-Haglund theorem has two conjectured extensions that have been verified computationally up to reasonably large values of n. The first conjecture I will discuss is mine; the second is due to Haglund and Loehr [36]. Define "higher" q,t-Catalan polynomials by the formula

$$C_n^{(m)}(q,t) = \langle e_n, \nabla^m e_n \rangle.$$

They are polynomials with positive coefficients by Theorem 3.5.16, and the following specializations can be derived similarly to the case $m = 1$:

$$C_n^{(m)}(q,1) = \sum_{\lambda \subseteq m\delta_n} q^{m\binom{n}{2}-|\lambda|};$$

$$C_n^{(m)}(q,q^{-1}) = q^{-m\binom{n}{2}} \frac{1}{[mn+1]_q} \begin{bmatrix} (m+1)n \\ n \end{bmatrix}_q.$$

From the discussion in §5.4.4 we see that $C_n^{(m)}(q,t)$ has a natural geometric interpretation as the Hilbert series of the space of sections of the line bundle $\mathcal{O}(m)$ on the zero-fiber Z_n of the Hilbert

scheme. More concretely, this space of sections is $J^m/(\mathbf{x},\mathbf{y})J^m$, where J is the ideal generated by all antisymmetric polynomials in $\mathbb{C}[\mathbf{x},\mathbf{y}]$, as in Proposition 6.1.5.

Conjecture 7.1.3. *Denote by $b^{(m)}(\lambda)$ the number of cells $x \in \lambda$ whose arm and leg satisfy $l(x) \leq a(x) \leq l(x)+m$. For the higher q,t-Catalan polynomials we have the formula*

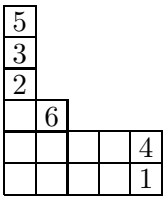
$$C_n^{(m)}(q,t) = \sum_{\lambda \subseteq m\delta_n} q^{m\binom{n}{2}-|\lambda|} t^{b^{(m)}(\lambda)}.$$

The other conjecture I want to discuss is Haglund and Loehr's conjecture for the Hilbert series of the whole diagonal coinvariant ring, or by Theorem 4.2.5, for the polynomial

$$D_n(q,t) \underset{\text{def}}{=} \langle e_1^n, \nabla e_n \rangle.$$

Recall that $D_n(q,1)$ enumerates parking functions by weight. Haglund and Loehr define an analog for parking functions of the statistic $b(\lambda)$ in Theorem 7.1.1.

We have seen that a parking function can be represented by a standard tableau on the skew shape $(\lambda + (1^n))/\lambda$, where $\lambda \subseteq \delta_n$. Here is example (15) again to illustrate this.



Call the *k-th diagonal* the set of cells (i,j) with $i+j=k$. Now let f be a parking function, T the corresponding tableau, $\lambda \subseteq \delta_n$ the corresponding shape, and define $b(f)$ to be the number of pairs of cells $x, y \in (\lambda + (1^n))/\lambda$ such that the label on x is less than the label on y, and either:

(i) x and y are on the same diagonal and y is in a higher row than x, or
(ii) x and y are on consecutive diagonals k and $k+1$, respectively, and y is in a lower row than x.

In the example above we have $b(f)=3$, for the pairs $\{1,3\}$, $\{3,4\}$ and $\{4,5\}$. For any given λ the maximum value of $b(f)$ is $b(\lambda)$, attained by the unique tableau T formed by labeling the cells in increasing order of diagonals, and from lowest row to highest within each diagonal.

Conjecture 7.1.4 ([36])**.** *The Hilbert series of the diagonal coinvariant ring is given by the sum*

$$D_n(q,t) = \sum_f q^{w(f)} t^{b(f)}$$

over parking functions f on $\{1,\ldots,n\}$, with weight $w(f) = \binom{n}{2} - |\lambda|$ and $b(f)$ as defined above.

7.2. **Extensions to other groups.** It it natural to ask which of the results about the Hilbert scheme, diagonal coinvariants, and $n!$ conjecture might hold and in what form with a more general finite group G in the role of S_n.

The first obvious possibility is to take for G a Coxeter group acting diagonally on two copies $\mathfrak{h} \oplus \mathfrak{h}$ of its defining representation, or more generally a complex reflection group acting on $\mathfrak{h} \oplus \mathfrak{h}^*$. In this setting only a limited generalization of the $n!$ conjecture seems to be possible, but it appears that the phenomena involving diagonal coinvariants generalize beautifully. Very recently, I. Gordon [34] has proved a conjecture of mine along these lines.

A second possibility is to take for G the wreath product $\Gamma_n = \Gamma \wr S_n$, where Γ is a finite subgroup of $SL_2(\mathbb{C})$. The possible groups Γ are classified by Dynkin diagrams of type ADE, via the *McKay correspondence* [64]. Associated to each affine ADE diagram and its fundamental weight at the affine node are *quiver varieties* of Nakajima [66], which provide crepant resolutions of $\mathbb{C}^{2n}/\Gamma_n$, with the Hilbert scheme as the special case when Γ is trivial. In this setting there appears to be a good generalization of the $n!$ conjecture.

Most interesting is the intersection of the above two settings, when $\Gamma = \mathbb{Z}/r\mathbb{Z}$ is cyclic, so $G = \Gamma \wr S_n$ is the complex reflection group $G(r, 1, n)$ with its doubled reflection representation $\mathfrak{h} \oplus \mathfrak{h}^*$. Here as for S_n we have the $\mathbb{T}^2$ action, with fixed points indexed by r-tuples of partitions. Conjecturally we expect analogs of the bundle P on the Hilbert scheme, with the graded characters of its fibers given by new *wreath Macdonald polynomials*.

Let me now explain some of these developments in more detail.

7.2.1. *Principal nilpotent pairs.* The *commuting variety* of a semisimple Lie algebra $\mathfrak{g}$ is the set

$$C(\mathfrak{g}) = \{(X, Y) \in \mathfrak{g} : [X, Y] = 0\}.$$

It is not known whether the equations $[X, Y] = 0$ generate the ideal of the commuting variety, so in principle we should distinguish the possibly non-reduced *commuting scheme* defined by these equations. It is known that $C(\mathfrak{g})$ is irreducible, of dimension $\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g}$. For $\mathfrak{gl}_n$ this is an old theorem of Motzkin and Taussky [65]; it was extended to all $\mathfrak{g}$ by Richardson [73]. Equivalently, the commuting regular semisimple pairs are dense in $C(\mathfrak{g})$.

One proves easily that the dimension of the Zariski tangent space to the commuting scheme at (X, Y) is equal to $\dim \mathfrak{g} + \dim \mathfrak{z}(X, Y)$, where $\mathfrak{z}(X, Y)$ is their common centralizer. Set

$$C_{\mathrm{reg}}(\mathfrak{g}) = \{(X, Y) \in C(\mathfrak{g}) : \dim \mathfrak{z} = \operatorname{rk} \mathfrak{g}\},$$

the nonsingular locus of the commuting scheme, which at least on this locus clearly coincides with the commuting variety. A point of $C(\mathfrak{gl}_n)$ is just a pair of commuting $n \times n$ matrices X, Y, and they make $\mathbb{C}^n$ into a $\mathbb{C}[x, y]$-module. Neubauer and Saltman [69] showed that $(X, Y) \in C_{\mathrm{reg}}(\mathfrak{gl}_n)$ if and only if $\mathbb{C}^n$ considered as a $\mathbb{C}[x, y]$-module is locally cyclic or cocylic at every point $(x, y) \in \mathbb{C}^2$. Their theorem implies that for every $(X, Y) \in C_{\mathrm{reg}}(\mathfrak{gl}_n)$, the ideal of relations $I = \{f(x, y) \in \mathbb{C}[x, y] : f(X, Y) = 0\}$ has $\dim_{\mathbb{C}} \mathbb{C}[x, y]/I = n$, yielding a morphism

$$C_{\mathrm{reg}}(\mathfrak{gl}_n) \to H_n \tag{103}$$

of $C_{\mathrm{reg}}(\mathfrak{gl}_n)$ onto the Hilbert scheme.

The points of $C_{\mathrm{reg}}(\mathfrak{gl}_n)$ lying over a $\mathbb{T}^2$-fixed point in H_n are exactly the *principal nilpotent pairs*, as defined more generally for any $\mathfrak{g}$ by Ginzburg.

Definition 7.2.1 ([33])**.** A *principal nilpotent pair* is an element $(e_1, e_2) \in C_{\mathrm{reg}}(\mathfrak{g})$ such that the orbit $\mathrm{Ad}(G)(e_1, e_2)$ is a fixed point for the action of $\mathbb{T}^2$ on the set of such orbits.

Here the $\mathbb{T}^2$ action is the obvious one on the two components of $(X, Y) \in C(\mathfrak{g})$.

The principal nilpotent pairs up to $\mathrm{Ad}(G)$ conjugacy have been classified by Elashvili and Panyushev [16]. For $\mathfrak{gl}_n$, two pairs dual to each other lie over each fixed point I_μ in the Hilbert scheme (or one self-dual pair if μ is rectangular). For other types they are much more rare, and the elements e_1, e_2 can belong only to very special nilpotent orbits.

Now let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and W the Weyl group. To each principal nilpotent pair $\mathbf{e}$ (apart from an exception in type E_7) Ginzburg associates a W-antisymmetric polynomial $\Delta_{\mathbf{e}} \in \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}]$ generalizing our polynomial Δ_μ in (77), which is the $\mathfrak{gl}_n$ case. Letting $J_{\mathbf{e}} \subseteq \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}]$

be the ideal of polynomials f such that $f(\partial)\Delta_{\mathbf{e}} = 0$, the analog of our ring R_μ would appear naively to be

$$R_{\mathbf{e}} \underset{\text{def}}{=} \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}]/J_{\mathbf{e}}.$$

It would be tempting to conjecture that $\dim R_{\mathbf{e}} = |W|$ and $R_{\mathbf{e}}$ affords the regular representation of W, but that is false for $\mathfrak{g} = \mathfrak{sp}_6$.

There seem to be two ways to fix this trouble. The first has been proposed by Kumar and Thomsen [54], who would replace the ring $R_{\mathbf{e}}$ by the coordinate ring $D_{\mathbf{e}}$ of the scheme-theoretic intersection of $\mathfrak{h} \oplus \mathfrak{h}$ with the closure $\overline{O}_{\mathbf{e}}$, where $O_{\mathbf{e}}$ is the orbit of $\mathbf{e}$ under a group $G^{\mathbf{e}}$ which is $\mathrm{Ad}(G)$ enlarged by certain outer automorphisms of $\mathfrak{g}$. By definition their ring $D_{\mathbf{e}}$ is a again a quotient of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}]$ by a doubly homogeneous W-invariant ideal, and they conjecture that $D_{\mathbf{e}}$ is Gorenstein and affords the regular representation of W. Unfortunately, it appears their conjecture cannot be true as stated since the properties of $D_{\mathbf{e}}$ would imply that $R_{\mathbf{e}} = D_{\mathbf{e}}$. However, I think a small modification may work.

Conjecture 7.2.2. *Let $\tilde{O}_{\mathbf{e}}$ be the normalization of the orbit closure $\overline{O}_{\mathbf{e}}$ defined by Kumar and Thomsen, when it is irreducible, or a suitable seminormalization when it is not. Then the coordinate ring $\tilde{D}_{\mathbf{e}}$ of the scheme-theoretic fiber product of $\tilde{O}_{\mathbf{e}}$ and $\mathfrak{h} \oplus \mathfrak{h}$ over $C(\mathfrak{g})$ is Gorenstein, has dimension $|W|$, and affords the regular representation of $|W|$.*

Observe that the original ring $D_{\mathbf{e}}$ is a subring of $\tilde{D}_{\mathbf{e}}$, the image of the canonical homomorphism $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}] \to \tilde{D}_{\mathbf{e}}$. It seems plausible to expect that $D_{\mathbf{e}} = R_{\mathbf{e}}$, so $D_{\mathbf{e}}$ should in fact be Gorenstein, but its dimension will be too small in general. In the case of $\mathfrak{gl}_n$, when $R_{\mathbf{e}} = R_\mu$ has the full dimension $n!$, the normalization step is unnecessary, and the Kumar-Thomsen conjecture should be correct in its original form. Even for $\mathfrak{gl}_n$ it is open.

There is another way to enlarge $R_{\mathbf{e}}$ to what should turn out to be the same ring $\tilde{D}_{\mathbf{e}}$ as in Conjecture 7.2.2. Restriction of functions defines a homomorphism of coordinate rings

$$\mathbb{C}[C(\mathfrak{g})] \to \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}],$$

and by an important theorem of Wallach [84] and Joseph [46], it induces an isomorphism of invariants

$$\mathbb{C}[C(\mathfrak{g})]^G \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}]^W.$$

Thus we get a ring homomorphism $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}]^W \to \mathbb{C}[C(\mathfrak{g})]$ and a G-invariant morphism of algebraic varieties $C(\mathfrak{g}) \to (\mathfrak{h} \oplus \mathfrak{h})/W$. Now define the *isospectral commuting variety* $X(\mathfrak{g})$ to be the *normalized*, *reduced* fiber product

$$\begin{array}{ccc} X(\mathfrak{g}) & \longrightarrow & \mathfrak{h} \oplus \mathfrak{h} \\ \big\downarrow & & \big\downarrow \\ C(\mathfrak{g}) & \longrightarrow & (\mathfrak{h} \oplus \mathfrak{h})/W. \end{array} \tag{104}$$

The Weyl group W acts on $X(\mathfrak{g})$, and the projection $X(\mathfrak{g}) \to C(\mathfrak{g})$ is finite of degree $|W|$ with W acting by the regular representation on its generic fibers.

Conjecture 7.2.3. *The isospectral commuting variety $X(\mathfrak{g})$ is Cohen-Macaulay and Gorenstein. In particular the coordinate ring of its fiber over each point of $C_{\mathrm{reg}}(\mathfrak{g})$ is a Gorenstein ring on which W acts by the regular representation.*

Note that the fiber ring over a point $(X, Y) \in C(\mathfrak{g})$ depends only on its $\mathrm{Ad}(G)$ orbit. The most interesting fibers are those over principal nilpotent pairs $\mathbf{e}$, where the coordinate ring $\hat{D}_{\mathbf{e}}$ of the fiber is doubly graded. Assuming the conjecture holds, the naive ring $R_{\mathbf{e}}$ is the image of the canonical map $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}] \to \hat{D}_{\mathbf{e}}$, and it seems reasonable to expect that $\hat{D}_{\mathbf{e}}$ is isomorphic to the ring $\tilde{D}_{\mathbf{e}}$ in Conjecture 7.2.2.

For the case of $\mathfrak{gl}_n$, Conjecture 7.2.3 in its full strength is open. Over the regular locus C_{reg}, however, it is true—this is equivalent to Theorem 5.2.1. The conjecture implies the famous conjecture generally attributed to Hochster that $C(\mathfrak{gl}_n)$ is Cohen-Macaulay. In the $\mathfrak{gl}_n$ case, I expect that the reduced fiber product in (104) is already normal. For other types it is not.

7.2.2. *Diagonal coinvariants and Gordon's theorem.* Let W be a Coxeter group and $\mathfrak{h}$ its (complexified) defining representation. Then $\mathfrak{h}$ is self-dual and it is best to identify the diagonal representation with the W-module $\mathfrak{h} \oplus \mathfrak{h}^*$, which has a natural symplectic structure. For $W = S_n$, we have seen that the coinvariant ring

$$C^W = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W_+)$$

has dimension $(n+1)^{n-1}$. It is natural for many reasons to predict that the analogous dimension formula in general should be $(h+1)^n$, where h is the Coxeter number of W and $n = \dim \mathfrak{h}$ is the rank, but this turns out not to be quite right. For W of type B_4, for example, the formula gives 9^4, but the actual dimension is $9^4 + 1$. In [37] I conjectured that after passing to a suitable quotient of the coinvariant ring, one should have the following result, which has now been proven by Gordon.

Theorem 7.2.4 (Gordon [34])**.** *There is a canonically defined doubly graded quotient ring R^W of the coinvariant ring C^W with the following properties.*

(i) $\dim R^W = (h+1)^n$, *where h is the Coxeter number and n is the rank;*
(ii) *The Hilbert series of R^W satisfies $H_{R^W}(t^{-1}, t) = t^{-hn/2}[h+1]_t^n$;*
(iii) *The image of $\mathbb{C}[\mathfrak{h}]$ in R^W is the classical coinvariant ring;*
(iv) *If W is a Weyl group, then $\varepsilon \otimes R^W$ is isomorphic as a W-module to the permutation representation on $Q/(h+1)Q$, where Q is the root lattice.*

Let us remark that (ii) obviously implies (i). In fact (ii) also implies (iii), because the image of $\mathbb{C}[\mathfrak{h}]$ is in any event a quotient of the classical coinvariant ring. If it were a proper quotient, then its socle would be killed and there would be no $t^{hn/2}$ term in $H_{R^W}(t^{-1}, t)$.

Note that $H_{R^W}(t^{-1}, t)$ is the Hilbert series of R^W in the grading that assigns degree 1 to the coordinates on $\mathfrak{h}$ and -1 to those on $\mathfrak{h}^*$, which is the natural grading from the symplectic point of view. In fact Gordon fully determines the graded character of R^W in this grading, which implies both (ii) and (iv). It seems to be difficult to understand the second grading using Gordon's approach. It also seems to be difficult to describe the ideal of R^W in C^W. In particular, for $W = S_n$, Gordon's theorem only implies that $(n+1)^{n-1}$ is a lower bound for $\dim R_n$.

Gordon proves Theorem 7.2.4 using representation theory of Cherednik algebras, more specifically their rational degenerations H_c which have been studied by Dunkl [15], Opdam and Rouqier [unpublished], Berest, Etingof and Ginzburg [5, 6], and Guay [unpublished] among others. To see how Cherednik algebras enter the picture, we first describe the coinvariant ring in a different way.

Consider the skew group algebra $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]W$, and let $e = \frac{1}{|W|}\sum_{w \in W} w$ be the invariant idempotent. A $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]W$-module is the same thing as a W-equivariant $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$-module, and the subalgebra $e\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]We$ is the ring of invariants $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. Regard $\mathbb{C}$ as a trivial $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$-module

annihilated by the ideal $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W_+$. Then the coinvariant ring can be described as

$$C^W = \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]We \otimes_{e\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]We} \mathbb{C}.$$

Alternatively, let $e_\varepsilon = \frac{1}{|W|}\sum_{w\in W}\varepsilon(w)w$ be the sign idempotent. Then we also have

$$\varepsilon \otimes C^W = \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]We_\varepsilon \otimes_{e_\varepsilon\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]We_\varepsilon} \mathbb{C}. \tag{105}$$

The *rational Cherednik algebra* H_c (depending on a parameter c) is the algebra of operators on $\mathbb{C}[\mathfrak{h}]$ generated by

(i) multiplication operators by coordinates x_i on $\mathfrak{h}$;
(ii) Dunkl-differential operators

$$y_i = \partial/\partial x_i - c\sum_{s\in S}\langle\alpha_s, x_i^*\rangle\frac{1-s}{\alpha_s},$$

where $\{x_i^*\}$ is a dual basis of coordinates on $\mathfrak{h}^*$, S is the set of reflections in W, and α_s is a linear form vanishing on the fixed hyperplane of s;
(iii) the group W.

The operators y_i commute. Thus H_c contains a copy of $\mathbb{C}[\mathfrak{h}]$ generated by the x_i's, a copy of $\mathbb{C}[\mathfrak{h}^*]$ generated by the y_i's, and a copy of $\mathbb{C}W$. It has a Poincaré-Birkhoff-Witt decomposition

$$H_c \cong \mathbb{C}[\mathfrak{h}]\otimes\mathbb{C}W\otimes\mathbb{C}[\mathfrak{h}^*] \tag{106}$$

and commutation relations

$$[y_i, x_j] = \delta_{ij} - c\sum_{s\in S}\langle\alpha_s, x_i^*\rangle\langle x_i, \alpha_s^\vee\rangle s,$$

where $\alpha_s^\vee$ is the coroot vector such that $sv = v - \langle\alpha_s, v\rangle\alpha_s^\vee$. Under the filtration of H_c assigning degree zero to $\mathbb{C}W$ and degree 1 to x_i, y_i, we have

$$\operatorname{gr} H_c \cong \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]W. \tag{107}$$

The decomposition (106) implies that for each irreducible W-module τ there is a standard "Verma" module $M_c(\tau)$ isomorphic to $\mathbb{C}[\mathfrak{h}]\otimes\tau$ as a $\mathbb{C}[\mathfrak{h}]W$ module and annihilated by $\mathbb{C}[\mathfrak{h}^*]_+$. It has a unique irreducible quotient $L_c(\tau)$. Put $\mathfrak{h}_k = \wedge^k\mathfrak{h}$, which is always an irreducible W-module.

Gordon proves that for the special value $c = (1+h)/h$ of the parameter, the following things happen. First, there is a decomposition

$$L_c(\mathfrak{h}_0) = \sum_k (-1)^k M_c(\mathfrak{h}_k)$$

in the Grothendieck group of H_c-modules. Second, the degree operator $d = \frac{1}{2}\sum_i(x_iy_i + y_ix_i)$ induces a grading on these modules, such that $\deg(x_i) = 1$, $\deg(y_i) = -1$, and $M_c(\mathfrak{h}_k)$ has the same grading as $\mathbb{C}[\mathfrak{h}]\otimes\mathfrak{h}_k$ with the generators in degree $-hn/2 + k(h+1)$. It follows that the Hilbert series of $L_c(\mathfrak{h}_0)$ is equal to

$$t^{-hn/2}[h+1]_t^n,$$

as we want to prove for R^W. The same computation also determines the character of $L_c(\mathfrak{h}_0)$ as a graded W-module. Finally, the subalgebra $e_\varepsilon H_c e_\varepsilon$ has a unique 1-dimensional unital module $\mathbb{C}$ (with degree zero), and $L_c(\mathfrak{h}_0)$ is the induced module

$$L_c(\mathfrak{h}_0) \cong H_c e_\varepsilon \otimes_{e_\varepsilon H_c e_\varepsilon} \mathbb{C}.$$

Passing to the associated graded using (107), and comparing with (105), we get a canonical surjection

$$C^W \to \varepsilon \otimes \operatorname{gr} L_c(\mathfrak{h}_0)$$

of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]W$-modules. Hence $R^W \underset{\text{def}}{=} \varepsilon \otimes \operatorname{gr} L_c(\mathfrak{h}_0)$ has the desired properties.

Shortly before Gordon's work appeared I had conjectured another way to pick out the correct quotient of C^W, as follows.

Conjecture 7.2.5. *Let $\tilde{R}^W$ be the quotient of C^W by the unique largest ideal $J \subseteq C^W$ such that $J \cap (C^W)^\varepsilon = 0$. Then $\tilde{R}^W$ has the properties in Theorem 7.2.4.*

Conjecture 7.2.5 is true for $W = S_n$, that is, we have $R_n^\varepsilon = \operatorname{soc} R_n$. This follows from [40, Theorem 4.1]. I can prove it for W of type B_n, and have verified it by computer for type D_4.

Proposition 7.2.6. *If Conjecture 7.2.5 holds for W, then the ring $\tilde{R}^W$ in the conjecture is the same as Gordon's ring R_W.*

Proof. Suppose Conjecture 7.2.5 holds. Then $\dim(C^W)^\varepsilon = \dim(\tilde{R}^W)^\varepsilon = \dim(R^W)^\varepsilon$. This implies that the defining ideal of Gordon's ring R^W is contained in the ideal J in the conjecture. But $\dim \tilde{R}^W = \dim R^W = (h+1)^n$, so $\tilde{R}^W = R^W$. □

7.2.3. *Quiver varieties.* In this section and the next, the role of S_n will be played by the wreath product $\Gamma_n = \Gamma \wr S_n$, where Γ is a finite subgroup of $SL_2(\mathbb{C})$. Then Γ^n acts on $\mathbb{C}^{2n}$ and is normalized by S_n; the wreath product Γ_n is their semidirect product. The presentation here is a synopsis of a preprint [41] that I hope to make available on the servers soon.

The quotient singularities $\mathbb{C}^{2n}/\Gamma_n$ have crepant resolutions by quiver varieties, generalizing the Hilbert scheme in the case $\Gamma = 1$. Our first goal will be to formulate a conjectured analog of the Cohen-Macaulay property of the isospectral Hilbert scheme, Theorem 5.2.1.

The possible groups Γ are classified by the McKay correspondence. Let $\chi_0, \chi_1, \dots, \chi_{r-1}$ be a list of the irreducible characters of Γ, with $\chi_0 = 1$, and let ζ be the character of the defining representation $\mathbb{C}^2$. The *McKay graph* is the graph on vertex set $I = \{0, 1, \dots, r-1\}$ with an edge $\{i, j\}$ if $\langle \chi_i \otimes \zeta, \chi_j \rangle \neq 0$. Since ζ is self-dual, this relation is symmetric in i and j. The McKay graphs turn out to be the affine Dynkin diagrams of type $\hat{A}$, $\hat{D}$ and $\hat{E}$, as follows.

- Type $\hat{A}_{r-1}$: $\Gamma = \mathbb{Z}/r\mathbb{Z}$ is cyclic.
- Type $\hat{D}_{r+2}$: Γ is the *binary dihedral group*, the preimage of the dihedral group D_{2r} by the double cover $SU_2(\mathbb{C}) \to SO_3(\mathbb{R})$.
- Type $\hat{E}_6$, $\hat{E}_7$, $\hat{E}_8$: the binary tetrahedral, octahedral and icosahedral groups.

Fix Γ and its associated affine Dynkin diagram. Let P be the weight lattice, P^+ the dominant weights, Q the root lattice, Q^+ the cone spanned by the simple roots $\alpha_0, \dots, \alpha_{r-1}$, and δ the smallest positive imaginary root. The corresponding root lattice Q_0 of finite type can be identified with $Q/\mathbb{Z}\delta$. To each character $\chi = \sum_i m_i \chi_i$ corresponds the element $\sum_i m_i \alpha_i \in Q^+$, and a key property of the McKay correspondence is that δ corresponds to the character 1_1^Γ of the regular representation.

Denote by $\mathfrak{M}_\theta^\Gamma(\lambda, \mu)$ the *quiver variety* associated by Nakajima [66] to the McKay graph I of Γ, elements $\lambda \in P^+$ and $\mu \in Q^+$, and a *stability condition* given by a linear function $\theta\colon Q \to \mathbb{R}$. The quiver varieties $\mathfrak{M}_\theta^\Gamma(\lambda, \mu)$ have many remarkable properties. For generic θ they are nonsingular and have a symplectic (and even a hyper-Kähler) structure. Nakajima has shown that for each λ the top

cohomology groups $\bigoplus_\mu H^{\mathrm{top}}(\mathfrak{M}_\theta^\Gamma(\lambda,\mu),\mathbb{C})$ carry the irreducible highest-weight representation $L(\lambda)$ of the Kac-Moody algebra attached to I, with $\mathfrak{M}_\theta^\Gamma(\lambda,\mu)$ contributing the weight space $L(\lambda)_{\lambda-\mu}$.

We will only be concerned with $\mathfrak{M}_\theta^\Gamma(\lambda,\mu)$ for $\lambda=\lambda_0$, the fundamental weight at the affine node of I, and $\theta(\alpha_i)=1$ for all i, the *standard* stability condition. The highest-weight module $L(\lambda_0)$ is the *basic representation*. The stabilizer of λ_0 in the affine Weyl group W is the finite Weyl group W_0, so every element of Q_0 has a unique representative $\nu_0\in Q$ such that $\lambda_0-\nu_0$ is W-conjugate to λ_0. The weight spaces $L(\lambda_0)_{\lambda_0-\nu_0}$ are one-dimensional. We write every $\nu\in Q$ uniquely as

$$\nu=\nu_0+n\delta,\quad n\in\mathbb{Z},$$

with ν_0 the distinguished representative of its coset. The weight space $L(\lambda_0)_{\lambda_0-\nu}$ is non-zero, and the quiver variety $\mathfrak{M}_{\mathbf{1}}^\Gamma(\lambda_0,\nu)$ is non-empty, if and only if $n\geq 0$. For such ν set

$$Y_{\Gamma,\nu}=\mathfrak{M}_{\mathbf{1}}^\Gamma(\lambda_0,\nu).$$

Proposition 7.2.7. *With $\nu=\nu_0+n\delta$ as above, the quiver variety $Y_{\Gamma,\nu}$ is a crepant resolution of $\mathbb{C}^{2n}/\Gamma_n$.*

One way to prove this is to use Nakajima's reflection functors [68] to identify $Y_{\Gamma,\nu}$ with $\mathfrak{M}_\theta^\Gamma(\lambda_0,n\delta)$ for a different stability condition θ. Now $\mathfrak{M}_\theta^\Gamma(\lambda_0,n\delta)$ is projective and birational over the affine quiver variety $\mathfrak{M}_0^\Gamma(\lambda_0,n\delta)$ and it is not hard to show that $\mathfrak{M}_0^\Gamma(\lambda_0,n\delta)\cong\mathbb{C}^{2n}/\Gamma_n$.

There is another, more illuminating, way to understand Proposition 7.2.7. The quiver varieties $Y_{\Gamma,\nu}$ are exactly the irreducible components of the fixed loci H_m^Γ in the Hilbert schemes H_m, for all possible m. This fact, which is well-known to the experts, is a consequence of the McKay correspondence and results of Kronheimer and Nakajima [53], Nakajima [67], and Crawley-Boevey [14]. It develops that these fixed loci also have the following geometrically explicit description.

Proposition 7.2.8. *For each distinguished coset representative ν_0 as above, there is a* unique Γ*-invariant ideal $I_{\nu_0}\subseteq R=\mathbb{C}[x,y]$ such that $\mathrm{char}(R/I_{\nu_0})=\chi_{\nu_0}$, the character of Γ corresponding to ν_0. Set $\nu=\nu_0+n\delta$ and $m=\chi_{\nu_0}(1)+n|\Gamma|$. The quiver variety $Y_{\Gamma,\nu}$ is the closure of the open set in H_m^Γ parametrizing subschemes $S\subseteq\mathbb{C}^2$ of the form $V(I_{\nu_0})\cup T$, where T is a union of n disjoint non-zero Γ-orbits.*

From the proposition we see that the image of $Y_{\Gamma,\nu}$ under the Chow morphism consists of algebraic cycles of the form $\chi_{\nu_0}(1)\cdot 0+C$, where $C=\sum_{i=1}^n\sum_{g\in\Gamma}gP_i$. But $\mathbb{C}^{2n}/\Gamma_n$ can be identified with the set of cycles C, and subtracting the terms $\chi_{\nu_0}(1)\cdot 0$ gives an isomorphism of the image of $Y_{\Gamma,\nu}$ onto $\mathbb{C}^{2n}/\Gamma_n$.

Definition 7.2.9. An ideal I_{ν_0} as in Proposition 7.2.8—*i.e.*, a Γ-invariant ideal in $\mathbb{C}[x,y]$ with no Γ-invariant deformations—is a Γ-*core*.

The motivation for this definition is the remarkable fact that for $\Gamma=\mathbb{Z}/r\mathbb{Z}$, the Γ-cores are exactly the monomial ideals I_μ such that the partition μ is an r-*core* in the classical sense (see §7.2.4, below).

In terms of the picture given by Proposition 7.2.8 we distinguish an open set $U\subseteq Y_{\Gamma,\nu}$ consisting of points with at most one degeneracy: either one orbit collapses to zero or two orbits coincide. For a given n, the quiver varieties $Y_{\Gamma,\nu}$ are all birational to $\mathbb{C}^{2n}/\Gamma_n$ and hence to each other.

Proposition 7.2.10. *For fixed n and $\nu=\nu_0+n\delta$, $\nu'=\nu_0'+n\delta$, the birational map $Y_{\Gamma,\nu}\sim Y_{\Gamma\nu'}$ is given by an isomorphism of the open set $U\subseteq Y_{\Gamma,\nu}$ described above with the corresponding open set $U'\subseteq Y_{\Gamma,\nu'}$. Moreover these open sets meet every divisor.*

The proposition identifies all the Picard groups $\mathrm{Pic}(Y_{\Gamma,\nu})$ for a given n with the Picard group $\mathrm{Pic}(U_{\Gamma,n})$ of their common open subvariety. We can regard the quiver varieties $Y_{\Gamma,\nu}$ as different projective completions of $U_{\Gamma,n}$ over $\mathbb{C}^{2n}/S_n$, distinguished by different ample cones in $\mathrm{Pic}(U_{\Gamma,n})$.

Each quiver variety $Y_{\Gamma,\nu}$ with $\nu = \nu_0 + n\delta$ carries a *tautological bundle* M whose fibers are $\mathbb{C}[x,y]\Gamma$-modules with character $n \cdot 1_1^\Gamma$ (the character corresponding to $n\delta$) and a distinguished Γ-invariant section. To define M, we once again identify $Y_{\Gamma,\nu}$ with $\mathfrak{M}_\theta^\Gamma(\lambda_0, n\delta)$ for a different stability condition θ; then M is the usual tautological bundle of quiver data.

We are now almost ready to formulate our conjectured extension of Theorem 5.2.1. It would be absurd to conjecture that $Y_{\Gamma,\nu}$ is isomorphic to the Hilbert scheme $\mathbb{C}^{2n}//\Gamma_n$. Not only do we get different crepant resolutions as ν_0 varies, but in general $\mathbb{C}^{2n}//\Gamma_n$ is not a crepant resolution (this already happens for $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and $n = 2$). Our conjecture instead will be that $Y_{\Gamma,\nu}$ is a *moduli space of stable constellations*.

Definition 7.2.11. Let G be a finite subgroup of $GL(V)$. A *G-constellation* is a G-equivariant $\mathbb{C}[V]$ module that affords the regular representation of V. Let $\theta\colon \mathcal{X}(G) \to \mathbb{R}$ be a linear function on characters of G with $\theta(1_1^G) = 0$. A constellation M is *θ-stable* if for every proper G-invariant $\mathbb{C}[V]$-submodule $N \subseteq M$ we have $\theta(N) < 0$. A *family of G-constellations* on a scheme Y is a locally free sheaf P of G-equivariant $\mathcal{O}_Y \otimes \mathbb{C}[V]$-modules whose fibers as a vector bundle over Y are G-constellations.

When $\theta(1) = 1$ and $\theta(\chi) < 0$ for all $\chi \neq 1$, a constellation is stable if and only if it is a quotient of $\mathbb{C}[V]$, in which case it is called a *G-cluster*. By standard geometric invariant theory techniques one constructs for generic θ a scheme M_θ projective over V/G that parametrizes θ-stable G-constellations, in the technical sense that it represents the relevant functor of families. The scheme M_θ is the *moduli scheme of θ-stable constellations*. The G-Hilbert scheme $V//G$ is an irreducible component of the moduli scheme of clusters. The definition and construction go back to Kronheimer [52], although the terminology of constellations and clusters seems to be due to Reid.

Note that if P is a family of Γ_n-constellations, normalized so that the line bundle of invariants P^{Γ_n} is trivial, then its sub-bundle of Γ_{n-1}-invariants $P^{\Gamma_{n-1}}$ is a family of $\mathbb{C}[x,y]\Gamma$-modules with a distinguished Γ-invariant section and its fibers have character $n \cdot 1_1^\Gamma$.

Proposition 7.2.12. *On the open set $U_{\Gamma,n} \subseteq Y_{\Gamma,\nu}$ there is a unique family of Γ_n-constellations P such that $P^{\Gamma_{n-1}}$ coincides with the tautological bundle M.*

We remark that although the open set $U_{\Gamma,n}$ is common to every $Y_{\Gamma,\nu}$, the restriction of the tautological bundle M to $U_{\Gamma,n}$ depends on which Weyl chamber in Q_0 contains ν_0).

Conjecture 7.2.13. *On every $Y_{\Gamma,\nu}$ there is a family of Γ_n-constellations P, unique by Proposition 7.2.12, such that $P^{\Gamma_{n-1}}$ coincides with M. Moreover, P is a family of θ-stable constellations for some θ.*

The proof of the Bridgeland-King-Reid theorem in [10] goes through almost verbatim with $V//G$ replaced by the component birational to V/G in a moduli space of stable G-constellations. Hence Conjecture 7.2.13 has the following consequence, which should be important for the study of the quiver varieties $Y_{\Gamma,\nu}$.

Corollary 7.2.14. *Assume Conjecture 7.2.13 holds. Then the functor*

$$\Phi = R\Gamma(P \otimes -)\colon D(Y_{\Gamma,\nu}) \to D^{\Gamma_n}(\mathbb{C}^{2n})$$

is an equivalence of categories.

Conjecture 7.2.13 is true for ν_0 sufficiently far from the walls of its Weyl chamber. In this case $Y_{\Gamma,\nu}$ coincides with the Hilbert scheme of points $\mathrm{Hilb}^n(X_\Gamma)$ on the unique minimal resolution X_Γ of $\mathbb{C}^2/\Gamma$, and the conjecture can be deduced from Corollary 5.2.2 applied to X_Γ^n/S_n.

I have checked Conjecture 7.2.13 by computer for $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and $n = 3$. This case implies that Conjecture 7.2.3 holds locally at the points of $X(\mathfrak{sp}_6)$ lying over principal nilpotent pairs in $\mathfrak{sp}_6$, confirming our improved analog of the $n!$ conjecture. Recall that among these pairs are counterexamples to the more naive analog of the $n!$ conjecture.

Further evidence for Conjecture 7.2.13 lies in phenomena it predicts for $\Gamma = \mathbb{Z}/r\mathbb{Z}$, as I will now explain.

7.2.4. *Cores, wreath Macdonald polynomials, and a new positivity conjecture.* For the rest of the discussion, fix r and $\Gamma = \mathbb{Z}/r\mathbb{Z}$, the group of 2×2 matrices

$$\begin{bmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{bmatrix}, \quad \omega = e^{2\pi i/r}.$$

In this case, the representation of Γ_n on $\mathbb{C}^{2n}$ splits into two invariant subspaces $\mathfrak{h} \oplus \mathfrak{h}^*$, with Γ_n acting as the complex reflection group $G(r, 1, n)$ on $\mathfrak{h}$.

The torus group $\mathbb{T}^2$ in (82) commutes with Γ and acts on $Y_{\Gamma,\nu}$. By Proposition 7.2.8, each Γ-core I_{ν_0} must be a monomial ideal. Using this to identify ν_0 with a partition, the $\mathbb{T}^2$-fixed points of $Y_{\Gamma,\nu}$ are monomial ideals I_μ for suitable partitions $\mu \supseteq \nu_0$. To see which partitions are involved, we first need to recall the combinatorial theory of r-cores and r-quotients (see James and Kerber [44] or Macdonald [61]).

A *ribbon* in a partition λ is a connected skew sub-diagram λ/ν containing no 2×2 rectangle. To each cell $x \in \lambda$ corresponds a ribbon of size equal to the hook length $h(x)$, running from the end of the leg of x to the end of its arm.

Definition 7.2.15. The partition λ is an *r-core* if it contains no ribbon of size r. The *r-core* $\mathrm{Core}_r(\lambda)$ of any λ is the partition that remains after one removes as many r-ribbons in succession as possible (the result is independent of choices made).

Here is an example for $r = 4$, with the numbered cells showing one possible sequence of 4-ribbon removals. The blank cells form the 4-core.

2	2	2				
4	4	2	1	1	1	
	4	4	3	3	1	
				3	3	

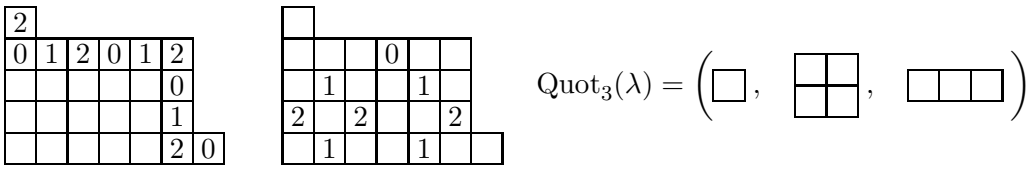
Define the *content* of a cell $x = (i, j) \in \lambda$ to be $c(x) = i - j$. Let y, z be the cells at the end of the arm and leg of x, and define the *row-residue* $\rho(x)$ and *column-residue* $\gamma(x)$ to be the residues of the contents $c(y)$, $c(z) \pmod r$. Observe that $h(x) \equiv 0 \pmod r$ if and only if $\gamma(x) \equiv \rho(x) + 1$. For each residue class i, the cells $x \in \lambda$ with $\gamma(x) \equiv i$ and $\rho(x) \equiv i - 1$ form an "exploded" copy of the diagram of a partition λ^i, with rows and columns not necessarily adjacent, and with a shifted origin.

Definition 7.2.16. The *r-quotient* of a partition λ is the sequence of partitions

$$\mathrm{Quot}_r(\lambda) \underset{\text{def}}{=} (\lambda^0, \ldots, \lambda^{r-1})$$

constructed as above.

Here is an example with $r = 3$. In the first picture the ends of rows and columns are labeled with their contents (mod 3). In the second, the cells forming the exploded diagrams are labeled by their $\gamma(x)$ value.



Proposition 7.2.17. *For any partition λ, if $\mathrm{Core}_r(\lambda) = \nu$ is its r-core and $\mathrm{Quot}_r(\lambda) = (\lambda^0, \ldots, \lambda^{r-1})$ is its r-quotient, then*

$$|\lambda| = |\nu| + r \sum_i |\lambda^i|.$$

For any fixed r-core ν_0, the map $\lambda \mapsto \mathrm{Quot}_r(\lambda)$ is a bijection from $\{\lambda : \mathrm{Core}_r(\lambda) = \nu_0\}$ onto the set of all r-tuples of partitions.

Now we can describe the fixed points of $Y_{\Gamma,\nu}$.

Proposition 7.2.18. *For $\Gamma = \mathbb{Z}/r\mathbb{Z}$, the Γ-cores are exactly the monomial ideals I_{ν_0}, where the partition ν_0 is an r-core, and the $\mathbb{T}^2$-fixed points of $Y_{\Gamma,\nu_0+n\delta}$ are the monomial ideals I_μ where $|\mu| = |\nu_0| + nr$ and $\mathrm{Core}_r(\mu) = \nu_0$. In particular, they are in natural bijective correspondence with r-tuples of partitions $(\mu^0, \ldots, \mu^{r-1})$ of total size $\sum_i |\mu^i| = n$.*

We have abused notation by writing ν_0 for both an r-core and a character of Γ. Explicitly, the character corresponding to ν_0 is $\sum_i m_i \chi_i$, where m_i is the number of cells $x \in \nu_0$ with content $c(x) \equiv i \pmod r$, and $\chi_i(\left[\begin{smallmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{smallmatrix}\right]) = \omega^{ik}$.

Recall the standard indexing of irreducible characters of Γ_n by r-tuples $(\lambda^0, \ldots, \lambda^{r-1})$ of partitions with $\sum_i |\lambda^i| = n$. If V^λ is an irreducible S_n-module and $W_i = \mathbb{C}$ considered as a Γ-module with character χ_i, then $V^\lambda \otimes W_i$ becomes a Γ_n-module via natural homomorphisms $\Gamma_n \to S_n$ and $\Gamma_n \to \Gamma$. Setting $k_i = |\lambda^i|$, the induced module

$$\left(\bigotimes_{i=0}^{r-1} (V^{\lambda^i} \otimes W_i) \right) \uparrow_{\Gamma_{k_0} \times \cdots \times \Gamma_{k_{r-1}}}^{\Gamma_n}$$

is irreducible, and we write $\chi^{(\lambda^0, \ldots, \lambda^{r-1})}$ for its character.

Assuming Conjecture 7.2.13 holds, the fiber of the family of constellations P over a fixed point I_μ in $Y_{\Gamma,\nu}$ is a doubly-graded Γ_n-equivariant $\mathbb{C}[\mathbf{x}, \mathbf{y}]$-module affording the regular representation of Γ_n, and we should expect its character to be an analog of the Macdonald polynomial $\tilde{H}_\mu(z; q, t)$. We could define the Frobenius series of a graded Γ_n-module as a symmetric function in r sets of variables, but it is simpler to work in the space $\mathcal{X}_{q,t}(\Gamma_n)$ of virtual characters with coefficients in $\mathbb{Q}(q, t)$. So, for a doubly graded Γ_n-module $A = \bigoplus_{i,j} A_{i,j}$, set

$$F_A(q, t) = \sum_{i,j} t^i q^j \operatorname{char} A_{i,j}.$$

Conjecture 7.2.13 and some plausible extra assumptions about the geometry of $Y_{\Gamma,\nu}$ lead to the following conjecture/definition. The first part of the conjecture is an analog for Γ_n characters of the

definition of Macdonald polynomials (Definition 3.5.2) and makes no explicit reference to quiver varieties.

Conjecture 7.2.19. *Fix r, $\Gamma = \mathbb{Z}/r\mathbb{Z}$ and an r-core ν_0. Let $\mathfrak{h} = \mathbb{C}^n$ be the defining representation of the complex reflection group $G(r,1,n) \cong \Gamma_n$.*

Existence: *there exists a basis $\{H_\mu(q,t)\}$ of $\mathcal{X}_{q,t}(\Gamma_n)$ indexed by partitions μ of size $|\mu| = |\nu_0| + nr$ with $\mathrm{Core}_r(\mu) = \nu_0$, characterized by the following properties.*

(i) $H_\mu(q,t) \otimes \sum_i (-q)^i \operatorname{char}(\wedge^i \mathfrak{h}) \in \mathbb{Q}(q,t)\{\chi^{\mathrm{Quot}_r(\lambda)} : \lambda \geq \mu,\ \mathrm{Core}_r(\lambda) = \nu_0\}$;

(ii) $H_\mu(q,t) \otimes \sum_i (-t)^{-i} \operatorname{char}(\wedge^i \mathfrak{h}) \in \mathbb{Q}(q,t)\{\chi^{\mathrm{Quot}_r(\lambda)} : \lambda \leq \mu,\ \mathrm{Core}_r(\lambda) = \nu_0\}$;

(iii) $\langle H_\mu(q,t), 1_{\Gamma_n} \rangle = 1$.

Terminology: *The characters $H_\mu(q,t)$ (assuming they exist) are* wreath Macdonald polynomials.

Positivity: *The wreath Macdonald polynomials have coefficients in $\mathbb{N}[q,t,q^{-1},t^{-1}]$.*

Geometry: *The wreath Macdonald polynomials are the graded characters of the fibers $P(I_\mu)$ of the family of Γ_n-constellations in Conjecture 7.2.13 on $Y_{\Gamma,\nu_0+n\delta}$,*

$$H_\mu(q,t) = F_{P(I_\mu)}(q,t).$$

I will close with a few remarks about this conjecture. Like Conjecture 7.2.13 (and for the same reasons), it is true for ν_0 sufficiently far from the walls of its Weyl chamber. The wreath Macdonald polynomials for such ν_0 depend only on the chamber and can be written explicitly in terms of the classical Macdonald polynomials $\tilde{H}_\mu(z;q,t)$. Note that although there are infinitely many r-cores ν_0, they induce for each n only finitely many different orderings on the characters $\chi^{\mathrm{Quot}_r(\lambda)}$, and the wreath Macdonald polynomials depend only on the ordering. Similarly, there are only finitely many non-isomorphic quiver varieties $Y_{\Gamma,\nu_0+n\delta}$ for each Γ and n.

The existence and positivity assertions in the conjecture are easy to check by computer, and I have done so for a fairly large number of cases. The geometry assertion is much harder to verify. The basic trouble is that there is a description analogous to the $n!$ conjecture only for the image of $\mathbb{C}[\mathbf{x},\mathbf{y}]$ in $P(I_\mu)$, which is nearly always a proper subalgebra. However, the existence and positivity can themselves be regarded as evidence for the geometry, especially because the geometry predicts other properties of the characters $H_\mu(q,t)$ that are confirmed by the computations.

Let q^{-k} be the lowest power of q that occurs in $H_\mu(q,t)$. We can regard $H_\mu(t) \underset{\text{def}}{=} q^k H_\mu(q,t)|_{q=0}$ as wreath Hall-Littlewood polynomials. They are characterized (for fixed ν_0) by suitable orthogonality and triangularity properties. The generalized Green functions for $G(r,1,n)$ defined by Shoji [79] satisfy the same orthogonality as our $H_\mu(t)$ but a different kind of triangularity. For us, the ordering of characters depends on ν_0 via the r-quotient, whereas Shoji uses an ordering based on generalized Lusztig symbols. Shoji's ordering is incompatible with ours in general, resulting in different polynomials, although some of them are (accidentally?) the same for small r and n.

Finally, when $r = 2$, Γ_n is the Weyl group of type B_n or C_n. It turns out that $\nu_0 = 0$, corresponding to the empty partition, is type C_n, while $\nu_0 = \alpha_0$, corresponding to the partition (1), is type B_n. The reason for this is that there are canonical $\mathrm{Ad}(G)$-invariant morphisms

$$\begin{aligned} C_{\mathrm{reg}}(\mathfrak{sp}_{2n}) &\to Y_{\Gamma,0+n\delta} \\ C_{\mathrm{reg}}(\mathfrak{so}_{2n+1}) &\to Y_{\Gamma,\alpha_0+n\delta}. \end{aligned} \tag{108}$$

The preimages of $\mathbb{T}^2$-fixed points in $Y_{\Gamma,\nu}$ are the principal nilpotent pairs in $C_{\mathrm{reg}}(\mathfrak{g})$. In contrast to (103), the above morphisms are not surjective, and $Y_{\Gamma,\nu}$ has many $\mathbb{T}^2$-fixed points that are not images of principal nilpotent pairs. One can show that in this situation, the restriction of

Conjecture 7.2.13 to the image of the morphism in (108) is equivalent to Conjecture 7.2.3, restricted to the part of $X(\mathfrak{g})$ lying above $C_{\mathrm{reg}}(\mathfrak{g})$.

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