MULTIGRADED HILBERT SCHEMES

MARK HAIMAN AND BERND STURMFELS

Abstract. We introduce the multigraded Hilbert scheme, which parametrizes all homogeneous ideals with fixed Hilbert function in a polynomial ring that is graded by any abelian group. Our construction is widely applicable, it provides explicit equations, and it allows us to prove a range of new results, including Bayer's conjecture on equations defining Grothendieck's classical Hilbert scheme and the construction of a Chow morphism for toric Hilbert schemes.

1. Introduction

The multigraded Hilbert scheme parametrizes all ideals in a polynomial ring which are homogeneous and have a fixed Hilbert function with respect to a grading by an abelian group. Special cases include Hilbert schemes of points in affine space [19], toric Hilbert schemes [27], Hilbert schemes of abelian group orbits [25], and Grothendieck's classical Hilbert scheme [13]. We show that the multigraded Hilbert scheme always exists as a quasiprojective scheme over the ground ring k. This result is obtained by means of a general construction which works in more contexts than just multigraded polynomial rings. It also applies to Quot schemes and to Hilbert schemes arising in noncommutative geometry; see *e.g.*, [1], [4]. Our results resolve several open questions about Hilbert schemes and their equations.

Our broader purpose is to realize the multigraded Hilbert scheme effectively, in terms of explicit coordinates and defining equations. These coordinates may either be global, in the projective case, or local, on affine charts covering the Hilbert scheme. A byproduct of our aim for explicit equations is, perhaps surprisingly, a high level of abstract generality. In particular, we avoid using Noetherian hypotheses, so our results are valid over any commutative ground ring k whatsoever.

Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring over a commutative ring k. Monomials x^u in S are identified with vectors u in $\mathbb{N}^n$. A grading of S by an abelian group A is a semigroup homomorphism $\deg\colon \mathbb{N}^n \to A$. This induces a decomposition

$$S = \bigoplus_{a \in A} S_a, \qquad \text{satisfying} \quad S_a \cdot S_b \subseteq S_{a+b},$$

where S_a is the k-span of all monomials x^u whose degree is equal to a. Note that S_a need not be finitely-generated over k. We always assume, without loss of generality, that the group A is generated by the elements $a_i = \deg(x_i)$ for $i = 1, 2, \ldots, n$. Let $A_+ = \deg(\mathbb{N}^n)$ denote the subsemigroup of A generated by $a_1, \ldots, a_n$.

A homogeneous ideal I in S is *admissible* if $(S/I)_a = S_a/I_a$ is a locally free k-module of finite rank (constant on $\operatorname{Spec} k$) for all $a \in A$. Its *Hilbert function* is

$$h_I\colon A \to \mathbb{N}, \quad h_I(a) = \operatorname{rk}_k (S/I)_a. \tag{1}$$

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Note that the support of h_I is necessarily contained in A_+. Fix any numerical function $h\colon A \to \mathbb{N}$ supported on A_+. We shall construct a scheme over k which parametrizes, in the technical sense below, all admissible ideals I in S with $h_I = h$.

Recall (*e.g.* from [13]) that every scheme Z over k is characterized by its *functor of points*, which maps the category of k-algebras to the category of sets as follows:

$$\underline{Z}\colon \underline{k\text{-Alg}} \to \underline{\text{Set}}, \quad \underline{Z}(R) = \operatorname{Hom}(\operatorname{Spec} R, Z). \tag{2}$$

Given our graded polynomial ring $S = k[x_1, \ldots, x_n]$ and Hilbert function h, the *Hilbert functor* $H_S^h\colon \underline{k\text{-Alg}} \to \underline{\text{Set}}$ is defined as follows: $H_S^h(R)$ is the set of homogeneous ideals $I \subseteq R \otimes_k S$ such that $(R \otimes S_a)/I_a$ is a locally free R-module of rank $h(a)$ for each $a \in A$. We shall construct the scheme which represents this functor.

Theorem 1.1. *There exists a quasiprojective scheme Z over k such that $\underline{Z} = H_S^h$.*

The scheme Z is called the *multigraded Hilbert scheme* and is also denoted H_S^h. It is projective if the grading is *positive*, which means that $x^0 = 1$ is the only monomial of degree 0. Note that if the grading is positive, then $A_+ \cap -(A_+) = \{0\}$.

Corollary 1.2. *If the grading of the polynomial ring $S = k[x_1, \ldots, x_n]$ is positive then the multigraded Hilbert scheme H_S^h is projective over the ground ring k.*

This corollary also follows from recent work of Artin and Zhang [1]. The approach of Artin and Zhang is non-constructive, and does not apply when the S_a are not finite over k and the Hilbert scheme is only quasiprojective. In our approach, the Noetherian and finite-generation hypotheses in [1] are replaced by more combinatorial conditions. This gives us sufficient generality to construct quasiprojective Hilbert schemes, and the proof becomes algorithmic, transparent and uniform, requiring no restrictions on the ground ring k, which need not even be Noetherian.

This paper is organized as follows. In Section 2 we present a general construction realizing Hilbert schemes as quasiprojective varieties. The main results in Section 2 are Theorems 2.2 and 2.3. In Section 3, we apply these general theorems to prove Theorem 1.1 and Corollary 1.2. The needed finiteness hypotheses are verified using Maclagan's finiteness theorem [21] for monomial ideals in S. Our main results in Section 3 are Theorems 3.6 and 3.16. These two theorems identify finite subsets D of the group A such that the degree restriction morphism $H_S^h \to H_{S_D}^h$ is a closed embedding (respectively an isomorphism), and they lead to explicit determinantal equations and quadratic equations for the Hilbert scheme H_S^h.

Section 4 concerns the classical Grothendieck Hilbert scheme which parametrizes ideals with a given Hilbert polynomial (as opposed to a given Hilbert function) in the usual $\mathbb{N}$-grading. The results of Gotzmann [16] can be interpreted as identifying the Grothendieck Hilbert scheme with our H_S^h, for a suitably chosen Hilbert function h, depending on the Hilbert polynomial. Our construction naturally yields two descriptions of the Hilbert scheme by coordinates and equations. The first reproduces Gotzmann's equations in terms of Plücker coordinates in two consecutive degrees. The second reproduces equations in terms of Plücker coordinates in just one degree. We prove a conjecture from Bayer's 1982 thesis [2] stating that Bayer's set-theoretic equations of degree n actually define the Hilbert scheme as a scheme.

In Section 5 we examine the case where h is the incidence function of the semigroup A_+, in which case H_S^h is called the *toric Hilbert scheme*. In the special cases when the grading is positive or when the group A is finite, this scheme was constructed by Peeva and Stillman [27] and Nakamura [25] respectively. We unify and extend results by these authors, and we resolve Problem 6.4 in [30] by constructing the natural morphism from the toric Hilbert scheme to the *toric Chow variety*.

Recent work by Santos [28] provides an example where both the toric Chow variety and the toric Hilbert scheme are *disconnected*. This shows that the multigraded Hilbert scheme H_S^h can be disconnected, in contrast to Hartshorne's classical connectedness result [17] for the Grothendieck Hilbert scheme.

In Section 6 we demonstrate that the results of Section 2 are applicable to a wide range of parameter spaces other than the multigraded Hilbert scheme; specifically, we construct Quot schemes and Hilbert schemes parametrizing ideals in the Weyl algebra, the exterior algebra and other noncommutative rings.

Before diving into the abstract setting of Section 2, we wish to first present a few concrete examples and basic facts concerning multigraded Hilbert schemes.

Example 1.3. Let $n = 2$ and $k = \mathbb{C}$, the complex numbers, and fix $S = \mathbb{C}[x, y]$. We conjecture that H_S^h is smooth and irreducible for any group A and any $h : A \to \mathbb{N}$.

(a) If $A = 0$ then H_S^h is the Hilbert scheme of $n = h(0)$ points in the affine plane $\mathbb{A}^2$. This scheme is smooth and irreducible of dimension $2n$; see [14].
(b) If $A = \mathbb{Z}$, $\deg(x)$, $\deg(y)$ are positive integers, and h has finite support, then H_S^h is an irreducible component in the fixed-point set of a $\mathbb{C}^*$-action on the Hilbert scheme of points; see e.g. [9]. This was proved by Evain [10].
(c) If $A = \mathbb{Z}$, $\deg(x) = \deg(y) = 1$ and $h(a) = 1$ for $a \geq 0$, then $H_S^h = \mathbb{P}^1$.
(d) More generally, if $A = \mathbb{Z}$, $\deg(x) = \deg(y) = 1$ and $h(a) = \min(m, a + 1)$, for some integer $m \geq 1$, then H_S^h is the Hilbert scheme of m points on $\mathbb{P}^1$.
(e) If $A = \mathbb{Z}$, $\deg(x) = -\deg(y) = 1$ and $h(a) = 1$ for all a, then $H_S^h = \mathbb{A}^1$.
(f) If $A = \mathbb{Z}^2$, $\deg(x) = (1, 0)$ and $\deg(y) = (0, 1)$, then H_S^h is either empty or a point. In the latter case it consists of a single monomial ideal.
(g) If $A = \mathbb{Z}/2\mathbb{Z}$, $\deg(x) = \deg(y) = 1$ and $h(0) = h(1) = 1$, then H_S^h is isomorphic to the cotangent bundle of the projective line $\mathbb{P}^1$.

Example 1.4. Let $n = 3$. This example is the smallest reducible Hilbert scheme known to us. We fix the $\mathbb{Z}^2$-grading of the polynomial ring $S = \mathbb{C}[x, y, z]$ given by

$$\deg(x) = (1, 0)\,,\ \ \deg(y) = (1, 1)\,,\ \ \deg(z) = (0, 1).$$

Consider the closed subscheme in the Hilbert scheme of nine points in $\mathbb{A}^3$ consisting of homogeneous ideals $I \subset S$ such that S/I has the bivariate Hilbert series

$$s^2t^2 \,+\, s^2t \,+\, st^2 \,+\, s^2 \,+ 2st \,+\, s \,+\, t \,+\, 1.$$

This multigraded Hilbert scheme is the reduced union of two projective lines $\mathbb{P}^1$ which intersect in a common torus fixed point. The universal family equals

$$\langle\, x^3,\, xy^2,\, x^2y,\, y^3,\, a_0x^2z - a_1xy,\, b_0xyz - b_1y^2,\, y^2z,\, z^2 \,\rangle \quad \text{with} \quad a_1b_1 \;=\; 0.$$

Here $(a_0 : a_1)$ and $(b_0 : b_1)$ are coordinates on two projective lines. This Hilbert scheme has three torus fixed points, namely, the three monomial ideals in the family.

In these examples we saw that if the Hilbert function h has finite support, say $m = \sum_{a\in A} h(a)$, then H_S^h is a closed subscheme of the Hilbert scheme of m points in $\mathbb{A}^n$. More generally, there is a canonical embedding of one multigraded Hilbert scheme into another when the grading and Hilbert function of the first refine those of the second. Let $\phi\colon A_0 \to A_1$ be a homomorphism of abelian groups. A grading $\deg_0\colon \mathbb{N}^n \to A_0$ *refines* $\deg_1\colon \mathbb{N}^n \to A_1$ if $\deg_1 = \phi \circ \deg_0$. In this situation, a function $h_0\colon A_0 \to \mathbb{N}$ *refines* $h_1\colon A_1 \to \mathbb{N}$ if $h_1(u) = \sum_{\phi(v)=u} h_0(v)$ for all $u \in A_1$.

Any admissible ideal $I \subseteq R \otimes S$ with Hilbert function h_0 for the grading $\deg_0$ is also admissible with Hilbert function h_1 for $\deg_1$. Hence the Hilbert functor $H_S^{h_0}$ is a subfunctor of $H_S^{h_1}$. The following assertion will be proved in Section 3.

Proposition 1.5. *If* $(\deg_0, h_0)$ *refines* $(\deg_1, h_1)$*, then the natural embedding of Hilbert functors described above is induced by an embedding of the multigraded Hilbert scheme* $H_S^{h_0}$ *as a closed subscheme of* $H_S^{h_1}$.

A nice feature of the multigraded Hilbert scheme, in common with other Hilbert schemes, is that its tangent space at any point has a simple description. We assume that k is a field and $I \in H_S^h(k)$. The S-module $\mathrm{Hom}_S(I, S/I)$ is graded by the group A, and each component $(\mathrm{Hom}_S(I, S/I))_a$ is a finite-dimensional k-vector space.

Proposition 1.6. *For k a field, the Zariski tangent space to the multigraded Hilbert scheme* H_S^h *at a point* $I \in H_S^h(k)$ *is canonically isomorphic to* $(\mathrm{Hom}_S(I, S/I))_0$.

Proof. Let $R = k[\epsilon]/\langle \epsilon^2 \rangle$. The tangent space at I is the set of points in $H_S^h(R)$ whose image under the map $H_S^h(R) \to H_S^h(k)$ is I. Such a point is an A-homogeneous ideal $J \subset R[\mathbf{x}] = k[\mathbf{x}, \epsilon]/\langle \epsilon^2 \rangle$ such that $J/\langle \epsilon \rangle$ equals the ideal I in $S = k[\mathbf{x}]$ and $R[\mathbf{x}]/J$ is a free R-module. Consider the map from $k[\mathbf{x}]$ to $\epsilon R[\mathbf{x}] \cong k[\mathbf{x}]$ given by multiplication by ϵ. This multiplication map followed by projection onto $\epsilon R[\mathbf{x}]/(J \cap \epsilon R[\mathbf{x}]) \cong k[\mathbf{x}]/I$ represents a degree zero homomorphism $I \to S/I$, and, conversely, every degree zero homomorphism $I \to S/I$ arises in this manner from some J. $\square$

2. A general framework for Hilbert schemes

Fix a commutative ring k and an arbitrary index set A called "degrees." Let

$$T \;=\; \bigoplus_{a \in A} T_a \tag{3}$$

be a graded k-module, equipped with a collection of operators $F = \bigcup_{a,b \in A} F_{a,b}$, where $F_{a,b} \subseteq \mathrm{Hom}_k(T_a, T_b)$. Given a commutative k-algebra R, we denote by $R \otimes T$ the graded R-module $\bigoplus_a R \otimes T_a$, with operators $\hat{F}_{a,b} = (1_R \otimes -)(F_{a,b})$. A homogeneous submodule $L = \bigoplus_a L_a \subseteq R \otimes T$ is an *F-submodule* if it satisfies $\hat{F}_{a,b}(L_a) \subseteq L_b$ for all $a, b \in A$. We may assume that F is closed under composition: $F_{bc} \circ F_{ab} \subseteq F_{ac}$ for all $a, b, c \in A$ and F_{aa} contains the identity map on T_a for all $a \in A$. In other words, (T, F) is a small category of k-modules, with the components T_a of T as objects and the elements of F as arrows.

Fix a function $h \colon A \to \mathbb{N}$. Let $H_T^h(R)$ be the set of F-submodules $L \subseteq R \otimes T$ such that $(R \otimes T_a)/L_a$ is a locally free R-module of rank $h(a)$ for each $a \in A$. If $\phi \colon R \to S$ is a homomorphism of commutative rings (with unit), then local freeness implies that $L' = S \otimes_R L$ is an F-submodule of $S \otimes T$, and $(S \otimes T_a)/L'_a$ is locally free of rank $h(a)$ for each $a \in A$. Defining $H_T^h(\phi) \colon H_T^h(R) \to H_T^h(S)$ to be the map sending L to L' makes H_T^h a functor $\underline{k\text{-Alg}} \to \underline{\mathrm{Set}}$, called the *Hilbert functor*.

If (T, F) is a graded k-module with operators, as above, and $D \subseteq A$ is a subset of the degrees, we denote by (T_D, F_D) the restriction of (T, F) to degrees in D. In the language of categories, (T_D, F_D) is the full subcategory of (T, F) with objects T_a for $a \in D$. There is an obvious natural transformation of Hilbert functors $H_T^h \to H_{T_D}^h$ given by restriction of degrees, that is, $L \in H_T^h(R)$ goes to $L_D = \bigoplus_{a \in D} L_a$.

Remark 2.1. Given an F_D-submodule $L \subseteq R \otimes T_D$, let $L' \subseteq R \otimes T$ be the F-submodule it generates. The assumption that F is closed under composition implies that $L'_a = \sum_{b \in D} F_{ba}(L_b)$. In particular, the restriction L'_D of L' is equal to L.

We show that, under suitable hypotheses, the Hilbert functor H_T^h is represented by a quasiprojective scheme over k, called the *Hilbert scheme*. Here and elsewhere we will abuse notation by denoting this scheme and the functor it represents by the same symbol, so we also write H_T^h for the Hilbert scheme.

Theorem 2.2. *Let (T, F) be a graded k-module with operators, as above. Suppose that $M \subseteq N \subseteq T$ are homogeneous k-submodules satisfying four conditions:*

(i) *N is a finitely generated k-module;*
(ii) *N generates T as an F-module;*
(iii) *for every field $K \in \underline{k\text{-}Alg}$ and every $L \in H_T^h(K)$, M spans $(K \otimes T)/L$; and*
(iv) *there is a subset $G \subseteq F$, generating F as a category, such that $GM \subseteq N$.*

Then H_T^h is represented by a quasiprojective scheme over k. It is a closed subscheme of the relative Grassmann scheme $G^h_{N \setminus M}$, which is defined below.

In hypothesis (iii), N also spans $(K \otimes T)/L$, so $\dim_K (K \otimes T)/L = \sum_{a \in A} h(a)$ is finite. Therefore Theorem 2.2 only applies when h has finite support. Our strategy in the general case is to construct the Hilbert scheme for a finite subset D of the degrees A and then to use the next theorem to refine it to all degrees.

Theorem 2.3. *Let (T, F) be a graded k-module with operators and $D \subseteq A$ such that $H^h_{T_D}$ is represented by a scheme over k. Assume that for each degree $a \in A$:*

(v) *there is a finite subset $E \subseteq \bigcup_{b \in D} F_{ba}$ such that $T_a / \sum_{b \in D} E_{ba}(T_b)$ is a finitely generated k-module; and*
(vi) *for every field $K \in \underline{k\text{-}Alg}$ and every $L_D \in H^h_{T_D}(K)$, if L' denotes the F-submodule of $K \otimes T$ generated by L_D, then $\dim(K \otimes T_a)/L'_a \leq h(a)$.*

Then the natural transformation $H_T^h \to H^h_{T_D}$ makes H_T^h a subfunctor of $H^h_{T_D}$, represented by a closed subscheme of the Hilbert scheme $H^h_{T_D}$.

We realize that conditions (i)–(vi) above appear obscure at first sight. Their usefulness will become clear as we apply these theorems in Section 3.

Sometimes the Hilbert scheme is not only quasiprojective over k, but projective.

Corollary 2.4. *In Theorem 2.2, in place of hypotheses* (i)–(iv)*, assume only that the degree set A is finite, and T_a is a finitely-generated k-module for all $a \in A$. Then H_T^h is projective over k.*

Proof. We can take $M = N = T$ and $G = F$. Then hypotheses (i)–(iv) are trivially satisfied, and the relative Grassmann scheme $G^h_{N \setminus M}$ in the conclusion is just the Grassmann scheme G^h_N. It is projective by Proposition 2.10, below. □

Remark 2.5. In Theorem 2.3, suppose in addition to hypotheses (v) and (vi) that D is finite and T_a is finitely generated for all $a \in D$. Then we can again conclude that H_T^h is projective, since it is a closed subscheme of the projective scheme $H^h_{T_D}$.

In what follows we review some facts about functors, Grassmann schemes, and the like, then turn to the proofs of Theorems 2.2 and 2.3. In Section 3 we use these theorems to construct the multigraded Hilbert scheme.

We always work in the category $\underline{\mathrm{Sch}/k}$ of schemes over a fixed ground ring k. We denote the functor of points of a scheme Z by $\underline{Z}$ as in (2).

Proposition 2.6 ([13, Proposition VI-2])**.** *The scheme Z is characterized by its functor $\underline{Z}$, in the sense that every natural transformation of functors $\underline{Y} \to \underline{Z}$ is induced by a unique morphism $Y \to Z$ of schemes over k.*

Our approach to the construction of Hilbert schemes will be to represent the functors in question by subschemes of Grassmann schemes. The theoretical tool we need for this is a representability theorem for a functor defined relative to a given scheme functor. The statement below involves the concepts of *open subfunctor*, see [13, §VI.1.1], and *Zariski sheaf*, introduced as "sheaf in the Zariski topology" at the beginning of [13, §VI.2.1]. Being a Zariski sheaf is a necessary condition for a functor $\underline{k\text{-Alg}} \to \underline{\text{Set}}$ to be represented by a scheme. See [13, Theorem VI-14] for one possible converse. Here is the relative representability theorem we will use.

Proposition 2.7. *Let $\eta\colon Q \to \underline{Z}$ be a natural transformation of functors $\underline{k\text{-Alg}} \to \underline{\text{Set}}$, where $\underline{Z}$ is a scheme functor and Q is a Zariski sheaf. Suppose that Z has a covering by open sets U_α such that each subfunctor $\eta^{-1}(\underline{U_\alpha}) \subseteq Q$ is a scheme functor. Then Q is a scheme functor, and η corresponds to a morphism of schemes.*

Proof. Let Y_α be the scheme whose functor is $\eta^{-1}(\underline{U_\alpha})$. The induced natural transformation $\eta^{-1}(\underline{U_\alpha}) \to \underline{U_\alpha}$ provides us with a morphism $\pi_\alpha\colon Y_\alpha \to U_\alpha$. For each α and β, the open subscheme $\pi_\alpha^{-1}(U_\alpha \cap U_\beta) \subseteq Y_\alpha$ has functor $\eta^{-1}(\underline{U_\alpha} \cap \underline{U_\beta})$. In particular, we have a canonical identification of $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$ with $\pi_\beta^{-1}(U_\alpha \cap U_\beta)$, and these identifications are compatible on every triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$. By the gluing lemma for schemes, there is a scheme Y with a morphism $\pi\colon Y \to Z$ such that for each α we have $Y_\alpha = \pi^{-1}(U_\alpha)$ and $\pi_\alpha = \pi|_{Y_\alpha}$.

Let R be a k-algebra and let ϕ be an element of $\underline{Y}(R)$, that is, a morphism $\phi\colon \operatorname{Spec} R \to Y$. Since the Y_α form an open covering of Y, there are elements f_i generating the unit ideal in R such that ϕ maps $U_{f_i} \subseteq \operatorname{Spec} R$ into some Y_{α_i}. Let $\phi_i\colon U_{f_i} \to Y_{\alpha_i}$ be the restriction of ϕ; it is an element of $\underline{Y_{\alpha_i}}(R_{f_i}) \subseteq Q(R_{f_i})$. For each i, j, the elements ϕ_i, ϕ_j restrict to the same morphism $\phi_{ij}\colon U_{f_i f_j} \to Y_{\alpha_i} \cap Y_{\alpha_j}$, and therefore have the same image in $Q(R_{f_i f_j})$. Since Q is a Zariski sheaf by hypothesis, the elements ϕ_i are all induced by a unique element $\hat{\phi} \in Q(R)$.

We have thus constructed a transformation $\xi\colon \underline{Y} \to Q$ sending $\phi \in \underline{Y}(R)$ to $\hat{\phi} \in Q(R)$, and it is clearly natural in R. We claim that ξ is a natural isomorphism. First note that $\hat{\phi}$ determines each ϕ_i by definition, and these determine ϕ since the U_{f_i} cover $\operatorname{Spec} R$. Hence ξ_R is injective. Now consider any k-algebra R and $\lambda \in Q(R)$. Then $\eta(\lambda) \in \underline{Z}(R)$ is a morphism $\operatorname{Spec} R \to Z$, and we can cover $\operatorname{Spec} R$ by open sets U_{f_i} such that $\eta(\lambda)$ maps each U_{f_i} into some U_{α_i}. This means that the image of λ in $Q(R_{f_i})$ belongs to $\eta^{-1}(\underline{U_{\alpha_i}})$, that is, to $\underline{Y_{\alpha_i}}$. Since $\underline{Y}$ is a Zariski sheaf and the U_{f_i} cover $\operatorname{Spec} R$, this implies that λ belongs to $\xi_R(\underline{Y}(R))$. Hence ξ is surjective. □

Corollary 2.8. *Under the hypotheses of Proposition 2.7, if the natural transformations $\eta^{-1}(\underline{U_\alpha}) \to \underline{U_\alpha}$ given by restricting η are induced by closed embeddings of schemes, then so is η.*

Proof. This just says that the condition for a morphism $\eta\colon Y \to Z$ to be a closed embedding is local on Z. Indeed, the result is valid with "closed embedding" replaced by any property of a morphism that is local on the base. □

Another useful characterization of natural transformations $\eta\colon Q \to \underline{Z}$ represented by closed subschemes of Z is as subfunctors defined by a closed condition. A condition on R-algebras is *closed* if there exists an ideal $I \subseteq R$ such that the condition holds for an R-algebra S if and only if the image of I in S is zero.

Let Z be a scheme over k and $\eta\colon Q \hookrightarrow \underline{Z}$ a subfunctor. We wish to decide whether η is represented by a closed embedding. Consider a k-algebra R and an element $\lambda \in \underline{Z}(R)$, or equivalently a morphism $\lambda\colon \operatorname{Spec} R \to Z$. Given this data, we define a condition $V_{R,\lambda}$ on R-algebras S, as follows. Let $\phi\colon R \to S$ be the ring homomorphism making S an R-algebra. Then S satisfies the condition $V_{R,\lambda}$ if the element $\underline{Z}(\phi)\lambda \in \underline{Z}(S)$ belongs to the subset $\eta_S(Q(S)) \subseteq \underline{Z}(S)$ defined by the subfunctor. We can now express the content of Proposition 2.7 and Corollary 2.8 as follows.

Proposition 2.9. *Let $\eta\colon Q \hookrightarrow \underline{Z}$ be a subfunctor, where $\underline{Z}$ is a scheme functor and Q is a Zariski sheaf. Then Q is represented by a closed subscheme of Z if and only if $V_{R,\lambda}$ is a closed condition for all $R \in \underline{k\text{-}Alg}$ and $\lambda \in \underline{Z}(R)$.*

Proof. First suppose that $Y \subseteq Z$ is a closed subscheme, and $Q = \underline{Y}$ is the corresponding subfunctor of $\underline{Z}$. Given $\lambda\colon \operatorname{Spec} R \to Z$, let $I \subseteq R$ be the ideal defining the scheme-theoretic preimage $\lambda^{-1}(Y) \subseteq \operatorname{Spec} R$. The condition $V_{R,\lambda}$ on an R-algebra S is that $\phi\colon R \to S$ factor through R/I, so it is a closed condition.

For the converse, using Proposition 2.7 and Corollary 2.8, it suffices to verify that $Q' = Q \cap \underline{U}$ is represented by a closed subscheme of U, for each $U = \operatorname{Spec} R$ in an affine open covering of Z. The inclusion $\lambda\colon U \hookrightarrow Z$ is an element $\lambda \in \underline{Z}(R)$. The subset $Q'(S) \subseteq \underline{U}(S)$ is the set of morphisms $\nu\colon \operatorname{Spec} S \to U$ such that $\lambda \circ \nu$ belongs to $\eta_S(Q(S))$. If $\phi\colon R \to S$ is the ring homomorphism underlying such a morphism ν, then $\lambda \circ \nu = \underline{Z}(\phi)\lambda$, so ν belongs to $Q'(S)$ if and only if the R-algebra S satisfies the condition $V_{R,\lambda}$. By hypothesis, the closed condition $V_{R,\lambda}$ is defined by an ideal $I \subseteq R$. Hence $Q'(S)$ is naturally identified with the set of ring homomorphisms $\phi\colon R \to S$ that factor through R/I. In other words, Q' is represented by the closed subscheme $V(I) \subseteq U = \operatorname{Spec} R$. □

Recall that an R-module W is *locally free of rank r* if there exist $f_1, \dots, f_k \in R$ generating the unit ideal, such that $W_{f_i} \cong R^r_{f_i}$ for each i. Let N be any finitely generated k-module. The Grassmann scheme G^r_N represents the *Grassmann functor*, defined as follows: for $R \in \underline{k\text{-Alg}}$, the set $G^r_N(R)$ consists of all submodules $L \subseteq R \otimes N$ such that $(R \otimes N)/L$ is locally free of rank r.

We review the description of the Grassmann scheme G^r_N in terms of coordinates, starting with the free module $N = k^m$, whose basis we denote by X. For this N we write G^r_m in place of G^r_N. Consider a subset $B \subseteq X$ with r elements. Let $G^r_{m\backslash B} \subseteq G^r_m$ be the subfunctor describing submodules $L \in R^m$ such that R^m/L is free with basis B. This subfunctor is represented by the affine space $\mathbb{A}^{r(m-r)} = \operatorname{Spec} k[\gamma^x_b : x \in X\backslash B, b \in B]$. Evaluated at $L \in G^r_{m\backslash B}(R)$, the coordinate $\gamma^x_b \in R$ is given by the coefficient of the basis vector b in the unique expansion of x modulo L. We also set $\gamma^x_b = \delta_{x,b}$ for $x \in B$. Passing to Plücker coordinates, one proves (see [13, Exercise VI-18]) that the Grassmann functor G^r_m is represented by a projective scheme over k, called the *Grassmann scheme*, and the subfunctors $G^r_{m\backslash B}$ are represented by open affine subsets which cover the Grassmann scheme G^r_m.

Next consider an arbitrary finitely-generated k-module $N = k^m/J$. For any k-algebra R, the module $R \otimes N$ is isomorphic to R^m/RJ. The Grassmann functor

G_N^r is naturally isomorphic to the subfunctor of G_m^r describing submodules $L \subseteq R^m$ such that $RJ \subseteq L$. If R^m/L has basis $B \subseteq X$, then the condition $RJ \subseteq L$ can be expressed as follows: for each $u \in J$, write $u = \sum_{x\in X} a_x^u \cdot x$, with $a_x^u \in k$. Then

$$\sum_{x\in X} a_x^u \cdot \gamma_b^x = 0 \quad \text{for all } u \in J \text{ and } b \in B. \tag{4}$$

It follows that, for each B, the intersection of subfunctors $G_{m\setminus B}^r \cap G_N^r \subseteq G_m^r$ is represented by the closed subscheme of $\operatorname{Spec} k[\gamma_b^x]$ defined by the k-linear equations in (4). The condition $RJ \subseteq L$ is local on R, so the subfunctor $G_N^r \subseteq G_m^r$ is a Zariski sheaf. Therefore Proposition 2.7 and Corollary 2.8 give the following result.

Proposition 2.10. *Let N be a finitely generated k-module. The Grassmann functor G_N^r is represented by a closed subscheme of the classical Grassmann scheme G_m^r, called the* Grassmann scheme of N. *In particular, it is projective over k.*

Now suppose that we are given a submodule $M \subseteq N$ (not necessarily finitely generated, as we are not assuming k is Noetherian). For any set B of r elements in M, we can choose a presentation of N in which the generators X contain B. The intersection of G_N^r with the standard open affine $G_{m\setminus B}^r$ defines an open affine subscheme $G_{N\setminus B}^r \subseteq G_N^r$. The affine scheme $G_{N\setminus B}^r$ parametrizes quotients $(R\otimes N)/L$ that are free with basis B. The union of the subschemes $G_{N\setminus B}^r$ over all r-element subsets $B \subseteq M$ is an open subscheme $G_{N\setminus M}^r$ of G_N^r. The corresponding subscheme functor describes quotients $(R\otimes N)/L$ that are locally free with basis contained in M. In other words, $L \in G_N^r(R)$ belongs to $G_{N\setminus M}^r(R)$ if and only if there are elements $f_1, \ldots, f_k$ generating the unit ideal in R, such that each $(R\otimes N/L)_{f_i}$ has a basis $B_i \subseteq M$. Equivalently, L belongs to $G_{N\setminus M}^r(R)$ if and only if M generates $(R\otimes N)/L$, since the latter is a local condition on R. The subfunctor $G_{N\setminus M}^r$ of the Grassmann functor G_N^r is called the *relative Grassmann functor*.

Proposition 2.11. *Let N be a finitely generated k-module and M a submodule. The functor $G_{N\setminus M}^r$ is represented by an open subscheme of G_N^r, called the* relative Grassmann scheme of $M \subseteq N$. *In particular, it is quasiprojective over k.*

Note that if $M = N$ then the relative Grassmann scheme $G_{N\setminus M}^r$ coincides with G_N^r and is therefore projective. If M is any submodule of N then the open subscheme $G_{N\setminus M}^r \subseteq G_N^r$ can be described in local affine coordinates as follows. Fix a set of r elements $B \subseteq N$ and consider the standard affine in G_N^r describing submodules L such that $(R\otimes N)/L$ has basis B. We form a matrix Γ with r rows, and columns indexed by elements $x \in M$, whose entries in each column are the coordinate functions γ_b^x for $b \in B$. Then $G_{N\setminus M}^r$ is described locally as the complement of the closed locus defined by the vanishing of the $r \times r$ minors of Γ.

The definitions and results on Grassmann schemes extend readily to homogeneous submodules of a finitely generated graded module $N = \bigoplus_{a\in A} N_a$, where A is a finite set of "degrees." Fix a function $h\colon A \to \mathbb{N}$. We define the graded Grassmann functor G_N^h by the rule that $G_N^h(R)$ is the set of homogeneous submodules $L \subseteq R\otimes N$ such that $(R\otimes N_a)/L_a$ is locally free of rank $h(a)$ for all $a \in A$. To give such a submodule L, it is equivalent to give each L_a separately. Thus G_N^h is naturally isomorphic to the product $\prod_{a\in A} G_{N_a}^{h(a)}$, and in particular it is projective over k. Similarly, the *relative graded Grassmann functor* $G_{N\setminus M}^h$, where $M \subseteq N$ is a homogeneous submodule, is represented by a quasiprojective scheme over k.

Remark 2.12. In the graded situation, G_N^h is a subfunctor of the ungraded Grassmann functor G_N^r, where $r = \sum_a h(a)$. Similarly, $G_{N\setminus M}^h$ is a subfunctor of $G_{N\setminus M}^r$. The corresponding morphisms of schemes, $G_N^h \to G_N^r$ and $G_{N\setminus M}^h \to G_{N\setminus M}^r$, are closed embeddings. To see this, observe that G_N^h is defined locally by the vanishing of the coordinates γ_b^x on G_N^r with $x \in N_a$, $b \in N_c$, for $a \neq c$. □

We will now prove the two theorems stated at the beginning of this section.

Proof of Theorem 2.2: We shall apply Proposition 2.7 to represent H_T^h in $G_{N\setminus M}^h$.

Step 1: H_T^h is a Zariski sheaf. Let $f_1, \ldots, f_k$ generate the unit ideal in R. To give a homogeneous submodule $L \subseteq R \otimes T$, it is equivalent to give a compatible system of homogeneous submodules $L_i \subseteq R_{f_i} \otimes T$. The homogeneous component L_a is locally free of rank $h(a)$ if and only if the same holds for each $(L_i)_a$.

Step 2: For all $R \in \underline{k\text{-}Alg}$ and $L \in H_T^h(R)$, M generates $(R \otimes T)/L$ as an R-module. Localizing at each $P \in \operatorname{Spec} R$, it suffices to prove this when (R, P) is a local ring. Then for all $a \in A$, the R-module $(R \otimes T_a)/L_a$ is free of finite rank $h(a)$. By Nakayama's Lemma, $RM_a = (R \otimes T_a)/L_a$ if and only if $KM_a = (K \otimes T_a)/L_a$, where $K = R/P$ is the residue field. The latter holds by hypothesis (iii).

Step 3: We have a canonical natural transformation $\eta\colon H_T^h \to G_{N\setminus M}^h$. It follows from Step 2 that the canonical homomorphism $R \otimes N \to (R \otimes T)/L$ is surjective. If L' denotes its kernel, it further follows that M generates $(R \otimes N)/L'$. Hence we have $L' \in G_{N\setminus M}^h(R)$, and the rule $\eta_R(L) = L'$ clearly defines a natural transformation. Note that $G_{N\setminus M}^h$ makes sense as a scheme functor by hypothesis (i).

Step 4: The functors $\eta^{-1}G_{N\setminus B}^h$ are represented by affine schemes. Let $B \subseteq M$ be any homogeneous subset with $|B_a| = h(a)$ for all $a \in A$, so $G_{N\setminus B}^h$ is a standard affine chart in $G_{N\setminus M}^h$. In functorial terms, $G_{N\setminus B}^h(R)$ describes quotients $(R \otimes N)/L'$ that are free with basis B. Hence $\eta^{-1}G_{N\setminus B}^h(R)$ consists of those $L \in H_T^h(R)$ such that $(R \otimes T)/L$ is free with basis B. Given such an L, we define coordinates $\gamma_b^x \in R$ for all $a \in A$ and all $x \in T_a$, $b \in B_a$ by requiring that $x - \sum_{b \in B_a} \gamma_b^x \cdot b$ is in L. For $x \in N$, the coordinates γ_b^x of L coincide with the Grassmann functor coordinates of $\eta_R(L)$, so there is no ambiguity of notation. In particular, they satisfy

$$\gamma_b^x = \delta_{x,b} \quad \text{for } x \in B. \tag{5}$$

They also clearly satisfy a syzygy condition similar to (4), for every linear relation $\sum_x c_x \cdot x = 0$, $c_x \in k$, holding among elements $x \in T_a$. Namely,

$$\sum_x c_x \cdot \gamma_b^x = 0 \quad \text{for all } a \in A,\, b \in B_a. \tag{6}$$

Finally, since L is an F-submodule, the coordinates γ_b^x satisfy

$$\gamma_b^{fx} = \sum_{b' \in B_a} \gamma_{b'}^x \gamma_b^{fb'} \quad \text{for all } a, c \in A \text{ and all } x \in T_a,\, f \in F_{ac},\, b \in B_c. \tag{7}$$

Conversely, suppose we are given elements $\gamma_b^x \in R$ satisfying equations (5)–(7). We fix attention on an individual degree a for the moment. The elements γ_b^x for $x \in T_a$, $b \in B_a$ can be viewed as the entries of a (typically infinite) matrix defining a homomorphism of free R-modules

$$\phi_a\colon R^{T_a} \to R^{B_a}. \tag{8}$$

Equation (6) ensures that ϕ_a factors through the canonical map $R^{T_a} \to R \otimes T_a$, inducing $\phi'_a\colon R \otimes T_a \to R^{B_a}$. Equation (5) ensures that ϕ'_a is the identity on B_a. Let L_a be the kernel of ϕ'_a. We conclude that $(R \otimes T_a)/L_a$ is free with basis B_a. Considering all degrees again, equation (7) ensures that the homogeneous R-submodule $L \subseteq R \otimes T$ thus defined is an F-submodule. We have given correspondences in both directions between elements $L \in \eta^{-1}G^h_{N\setminus B}(R)$ and systems of elements $\gamma^x_b \in R$ satisfying (5)–(7). These two correspondences are mutually inverse and natural in R. By [13, §I.4], this shows that $\eta^{-1}G^h_{N\setminus B}$ is represented by an affine scheme over k.

Step 5. It now follows from Proposition 2.7 that H^h_T is represented by a scheme over $G^h_{N\setminus M}$, the morphism $H^h_T \to G^h_{N\setminus M}$ being given by the natural transformation η from Step 3. Up to this point, we have only used hypotheses (i) and (iii).

Step 6. The morphism corresponding to $\eta\colon H^h_T \to G^h_{N\setminus M}$ is a closed embedding. It is enough to prove this locally for the restriction of η to the preimage of $G^h_{N\setminus B}$. This restriction corresponds to the morphism of affine schemes given by identifying the coordinates γ^x_b on $G^h_{N\setminus B}$ with those of the same name on $\eta^{-1}G^h_{N\setminus B}$. To show that it is a closed embedding, we must show that the corresponding ring homomorphism is surjective. In other words, we claim that the elements γ^x_b with $x \in N$ generate the algebra $k[\{\gamma^x_b\}]/I$, where I is the ideal generated by (5)–(7). Consider the subalgebra generated by the γ^x_b with $x \in N$. Let $g \in G$. If γ^x_b belongs to the subalgebra for all $b \in B$, then so does γ^{gx}_b, by equation (7) and hypothesis (iv). Since G generates F, and N generates T as an F-module by hypothesis (ii), we conclude that γ^x_b lies in the subalgebra for all x. Theorem 2.2 is now proved. □

A description of the Hilbert scheme in terms of affine charts is implicit in the proof above. There is a chart for each homogeneous subset B of M with $h(a)$ elements in each degree a, and the coordinates on that chart are the γ^x_b for homogeneous elements x generating N. Local equations are derived from (5)–(7).

Proof of Theorem 2.3: We will show that Proposition 2.9 applies to $H^h_T \to H^h_{T_D}$.

Step 1: For $L_D \in H^h_{T_D}(R)$, let $L' \subseteq R \otimes T$ be the F-submodule generated by L_D. Then the R-module $(R \otimes T_a)/L'_a$ is finitely generated in each degree $a \in A$. Take E as in (v) and let Y be a finite generating set of the k-module $T_a/\sum_{b\in D} E_{ba}(T_b)$. Since E is finite, the sum can be taken over b in a finite set of degrees $D' \subseteq D$.

For $b \in D'$, the R-module $(R \otimes T_b)/L'_b$ is locally free of rank $h(b)$, and hence generated by a finite set M_b. For all $x \in R \otimes T_b$ there exist coefficients $\gamma^x_v \in R$ (not necessarily unique, as M_b need not be a basis) such that $x \equiv \sum_{v\in M_b} \gamma^x_v \cdot v$ (mod L'_b). For all $g \in E_{ba}$ we have $gx \equiv \sum_{v\in M_b} \gamma^x_v \cdot gv$ (mod L'_a). This shows that the finite set $Z = \bigcup_{b\in D', g\in E_{ba}} g(M_b)$ generates the image of $R \otimes \sum_{b\in D} E_{ba}(T_b)$ in $(R \otimes T_a)/L'_a$, and therefore $Y \cup Z$ generates $(R \otimes T_a)/L'_a$.

Step 2: H^h_T is a subfunctor of $H^h_{T_D}$. Equivalently, for all k-algebras R, the map $H^h_T(R) \to H^h_{T_D}(R)$, $L \mapsto L_D$ is injective. We will prove that if $L' \subseteq R \otimes T$ is the F-submodule generated by L_D, then $L' = L$. Localizing at a point $P \in \operatorname{Spec} R$, we can assume that (R, P) is local, and hence the locally free modules $(R \otimes T_a)/L_a$ are free. Fix a degree $a \in A$, and let B_a be a free module basis of $(R \otimes T_a)/L_a$. Then B_a is also a vector space basis of $(K \otimes T_a)/(K \otimes L_a)$, where $K = R/P$ is the residue field. In particular, $\dim(K \otimes T_a)/(K \otimes L_a) = |B_a| = h(a)$. By (vi) we have $\dim(K \otimes T_a)/(K \cdot L'_a) \le h(a)$, and hence $K \cdot L'_a = K \otimes L_a$, since $L' \subseteq L$. By Step 1, the R-module $(R \otimes T_a)/L'_a$ is finitely generated, so Nakayama's Lemma implies that B_a generates $(R \otimes T_a)/L'_a$. Since B_a is independent modulo $L_a \supseteq L'_a$, it follows that $L'_a = L_a$.

Step 3: The condition that $(S \otimes T_a)/L'_a$ be locally free of rank $h(a)$ is closed. More precisely, fix a k-algebra R and $L_D \in H^h_{T_D}(R)$. Given an R-algebra $\phi\colon R \to S$, let $L' \subseteq S \otimes T$ be the F-submodule generated by $H^h_{T_D}(\phi)L_D = S \otimes_R L_D$. Then the condition that $(S \otimes T_a)/L'_a$ be locally free of rank $h(a)$ is a closed condition on S. To see this, let L^0 be the F-submodule of $R \otimes T$ generated by L_D, that is, the L' for the case $S = R$. By Step 1, $(R \otimes T_a)/L^0_a$ is finitely generated, say by a set X. By (vi) and Nakayama's Lemma, $(R_P \otimes T_a)/(L^0_a)_P$ is generated by at most $h(a)$ elements of X, for every $P \in \operatorname{Spec} R$. For every subset $B \subseteq X$ with $|B| = h(a)$ elements, the set of points $P \in \operatorname{Spec} R$ where B generates $(R_P \otimes T_a)/(L^0_a)_P$ is an open set U_B, and these open sets cover $\operatorname{Spec} R$. The property that a condition on R-algebras is closed is local with respect to the base R. Therefore, replacing R by some localization R_f, we can assume that a single set B with $h(a)$ elements generates $(R \otimes T_a)/L^0_a$. Then B also generates $(S \otimes T_a)/L'_a$ for every R-algebra S.

A presentation of the S-module $(S \otimes T_a)/L'_a = S \otimes_R ((R \otimes T_a)/L^0_a)$ is given by the generating set B, modulo those relations on $b \in B$ that hold in $(R \otimes T_a)/L^0_a$:

$$\sum_{b \in B} c_b \cdot b \equiv 0 \pmod{L^0_a}, \quad c_b \in R. \tag{9}$$

Thus $(S \otimes T_a)/L'_a$ is locally free of rank $h(a)$ if and only if it is free with basis B, if and only if all coefficients c_b of all syzygies in (9) vanish in S, *i.e.*, $\phi(c_b) = 0$. This condition is closed, with defining ideal $I \subseteq R$ generated by all the coefficients c_b.

Step 4: The subfunctor $H^h_T \to H^h_{T_D}$ is represented by a closed subscheme. By Step 2, H^h_T is a subfunctor, and by Step 1 in the proof of Theorem 2.2, it is a Zariski sheaf. In Step 2 we saw that $L_D \in H^h_{T_D}(S)$ is in the image of $H^h_T(S)$ if and only if the F-submodule L' it generates belongs to $H^h_T(S)$. By Step 3, this is a closed condition, since it is the conjunction of the conditions that $(S \otimes T_a)/L'_a$ be locally free of rank $h(a)$, for all $a \in A$. Theorem 2.3 now follows from Proposition 2.9. $\square$

The algorithmic problem arising from Theorem 2.3 is to give equations on $H^h_{T_D}$ which define the closed subscheme H^h_T. We assume that we already have a description of an affine open subset $U \subseteq H^h_{T_D}$ as $\operatorname{Spec} R$ for some k-algebra R (see the paragraph following the proof of Theorem 2.2 above). The embedding of $U = \operatorname{Spec} R$ into $H^h_{T_D}$ corresponds to a *universal element* $L \in H^h_{T_D}(R)$. The ideal $I \subseteq R$ defining the closed subscheme $H^h_T \cap U$ is generated by separate contributions from each degree a, determined as follows. Construct the finite set $X = Y \cap Z \subseteq T_a$ in Step 1, and compute the syzygies of X modulo L^0_a, where $L^0 \subseteq R \otimes T$ is the F-submodule generated by L. These syzygies are represented by the (perhaps infinitely many) rows of a matrix Γ over R, with columns indexed by the finite set X. The content of hypothesis (vi) is that the minors of size $|X| - h(a)$ in Γ generate the unit ideal in R. The contribution to I from degree a is the *Fitting ideal* $I_{|X|-h(a)+1}(\Gamma)$ generated by the minors of size $|X| - h(a) + 1$. In fact, the vanishing of these minors, together with the fact that $I_{|X|-h(a)}(\Gamma)$ is the unit ideal, is precisely the condition that the submodule $L'_a \subseteq R \otimes T_a$ generated by the rows of Γ should have $(R \otimes T_a)/L'_a$ locally free of rank $h(a)$.

If k is Noetherian, so $H^h_{T_D}$ is a Noetherian scheme, then H^h_T must be cut out as a closed subscheme by the equations coming from a finite subset $E \subseteq A$ of the degrees. As we shall see, this is also true when T is a multigraded polynomial ring, even if the base ring k is not Noetherian. Finding such a set E amounts to finding an isomorphism $H^h_T \cong H^h_{T_E}$. Satisfactory choices of D and E for multigraded Hilbert schemes will be discussed in the next section.

Here is a simple example, taken from [2, §VI.1], to illustrate our results so far.

Example 2.13. Let $A = \{3, 4\}$, $T_3 \simeq k^4$ with basis $\{x^3, x^2y, xy^2, y^3\}$, $T_4 \simeq k^5$ with basis $\{x^4, x^3y, x^2y^2, xy^3, y^4\}$, and $F = F_{3,4} = \{x, y\}$, *i.e.*, the operators are multiplication by variables. Fix $h(3) = h(4) = 1$, and $D = \{3\}$. Then $H^h_{T_D}$ is the projective space $\mathbb{P}^3$ parametrizing rank 1 quotients of T_3, where $(c_{123} : c_{124} : c_{134} : c_{234}) \in \underline{\mathbb{P}^3}(R)$ corresponds to the R-module $L_D = L_3$ generated by the 2×2-minors of

$$\begin{pmatrix} c_{234} & -c_{134} & c_{124} & -c_{123} \\ x^3 & x^2y & xy^2 & y^3 \end{pmatrix}. \tag{10}$$

The Hilbert scheme H^h_T is the projective line $\mathbb{P}^1$ embedded as the twisted cubic curve in $H^h_{T_D} \simeq \mathbb{P}^3$ defined by the quadratic equations

$$c_{134}c_{124} - c_{123}c_{234} = c_{124}^2 - c_{123}c_{134} = c_{134}^2 - c_{124}c_{234} = 0. \tag{11}$$

3. Constructing the multigraded Hilbert scheme

We now take up our primary application of Theorems 2.2 and 2.3, the construction of multigraded Hilbert schemes. Let $S = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a polynomial ring over k, with a multigrading $S = \bigoplus_a S_a$ induced by a degree function $\deg\colon \mathbb{N}^n \to A$, with $\deg(x^u) = \deg(u)$, as in the introduction. Here A is an abelian group, or the subsemigroup A_+ generated by $\deg(x_i) = a_i$ for $1 = 1, \ldots, n$. As our k-module with operators (T, F) we take $T = S$, with F the set of all multiplications by monomials. More precisely, F_{ab} consists of multiplications by monomials of degree $b - a$, for all $a, b \in A$. Then an F-submodule $L \subseteq R \otimes S$ is an ideal of $R \otimes S = R[\mathbf{x}]$ which is homogeneous with respect to the A-grading.

Fix a Hilbert function $h\colon A \to \mathbb{N}$, and let H^h_S be the Hilbert functor. For any k-algebra R, the set $H^h_S(R)$ consists of admissible homogeneous ideals $L \subseteq R \otimes S$ with Hilbert function h. Theorem 1.1 states that the functor H^h_S is represented by a quasiprojective scheme. For the proof we need two combinatorial results.

Lemma 3.1 (Maclagan [21])**.** *Let C be a set of monomial ideals in $k[\mathbf{x}]$ which is an antichain, that is, no ideal in C contains another. Then C is finite.*

Let $I \subseteq k[\mathbf{x}]$ be a monomial ideal and $\deg\colon \mathbb{N}^n \to A$ a multigrading. The monomials not in I are called the *standard monomials* for I. The value $h_I(a)$ of the Hilbert function h_I at $a \in A$ is the number of standard monomials in degree a.

Proposition 3.2. *Given a multigrading $\deg\colon \mathbb{N}^n \to A$ and a Hilbert function $h\colon A \to \mathbb{N}$, there is a finite set of degrees $D \subseteq A$ with the following two properties:*

- (g) *Every monomial ideal with Hilbert function h is generated by monomials of degree belonging to D, and*
- (h) *every monomial ideal I generated in degrees D satisfies: if $h_I(a) = h(a)$ for all $a \in D$, then $h_I(a) = h(a)$ for all $a \in A$.*

Our labels for these properties are mnemonics for *generators* and *Hilbert function.*

Proof. Let C be the set of all monomial ideals with Hilbert function h. By Lemma 3.1, C is finite. Let D_0 be the set of all degrees of generators of ideals in C. Now let C_0 be the set of monomial ideals that are generated in degrees in D_0 and whose Hilbert function agrees with h on D_0. By Lemma 3.1 again, C_0 is finite.

If $C_0 = C$, then D_0 is the required D. Otherwise, for each ideal $I \in C_0 \backslash C$, we can find a degree a with $h_I(a) \neq h(a)$. Adjoining finitely many such degrees to D_0, we obtain a set D_1 such that every monomial ideal generated in degrees D_0 and having Hilbert function h in degrees D_1 belongs to C. Now we define C_1 in terms of D_1 as we defined C_0 in terms of D_0, namely, C_1 is the set of monomial ideals generated in degrees D_1 and with Hilbert function h on D_1. By construction, we have $C_1 \cap C_0 = C$. Iterating this process, we get a sequence $C_0, C_1, C_2, \ldots$ of sets of monomial ideals with $C_i \cap C_{i+1} = C$ for all i, and finite sets of degrees $D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$. Here D_i are the degrees such that every monomial ideal generated in degrees D_{i-1} and with Hilbert function h in degrees D_i belongs to C, and C_i are the monomial ideals generated in degrees D_i and with Hilbert function h on D_i. We claim that this sequence terminates with $C_k = C$ for some k.

Given an ideal $I_j \in C_j$, its *ancestor* in C_i for $i < j$ is the ideal I_i generated by the elements of degrees D_i in I_j. We say that I_j is a *descendant* of its ancestors. If I_j is a descendant of I_i, then $I_i \subseteq I_j$, and $I_i \in C$ implies $I_i = I_j$. Suppose, contrary to our claim, that $C_k \neq C$ for all k. Since C_0 is finite, there is an $I_0 \in C_0 \backslash C$ with descendants in $C_k \backslash C$ for infinitely many k, and hence for all $k > 0$. Among its descendants in C_1 must be one, call it I_1, with descendants in $C_k \backslash C$ for all $k > 1$. Iterating, we construct a sequence $I_0, I_1, \ldots$ with $I_k \in C_k$ and I_{k+1} a descendant of I_k. By the ascending chain condition, $I_k = I_{k+1}$ for some k. But this implies $I_k \in C$, a contradiction. We conclude that $C_k = C$ for some k, and $D = D_k$ is the required set of degrees. □

Lemma 3.3. *Given a multigrading* $\deg\colon \mathbb{N}^n \to A$*, let* $D \subseteq A$ *be a subset of the degrees and* $J = \langle x^u : \deg(u) \in D \rangle$ *the ideal generated by all monomials with degree in* D*. If* $a \in A$ *is a degree such that* $h_J(a)$ *is finite, then there is a finite set of monomials* $E \subseteq \bigcup_{b \in D} F_{ba}$ *such that* $S_a / \sum_{b \in D} E_{ba}(S_b)$ *is finitely generated.*

Proof. Choose an expression for each minimal monomial in J_a as $x^v x^u$ for some $x^u \in S_b$, $b \in D$, and let E be the set of monomials x^v that occur. For all $x^r \in J_a$, we have $x^r = x^q x^v x^u$ for some minimal $x^v x^u \in J_a$, and $\deg(q) = 0$. Hence $x^r = x^v(x^q x^u) \in E_{ba}(S_b)$. This shows that set of all standard monomials of degree a for J spans $S_a / \sum_{b \in D} E_{ba}(S_b)$. This set is finite, by hypothesis. □

We are now ready to construct the multigraded Hilbert scheme. In our proof, the condition (h) in Proposition 3.2 will be replaced by the following weaker condition.

(h′) every monomial ideal I generated in degrees D satisfies: if $h_I(a) = h(a)$ for all $a \in D$, then $h_I(a) \leq h(a)$ for all $a \in A$.

Proposition 3.2 holds verbatim for "(g) and (h′)" instead of "(g) and (h)". We fix a *term order* on $\mathbb{N}^n$, so that each ideal $L \subseteq K[\mathbf{x}]$, with $K \in \underline{k\text{-Alg}}$ a field, has an *initial monomial ideal* $\mathrm{in}(L)$. The Hilbert function of $\mathrm{in}(L)$ equals that of L.

Proof of Theorem 1.1: By definition, F is the system of operators on $S = k[\mathbf{x}]$ given by multiplication by monomials. We first verify the hypotheses of Theorem 2.2 for (S_D, F_D), where $D \subseteq A$ is any finite subset of the degrees. Let C be the set of monomial ideals generated by elements of degrees in D, and with Hilbert function agreeing with h on D. By Lemma 3.1, the set C is finite. Let M' be the union over all $I \in C$ of the set of standard monomials for I in degrees D. Then M' is a finite set of monomials which spans the free k-module S_D/I_D for all $I \in C$.

The monomials of degree zero in S form a finitely generated semigroup. Let G_0' be a finite generating set for this semigroup, so that S_0 is the k-algebra generated by G_0'. Every component S_a is a finitely generated S_0-module. For each $a \in A$, let F_a' be a finite set of monomials generating S_a as an S_0-module. Then every monomial of degree a is the product of a monomial in F_a' and zero or more monomials in G_0'. For $b, c \in D$, let $G_{bc} \subseteq F_{bc}$ consist of multiplications by monomials in F_{c-b}', if $b \neq c$, or in G_0', if $b = c$. Then $G = \bigcup_{b,c} G_{bc}$ is finite and generates F_D as a category.

Our construction is based on the following finite set of monomials:

$$N' = GM' \cup \bigcup_{a \in D} F_a'. \tag{12}$$

Setting $M = kM'$, $N = kN'$, it is obvious that M, N and G satisfy hypotheses (i), (ii) and (iv) of Theorem 2.2. For (iii), fix a field $K \in \underline{k\text{-Alg}}$ and an element $L_D \in H^h_{S_D}(K)$. Let $L \subseteq K \otimes S$ be the ideal generated by L_D and I the monomial ideal generated by $\mathrm{in}(L)_D$. Equivalently, I is the ideal generated by the leading monomials of elements of L_D. Therefore I belongs to C and its standard monomials span $(K \otimes S_D)/L_D$. We conclude that M' spans $(K \otimes S_D)/L_D$, which proves (iii). We have now shown that $H^h_{S_D}$ is represented by a quasiprojective scheme for every finite set of degrees D.

It remains to verify hypotheses (v) and (vi) of Theorem 2.3 for a suitable choice of D. Let D be any finite subset of A that satisfies the conditions (g) and (h$'$). We assume that there exists a monomial ideal I generated in degrees D and satisfying $h_I(a) = h(a)$ for all $a \in D$. Otherwise the Hilbert functor and Hilbert scheme are empty, so the result holds vacuously. By condition (h$'$), $h_I(a)$ is finite for all $a \in A$. The ideal J in Lemma 3.3 contains I, so $h_J(a)$ is also finite. For hypothesis (v), we can therefore take E as given by Lemma 3.3.

For (vi), we fix K and $L_D \in H^h_{S_D}(K)$ as we did for (iii), and again let L be the ideal generated by L_D and I the ideal generated by $\mathrm{in}(L)_D$. Our assumption on D implies that the Hilbert function of I satisfies $h_I(a) \leq h(a)$ for all $a \in A$. Since $I \subseteq \mathrm{in}(L)$ it follows that $h_L(a) = h_{\mathrm{in}(L)}(a) \leq h_I(a) \leq h(a)$ for all $a \in A$. This establishes hypothesis (vi). We have proved that the Hilbert functor H^h_S is represented by a closed subscheme of $H^h_{S_D}$. $\square$

Our ultimate goal is to compute the scheme H^h_S effectively. One key issue is to identify suitable finite sets of degrees. A subset D of the abelian group A is called *supportive* for h if D is finite and the conditions (g) and (h$'$) are satisfied. The last two paragraphs in the proof of Theorem 1.1 establish the following result.

Corollary 3.4. *Take S and $h\colon A \to \mathbb{N}$ as in Theorem 1.1. If the set of degrees $D \subseteq A$ is supportive then the canonical morphism $H^h_S \to H^h_{S_D}$ is a closed embedding.*

Remark 3.5. Corollary 1.2 follows immediately from this result and Remark 2.5. Using Remark 2.12, Proposition 1.5 also follows.

Consider one further condition on Hilbert functions and subsets of degrees:

(s) For every monomial ideal I with $h_I = h$, the syzygy module of I is generated by syzygies $x^u x^{v'} = x^v x^{u'} = \mathrm{lcm}(x^u, x^v)$ among generators x^u, x^v of I such that $\deg \mathrm{lcm}(x^u, x^v) \in D$ (*i.e.*, all minimal S-pairs have their degree in D).

A subset D of A is called *very supportive* for a given Hilbert function $h\colon A \to \mathbb{N}$ if D is finite and the conditions (g), (h) and (s) are satisfied. It follows from Proposition 3.2 that a very supportive set of degrees always exists.

Theorem 3.6. *Take S and $h\colon A \to \mathbb{N}$ as in Theorem 1.1. If the set of degrees $D \subseteq A$ is very supportive then the canonical morphism $H_S^h \to H_{S_D}^h$ is an isomorphism.*

Example 3.7. Let $S = k[x, y, z]$ with the $\mathbb{Z}$-grading $\deg(x) = \deg(y) = 1$ and $\deg(z) = -1$ and fix the Hilbert function $h(a) = 2$ for all $a \in \mathbb{Z}$. This example is typical in that both the support of h and the set of monomials in any fixed degree are infinite. There are eight monomial ideals with this Hilbert function:

$$\langle x^2z^2, y\rangle, \langle x^2, yz\rangle, \langle x^2z, xy, yz\rangle, \langle x^2z, y^2, yz\rangle,$$
$$\langle y^2z^2, x\rangle, \langle y^2, xz\rangle, \langle y^2z, xy, xz\rangle, \langle y^2z, x^2, xz\rangle.$$

The set $D = \{0, 1, 2\}$ is very supportive, so the Hilbert scheme H_S^h is isomorphic to $H_{S_D}^h$. It can be checked that this scheme is smooth of dimension 4 over $\operatorname{Spec} k$. □

For the proof of Theorem 3.6 we need a variant of Gröbner bases for ideals in the polynomial ring over a local ring R. Let (R, P) be a local ring satisfying

$$\bigcap_m P^m = 0. \tag{13}$$

This holds for example if R is complete or Noetherian. Let $R[\mathbf{x}] = R[x_1, \ldots, x_n]$ and fix a term order on $\mathbb{N}^n$. This induces a lexicographic order $<$ on the set $(-\mathbb{N}) \times \mathbb{N}^n$, in which $(-d, e) < (-d', e')$ if $-d < -d'$ or if $d = d'$ and $e < e'$ in the given term order. The lexicographic order is not well-ordered, but has the property that if

$$(-d_1, e_1) > (-d_2, e_2) > \cdots$$

is an infinite strictly decreasing chain, then the sequence $d_1, d_2, \ldots$ is unbounded.

Definition 3.8. The *order* $\operatorname{ord}(a)$ of a nonzero element $a \in R$ is the unique integer m such that $a \in P^m \backslash P^{m+1}$, which exists by (13). The *initial term* $\operatorname{in}(p)$ of a nonzero polynomial $p \in R[\mathbf{x}]$ is the term ax^e of p for which the pair $(-\operatorname{ord}(a), e) \in (-\mathbb{N}) \times \mathbb{N}^n$ is lexicographically greatest.

The definition of initial term is compatible with the following filtration of $R[\mathbf{x}]$ by R-submodules: given $(-d, e) \in (-\mathbb{N}) \times \mathbb{N}^n$, we define $R[\mathbf{x}]_{\leq(-d,e)}$ to be the set of polynomials p such that for every term bx^h of p, we have $(-\operatorname{ord}(b), h) \leq (-d, e)$. We also define $R[\mathbf{x}]_{<(-d,e)}$ in the obvious analogous way. Then $\operatorname{in}(p) = ax^e$, with $\operatorname{ord}(a) = d$, if and only if $p \in R[\mathbf{x}]_{\leq(-d,e)} \backslash R[\mathbf{x}]_{<(-d,e)}$.

Consider a set of nonzero polynomials $F \subseteq R[\mathbf{x}]$ satisfying the restriction:

$$\text{for all } f \in F, \text{ the initial term of } f \text{ has coefficient } 1. \tag{14}$$

Let $I = \langle F\rangle$ be the ideal in $R[\mathbf{x}]$ generated by F.

Definition 3.9. A set F satisfying (14) is a *Gröbner basis* of the ideal $I = \langle F\rangle$ if for all nonzero $p \in I$, the initial term $\operatorname{in}(p)$ belongs to the monomial ideal generated by the set of initial terms $\operatorname{in}(F) = \{\operatorname{in}(f) : f \in F\}$.

In general, we do not have $\operatorname{in}(pq) = \operatorname{in}(p)\operatorname{in}(q)$, but this does hold when $\operatorname{in}(p) = ax^e$, $\operatorname{in}(q) = bx^h$ with $\operatorname{ord}(ab) = \operatorname{ord}(a) + \operatorname{ord}(b)$. Thus condition (14) implies $\operatorname{in}(pf) = \operatorname{in}(p)\operatorname{in}(f)$, and in particular

$$\operatorname{in}(ax^e f) = ax^e \operatorname{in}(f) \qquad \text{for } f \in F. \tag{15}$$

Without this condition, even a one-element set F could fail to be a Gröbner basis.

If F is a Gröbner basis, the standard monomials for the monomial ideal $\langle \operatorname{in}(F)\rangle$ are R-linearly independent modulo I, since every nonzero element of I has an initial term belonging to $\langle \mathrm{in}(F)\rangle$. There is a reformulation of the Gröbner basis property in terms of a suitably defined notion of F-reducibility.

Definition 3.10. A polynomial $p \in R[\mathbf{x}]$ is *F-reducible* if $p = 0$ or if $\mathrm{in}(p) = ax^e$, $d = \mathrm{ord}(a)$ and for all $m \geq 0$ there exists an expression

$$p \equiv \sum b_i x^{h_i} f_i \qquad (\mathrm{mod}\ P^m R[\mathbf{x}]) \tag{16}$$

with $f_i \in F$ and $b_i x^{h_i} f_i \in R[\mathbf{x}]_{\leq(-d,e)}$.

An F-reducible polynomial belongs to $\bigcap_m (P^m R[\mathbf{x}]+I)$ but not necessarily to I.

Proposition 3.11. *A set F satisfying* (14) *is a Gröbner basis of $I = \langle F\rangle$ if and only if every element $p \in I$ is F-reducible.*

Proof. Suppose every $p \in I$ is F-reducible. Given $p \in I \setminus \{0\}$, we are to show $\mathrm{in}(p) \in \langle \mathrm{in}(F)\rangle$. Let $\mathrm{in}(p) = ax^e$, $d = \mathrm{ord}(a)$. If $x^e \notin \langle \mathrm{in}(F)\rangle$, then no summand in (16) has e as the exponent of its initial term, and hence every summand belongs to $R[\mathbf{x}]_{<(-d,e)}$. For $m > d$, this implies $p \in R[\mathbf{x}]_{<(-d,e)}$, a contradiction.

For the converse, fix an arbitrary m, and suppose $p \in I$ has no expression of the form (16) for this m. In particular, $p \notin P^m R[\mathbf{x}]$, so $\mathrm{in}(p) = ax^e$, with $\mathrm{ord}(a) < m$. Since $d = \mathrm{ord}(a)$ is bounded above for all such p, we may assume we have chosen p so that $(-d, e)$ is minimal. By hypothesis there is some $f \in F$ such that $\mathrm{in}(f)$ divides x^e, say $x^e = x^h \,\mathrm{in}(f)$. Then $q = p - ax^h f$ has an expression of the form (16) for this m, by the minimality assumption. But then so does p. $\square$

Remark 3.12. Suppose $R[\mathbf{x}]$ is given a multigrading $\deg\colon \mathbb{N}^n \to A$, and F consists of homogeneous polynomials. Then Proposition 3.11 holds in each degree separately: if every nonzero $p \in I_a$ has $\mathrm{in}(p) \in \langle \mathrm{in}(F)\rangle$, then every $p \in I_a$ is F-reducible.

To each $f, g \in F$, there is an associated binomial syzygy $x^u \,\mathrm{in}(f) = x^v \,\mathrm{in}(g) = \mathrm{lcm}(\mathrm{in}\, f, \mathrm{in}\, g)$. We define the corresponding *S-polynomial* as usual to be

$$S(f, g) = x^u f - x^v g.$$

Now we have a version of the Buchberger criterion for F to be a Gröbner basis.

Proposition 3.13. *Let B be a set of pairs $(f, g) \in F \times F$ whose associated binomial syzygies generate the syzygy module of the initial terms* $\mathrm{in}(f)$*, $f \in F$. If $S(f, g)$ is F-reducible for all $(f, g) \in B$, then F is a Gröbner basis.*

Proof. Fix $m \geq 0$. We will show that every $p \in I + P^m R[\mathbf{x}]$ has an expression of the form (16) satisfying the conditions in Definition 3.10 for this m. We can assume $p \notin P^m R[\mathbf{x}]$, so $\mathrm{in}(p) = ax^e$ with $d = \mathrm{ord}(a) < m$. Certainly p has *some* expression of the form (16), perhaps not satisfying $b_i x^{h_i} f_i \in R[\mathbf{x}]_{\leq(-d,e)}$. Set $x^{e_i} = \mathrm{in}(f_i)$ and let $(-d', e')$ be the maximum of $(-\mathrm{ord}(b_i), h_i + e_i)$ over all terms in our expression for p. Since $p \notin R[\mathbf{x}]_{<(-d,e)}$, we must have $(-d', e') \geq (-d, e)$. In particular, d' is bounded, so we can assume our chosen expression for p minimizes $(-d', e')$. We are to show that $(-d', e') = (-d, e)$.

Suppose to the contrary that $(-d', e') > (-d, e)$. Then we have $p \in R[\mathbf{x}]_{<(-d',e')}$, and every summand in (16) for which $(-\mathrm{ord}(b_i), h_i + e_i) \neq (-d', e')$ is also in $R[\mathbf{x}]_{<(-d',e')}$. Let J be the set of indices j for which $(-\mathrm{ord}(b_i), h_i + e_i) = (-d', e')$. The partial sum over these indices in (16) must be in $R[\mathbf{x}]_{<(-d',e')}$, so $\mathrm{ord}(\sum_J b_i) > d'$. The Buchberger criterion implies that for all indices $j, k \in J$, $x^{h_j} f_j - x^{h_k} f_k$ is a sum of monomial multiples $x^u S(f, g)$ of F-reducible S-polynomials, all satisfying

$x^u \operatorname{lcm}(\operatorname{in} f, \operatorname{in} g) = x^{e'}$, and hence $x^u S(f,g) \in R[\mathbf{x}]_{\leq(0,e')}$. Their $x^{e'}$ terms cancel, so in fact they belong to $R[\mathbf{x}]_{<(0,e')}$. Being F-reducible, each $x^u S(f,g)$ has an expression of the form (16) with every term belonging to $R[\mathbf{x}]_{<(0,e')}$, and hence so does $x^{h_j} f_j - x^{h_k} f_k$. Renaming indices so that $1 \in J$, we have

$$\sum_J b_i x^{h_i} f_i = (\textstyle\sum_J b_i) x^{h_1} f_1 + \displaystyle\sum_J b_i (x^{h_i} f_i - x^{h_1} f_1).$$

The first term on the right belongs to $R[\mathbf{x}]_{<(-d',e')}$, since $h_1 + e_1 = e'$ and $\operatorname{ord}(\sum_J b_i) > d'$. In the second term we can replace replace $b_i(x^{h_i} f_i - x^{h_1} f_1)$ with an expression of the form (16) with all terms in $R[\mathbf{x}]_{<(-d',e')}$. Adding the remaining terms of our original expression for p, we get a new expression with every term in $R[\mathbf{x}]_{<(-d',e')}$. This contradicts the assumption that $(-d', e')$ was minimal. $\square$

In order to apply the above results, we unfortunately need the hypothesis (13), which may fail in a non-Noetherian ring. We can still manage to avoid Noetherian hypotheses in Theorem 3.6 by the device of reduction to the ground ring $\mathbb{Z}$. For this we need one last lemma, and then we will be ready to prove our theorem.

Lemma 3.14. *Take S and $h\colon A \to \mathbb{N}$ as in Theorem 1.1. Then $H^h_{S_D} \cong (\operatorname{Spec} k) \times_{\mathbb{Z}} H^h_{\mathbb{Z}[\mathbf{x}]_D}$, for any subset D of the degrees. In particular, $H^h_S \cong (\operatorname{Spec} k) \times_{\mathbb{Z}} H^h_{\mathbb{Z}[\mathbf{x}]}$.*

Proof. For simplicity, we only consider the case $D = A$, $S_D = S$. The proof in the general case is virtually identical. Let $\hat{R}$ denote R viewed as a $\mathbb{Z}$-algebra. Then

$$H^h_S(R) = H^h_{\mathbb{Z}[\mathbf{x}]}(\hat{R}), \tag{17}$$

since we have the same set of ideals in $R[\mathbf{x}] = R \otimes_k S = \hat{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{x}]$ on each side. For any $X \in \underline{\mathrm{Sch}/\mathbb{Z}}$ and $R \in \underline{k\text{-Alg}}$, the k-morphisms $\operatorname{Spec} R \to (\operatorname{Spec} k) \times_{\mathbb{Z}} X$ are in natural bijection with the $\mathbb{Z}$-morphisms $\operatorname{Spec} R \to X$. In other words, we have a natural isomorphism $\underline{(\operatorname{Spec} k) \times X}(R) \cong \underline{X}(\hat{R})$. Together with (17), this shows that the k-schemes H^h_S and $(\operatorname{Spec} k) \times_{\mathbb{Z}} H^h_{\mathbb{Z}[\mathbf{x}]}$ have isomorphic scheme functors. $\square$

Proof of Theorem 3.6. By Lemma 3.14, it is enough to prove the theorem in the case $k = \mathbb{Z}$. Since D is supportive, the natural map $H^h_S \to H^h_{S_D}$ is a closed embedding. It suffices to verify that it is an isomorphism locally on $H^h_{S_D}$. Specifically, let $U = \operatorname{Spec} R \subseteq H^h_{S_D}$ be an affine open subset. The closed embedding $U \cap H^h_S \hookrightarrow U$ is given by a ring homomorphism $R \to R/I$, and we are to show that $I = 0$. Localizing at $P \in \operatorname{Spec} R$, it suffices to show that $I_P = 0$. The composite morphism

$$\operatorname{Spec} R_P \to \operatorname{Spec} R \to H^h_{S_D}$$

is an element of $H^h_{S_D}(R_P)$. We will show that it belongs to the image of the map $H^h_S(R_P) \hookrightarrow H^h_{S_D}(R_P)$. This implies that the morphism $\operatorname{Spec} R_P \to \operatorname{Spec} R$ factors through $\operatorname{Spec} R/I$, that is, the localization homomorphism $R \to R_P$ factors through R/I and therefore through $(R/I)_P$. This yields a left inverse $(R/I)_P \to R_P$ to the projection $R_P \to (R/I)_P$, so $I_P = 0$. Note that since we are assuming $k = \mathbb{Z}$, and we have already shown that $H^h_{S_D}$ is quasiprojective over k, the local ring R_P is Noetherian, and hence satisfies (13).

We will show that the inclusion $H^h_S(R) \hookrightarrow H^h_{S_D}(R)$ is surjective whenever R is a local ring satisfying (13). Let $L_D \subseteq R[\mathbf{x}]_D$ be an element of $H^h_{S_D}(R)$ and let $L' \subseteq R[\mathbf{x}]$ be the ideal generated by L_D. By Remark 2.1, we have $L'_D = L_D$. In the proof of Theorem 1.1 we saw that the conditions of Theorem 2.3 hold. We conclude as in the proof of Theorem 2.3 that $R[\mathbf{x}]_a/L'_a$ is a finitely-generated R-module for all $a \in A$. Let $K = R/P$ denote the residue field of R. Then KL' is the ideal in $K[\mathbf{x}]$ generated by $K\otimes L_D$. Fix a term order on $\mathbb{N}^n$ and let J be the monomial ideal generated by the initial terms $\{\mathrm{in}(p) : p \in K\otimes L_D\}$. For $a \in D$, $\dim K[\mathbf{x}]_a/(K\otimes L_a) = h(a)$. Hence J has Hilbert function agreeing with h on D, and by conditions (g) and (h) in the definition of "very supportive," J has Hilbert function h.

The standard monomials for J span $K[\mathbf{x}]/KL'$. By Nakayama's Lemma, applied to each $R[\mathbf{x}]_a/L'_a$ separately, it follows that these standard monomials generate $R[\mathbf{x}]/L'$ as an R-module. What remains to be shown is that they generate $R[\mathbf{x}]/L'$ freely. Then L' is the required preimage of L_D in $H^h_S(R)$. For each generator x^u of the monomial ideal J, there is an element of KL' with initial term x^u. Let $f \in L'$ be a representative of this element modulo $PR[\mathbf{x}]$. The coefficient of x^u in f is a unit in R, so we can assume it is 1. Then $\mathrm{in}(f) = x^u$. Let F be the set of polynomials f obtained this way.

For $a \in D$, $R[\mathbf{x}]_a/L'_a$ is free with basis the standard monomials of degree a. Given any monomial $x^u \in R[\mathbf{x}]_a$, its unique expansion modulo L'_a by standard monomials belongs to $R[\mathbf{x}]_{\leq(0,u)}$. To see this, observe that the expansion in $K[\mathbf{x}]$ of x^u modulo KL' contains only terms x^v with $v \leq u$. It follows that the expansion of bx^u belongs to $R[\mathbf{x}]_{\leq(-\operatorname{ord}(b),u)}$. Consider a nonzero element $p \in L'_a$, with $\mathrm{in}(p) = bx^e$. Replacing all remaining terms of p with their standard expansions, we get a polynomial $q \equiv p \pmod{L'}$. At worst, this can change the coefficient of x^e by an element of $P^{\operatorname{ord}(b)+1}$, so $\mathrm{in}(q) = b'x^e$ for some b'. All remaining terms of q are standard, and $q \in L' \setminus \{0\}$, so we must have $x^e \in J = \langle \mathrm{in}(F)\rangle$. By Remark 3.12, we deduce that every $p \in L'_a$ is F-reducible. In particular, $S(f,g)$ is F-reducible whenever the generators $\mathrm{in}(f)$ and $\mathrm{in}(g)$ of J participate in one of the syzygies referred to in condition (s) for the very supportive set D. This shows that F is a Gröbner basis for $I = \langle F\rangle$.

Now, $I \subseteq L'$, and both $R[\mathbf{x}]_D/I_D$ and $R[\mathbf{x}]_D/L'_D$ are free with basis the standard monomials in degrees D, so $I_D = L'_D$. Both I and L' are generated in degrees D, so $I = L'$. Hence the standard monomials are R-linearly independent modulo L'. $\square$

When the grading is positive and the Hilbert scheme is projective, the preceding results lead to an explicit description of the multigraded Hilbert scheme H^h_S by equations in Plücker coordinates, although the number of variables and equations involved may be extremely large. We write $a \leq b$ for degrees $a, b \in A$ if $b - a \in A_+$. Since our grading is positive, this is a partial ordering on the degrees. For any finite set of degrees $D \subseteq A$, the Hilbert functor $H^h_{S_D}$ is defined as a subfunctor of the Grassmann functor $G^h_{S_D}$ by the conditions on $L \in H^h_{S_D}(R)$:

$$\text{for all } a < b \in D \text{ and all } x^u \text{ with } \deg(u) = b - a\text{: } x^u L_a \subseteq L_b. \tag{18}$$

For a positive grading, there are finitely many monomials in each degree. Each member of the above finite system of inclusions translates into well-known quadratic equations in terms of Plücker coordinates on $G^{h(a)}_{S_a} \times G^{h(b)}_{S_b}$. Together these equations describe the Hilbert scheme $H^h_{S_D}$ as a closed subscheme of $G^h_{S_D}$. We call (18) the *natural quadratic equations*.

Corollary 3.15. *If the grading is positive and $D \subseteq A$ is very supportive for $h\colon A \to \mathbb{N}$ then the Hilbert scheme H^h_S is defined by the natural quadratic equations* (18).

Let $D \subseteq E$ be two finite sets of degrees, where D is supportive and E is very supportive. Then our problem is to write down equations for the image of the closed embedding of $H_S^h \simeq H_{S_E}^h$ into $H_{S_D}^h$ given by Corollary 3.4. Each degree $e \in E \backslash D$ contributes to these equations, which we have already described in the discussion following the proof of Theorem 2.3 as the Fitting ideal for a certain matrix Γ. In the positively graded case, this matrix is finite and we can describe it explicitly. The columns of Γ correspond to the monomials of degree e. For each degree $d \in D$, $d < e$, and each set B consisting of $h(d)+1$ monomials of degree d, there is an element $\sum_{b \in B} \gamma_{B \backslash \{b\}} \cdot b$ of L_d, where $\gamma_{B\backslash\{b\}}$ denotes the Plücker coordinate on $G_{S_d}^{h(d)}$ indexed by the set of $h(d)$ monomials $B \setminus \{b\}$. Equation (10) in Example 2.13 illustrates this. Multiply each such generator of L_d by a monomial x^u of degree $e-d$ to get a homogeneous polynomial of degree e in $\mathbf{x}$ with coefficients that are Plücker coordinates. The vector of coefficients gives a row of Γ, which is the matrix of all rows obtained in this way. Setting $r = \operatorname{rk} S_e = \binom{n+e-1}{e}$, the minors

$$I_{r-h(e)+1}(\Gamma) \tag{19}$$

are the *natural determinantal equations* contributed by the degree e.

Theorem 3.16. *If $D \subseteq A$ is supportive for $h\colon A \to \mathbb{N}$, then the Hilbert scheme H_S^h is defined by the natural quadratic equations* (18) *and the natural determinantal equations* (19)*, where e runs over $E \setminus D$, for a very supportive superset E of D.*

4. The Grothendieck Hilbert scheme

In this section we relate our construction to Grothendieck's classical Hilbert scheme. Expressing the latter as a special case of the multigraded Hilbert scheme, our natural quadratic equations will become Gotzmann's equations [16], while the natural determinantal equations become those of Iarrobino and Kleiman [20]. Dave Bayer in his thesis [2, §VI.1] proposed a more compact system of determinantal equations, each having degree n in Plücker coordinates, and he conjectured that they also define Grothendieck's Hilbert scheme. Here we prove Bayer's conjecture.

The *Grothendieck Hilbert scheme* Hilb_{n-1}^g represents the functor of flat families $X \subseteq \mathbb{P}^{n-1}(R)$, $R \in \underline{k\text{-Alg}}$, with a specified Hilbert polynomial g. The homogeneous coordinate ring of $\mathbb{P}^{n-1}(R)$ is $R[\mathbf{x}] = R[x_1, \ldots, x_n]$, and the ideal of X is a saturated homogeneous ideal $L \subseteq R[\mathbf{x}]$ such that in sufficiently large degrees, $R[\mathbf{x}]/L$ is locally free with Hilbert function g. Let $d_0 = d_0(g,n)$ denote the *Gotzmann number* [20, Definition C.12]. Gotzmann [16] proved: (1) every saturated ideal with Hilbert polynomial g has Hilbert function g in degrees $d \geq d_0$, and (2) every ideal with Hilbert function g in degrees $d \geq d_0$ coincides in these degrees with its saturation.

Lemma 4.1. *Grothendieck's Hilbert scheme Hilb_{n-1}^g is isomorphic to the multigraded Hilbert scheme H_S^h, where $S = k[\mathbf{x}]$ with the standard $\mathbb{Z}$-grading, with Hilbert function h defined by $h(d) = g(d)$ for $d \geq d_0$, $h(d) = \binom{n+d-1}{d}$ for $d < d_0$.*

Proof. The ideals described by the functor H_S^h are the truncations to degrees $d \geq d_0$ of the ideals described by the Grothendieck functor Hilb_{n-1}^g. A natural bijection between the two is given by truncation in one direction and saturation in the other. Hence both schemes represent the same functor. $\square$

The Gotzmann number d_0 equals the maximum of the Castelnuovo-Mumford regularity of any saturated monomial ideal I with Hilbert polynomial g [20, Proposition C.24]. The set of such ideals is finite by Lemma 3.1. For a monomial ideal, the regularity of I is a purely combinatorial invariant, equal to the maximum over all i and all minimal i-th syzygies of $d-i$, where d is the degree of the syzygy. The regularity will not exceed d_0 if I is replaced by its truncation to degrees $\geq d_0$. It follows that for every monomial ideal I generated in degree d_0 and with Hilbert polynomial g, the Hilbert function of I coincides with g in degrees $\geq d_0$, and I has a linear free resolution. In particular the minimal S-pairs of I have degree d_0+1. These considerations show that Gotzmann's Regularity Theorem and Persistence Theorem can be rephrased in the language of the previous section as follows:

Proposition 4.2. *Let g be a Hilbert polynomial, defining d_0 and h as above. Then $D=\{d_0\}$ is supportive and $E=\{d_0,d_0+1\}$ is very supportive for $H_S^h=\mathrm{Hilb}^g_{n-1}$.*

We can now write equations for the Grothendieck Hilbert scheme in two possible ways. The set $E=\{d_0,d_0+1\}$ gives an embedding into a product of Grassmannians

$$H_S^h = H_{S_E}^h \hookrightarrow G_{S_{d_0}}^{h(d_0)} \times G_{S_{d_0+1}}^{h(d_0+1)}. \tag{20}$$

This is the embedding described by Gotzmann in [16, Bemerkung (3.2)]; see also [20, Theorem C.29]. It is defined scheme-theoretically by the *natural quadratic equations* given in (18). We illustrate these equations with a simple example.

Example 4.3. Take $S=k[x,y,z]$ with Hilbert function $h(0)=1$ and $h(d)=2$ for $d\geq 1$. Our Hilbert scheme H_S^h coincides with the Grothendieck Hilbert scheme of two points in the projective plane $\mathbb{P}^2$. The Gotzmann number is $d_0=2$. The pair $E=\{2,3\}$ is very supportive and gives the embedding (20) into the product of Grassmannians $G_6^2\times G_{10}^2$. The Plücker coordinates for the Grassmannian G_6^2 (resp. G_{10}^2) are ordered pairs of quadratic (resp. cubic) monomials in x,y,z. These define the Plücker embeddings $G_6^2\hookrightarrow\mathbb{P}^{14}$ and $G_{10}^2\hookrightarrow\mathbb{P}^{44}$. The Hilbert scheme H_S^h is the closed subscheme of $G_6^2\times G_{10}^2$ defined by 600 bilinear equations as in (18). There are 180 two-term relations such as

$$[xy^2,xyz]\cdot[yz,xy]+[xy^2,y^2z]\cdot[xy,xz]=0,$$

and 420 three-term relations such as

$$[x^2z,xy^2]\cdot[xz,yz]+[x^2z,xyz]\cdot[yz,xy]+[x^2z,y^2z]\cdot[xy,xz]=0.$$

The validity of these equations is easily checked for subschemes of $\mathbb{P}^2$ consisting of two distinct reduced points $(x_1:y_1:z_1)$ and $(x_2:y_2:z_2)$. Just replace each bracket by the corresponding 2×2 determinant, as in $[x^2z,xy^2]\mapsto x_1^2z_1x_2y_2^2-x_2^2z_2x_1y_1^2$. □

In the remainder of this section we will study not the Gotzmann embedding (20) but the other (more efficient) embedding given by Proposition 4.2. Namely, the supportive singleton $D=\{d_0\}$ defines the embedding into a single Grassmannian

$$H_S^h \hookrightarrow H_{S_D}^h = G_{S_{d_0}}^{h(d_0)}. \tag{21}$$

This is the embedding described in Bayer's thesis [2, §VI.1] and in [20, Prop. C.28]. It follows from Theorem 3.16 that the Hilbert scheme is defined as a closed subscheme of the Grassmannian by the *natural determinantal equations* (19). Iarrobino and Kleiman proved this in the present case in [20, Proposition C.30], so we refer to the equations (19) for the Grothendieck Hilbert scheme as the *Iarrobino–Kleiman* *equations.* Note that the Iarrobino–Kleiman equations for the embedding (21) are homogeneous polynomials of degree $\binom{n+d_0}{d_0+1}-h(d_0+1)+1$ in the Plücker coordinates.

We now present a third system of homogeneous equations for the Grothendieck Hilbert scheme, which Bayer proved define it set-theoretically. Like the Iarrobino–Kleiman equations, Bayer's equations are homogeneous equations in the Plücker coordinates on the single Grassmannian $G^{h(d_0)}_{S_{d_0}}$. However, Bayer's equations are more compact: their degree always equals n, the number of variables, independently of g, h and d_0 [2, p. 144]. Bayer conjectured that his equations define the correct scheme structure [2, p. 134]. We will prove this conjecture.

Theorem 4.4. *Grothendieck's Hilbert scheme parametrizing subschemes of* $\mathbb{P}^{n-1}$ *with any fixed Hilbert polynomial is defined in the Grassmannian embedding* (21) *by Bayer's equations, which are homogeneous of degree* n *in the Plücker coordinates.*

Although the Bayer equations define the same subscheme of the Grassmannian as the Iarrobino–Kleiman equations, they do not generate the same homogeneous ideal. This phenomenon is hardly surprising, since any projective scheme can be defined by many different homogeneous ideals. Even the Bayer equations are often not the simplest ones: the common saturation of both ideals frequently contains equations of degree less than n. This happens for Example 4.3, which will be reexamined below, and it happens for [20, Example C.31], where the Iarrobino–Kleiman equations have degree 25 while the Bayer equations have degree 3.

The best way to introduce Bayer's equations and relate them to the Iarrobino–Kleiman equations is with the help of Stiefel coordinates on the Grassmannian. For the remainder of this section we use the following abbreviations:

$$d = d_0;\ h = h(d);\ h' = h(d+1);\ r = \binom{n+d-1}{d};\ r' = \binom{n+d}{d+1}.$$

As before, G^h_r denotes the Grassmann scheme parametrizing quotients of rank h of S_d. We digress briefly to review the relationship between local coordinates, Stiefel coordinates, and Plücker coordinates.

Recall from Section 2 that the Grassmannian G^h_r is covered by affine charts $G^h_{r\backslash B}$, whose functor $\underline{G^h_{r\backslash B}}(R)$ describes free quotients R^r/L with basis B, where B is an h-element subset of some fixed basis X of k^r. Here we identify k^r with S_d, and X is the set of all monomials of degree d. At a point $L \in \underline{G^h_{r\backslash B}}(R)$, the local (affine) coordinates γ^x_b take unique values in R such that

$$x \ \equiv \ \sum_{b\in B} \gamma^x_b \cdot b \pmod{L} \quad \text{for all } x \in X \setminus B.$$

Consider the $(r-h)\times r$ matrix Γ with columns indexed by the elements of X, constructed as follows. Index the rows of Γ by the elements of $X\backslash B$. In the column indexed by $b \in B$, put the coordinates $-\gamma^x_b$ of L, for $x \in X\backslash B$. In the complementary square submatrix with columns indexed by $X \setminus B$, put an $(r-h)\times(r-h)$ identity matrix. Then the rows of Γ span the submodule $L \subseteq R^r$.

More invariantly, if we insist that R^r/L be free, not just locally free, but do not choose the basis B in advance, we can always realize L as the row space of some $(r-h)\times r$ matrix Ω, at least one of whose maximal minors is invertible in R. The entries of Ω are the *Stiefel coordinates* of L. They are well-defined up to change of basis in L, that is, up to multiplication of Ω on the left by matrices in $GL_{r-h}(R)$. A little more generally, we can regard any $(r-h)\times r$ matrix Ω whose maximal minors generate the unit ideal in R as the matrix of Stiefel coordinates for its row-space $L \subseteq R^r$, as R^r/L will then be locally free of rank h.

When R^r/L is locally free of rank h, its top exterior power $\wedge^h(R^r/L)$ is a rank-1 locally free quotient of $\wedge^h(R^r)$, corresponding to an element of $\underline{\mathbb{P}^{\binom{r}{h}-1}}(R)$. The Plücker embedding $G_r^h \hookrightarrow \mathbb{P}^{\binom{r}{h}-1}$ is given in scheme functor terms by the natural transformation sending L to the kernel of $\wedge^h(R^r) \to \wedge^h(R^r/L)$. The homogeneous coordinates on $\mathbb{P}^{\binom{r}{h}-1}$ are *Plücker coordinates.* They are indexed by exterior products of the elements of X and denoted

$$[x_1, \dots, x_h]. \tag{22}$$

In terms of Stiefel coordinates, we can identify $[x_1, \dots, x_h]$ with the maximal minor of Ω whose columns are indexed by $x_1, \dots, x_h$, up to a sign depending on the order of the monomials in the bracket.

Some caution is due when using Stiefel and Plücker coordinates in the scheme functor setting: for an arbitrary $L \in \underline{G^h_{r\setminus B}}(R)$, the matrix Ω of Stiefel coordinates need not exist, as L may not be generated by $r-h$ elements. This difficulty arises even for homogeneous coordinates on projective space (the special case $h = r-1$). Nevertheless, for the purpose of determining the ideal of a closed subscheme $H \subseteq G_r^h$, it suffices to consider the restriction of the scheme functors involved to *local* rings R. Stiefel and Plücker coordinates then make sense for any R-valued point L. Throughout the rest of this section, R will always denote a local ring.

The basic observation leading to the Bayer equations is that when a subscheme of G_r^h is defined by nice enough equations in Stiefel coordinates, they can sometimes be converted to equations of much lower degree in Plücker coordinates. For instance, the submodule $L_2 \subseteq R[x,y,z]_2$ in Example 4.3 is spanned by four quadrics,

$$\begin{gathered}
a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2,\\
b_1x^2 + b_2xy + b_3xz + b_4y^2 + b_5yz + b_6z^2,\\
c_1x^2 + c_2xy + c_3xz + c_4y^2 + c_5yz + c_6z^2,\\
d_1x^2 + d_2xy + d_3xz + d_4y^2 + d_5yz + d_6z^2.
\end{gathered}$$

The matrix Ω is the 4×6 matrix of coefficients, which are the Stiefel coordinates. The fifteen 4×4 minors of Ω are identified with the fifteen Plücker coordinates on G_6^2. Some care is required with the signs; for instance,

$$[yz, z^2] = \det\begin{pmatrix} a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4\\ c_1 & c_2 & c_3 & c_4\\ d_1 & d_2 & d_3 & d_4\end{pmatrix}, \quad [y^2, z^2] = -\det\begin{pmatrix} a_1 & a_2 & a_3 & a_5\\ b_1 & b_2 & b_3 & b_5\\ c_1 & c_2 & c_3 & c_5\\ d_1 & d_2 & d_3 & d_5\end{pmatrix}. \tag{23}$$

Returning to the general discussion, observe that the image $\mathbf{x}\cdot L_d$ of $L_d \subseteq R[\mathbf{x}]_d$ is spanned by $x_1L_d, x_2L_d, \dots, x_nL_d$ inside $R[\mathbf{x}]_{d+1} = R^{r'}$. We may represent $\mathbf{x}\cdot L_d$ by a matrix $\widehat{\Omega}$ with $n(r-h)$ rows and r' columns. The matrix $\widehat{\Omega}$ contains n copies of the matrix Ω and is otherwise zero. The columns of $\widehat{\Omega}$ are labeled by the monomials in $R[\mathbf{x}]_{d+1}$ in lexicographic order. In our running example, we have

$$\widehat{\Omega} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & 0 & 0 & 0 & 0 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & a_3 & 0 & a_4 & a_5 & a_6 & 0 \\ 0 & b_1 & 0 & b_2 & b_3 & 0 & b_4 & b_5 & b_6 & 0 \\ 0 & c_1 & 0 & c_2 & c_3 & 0 & c_4 & c_5 & c_6 & 0 \\ 0 & d_1 & 0 & d_2 & d_3 & 0 & d_4 & d_5 & d_6 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 & 0 & a_4 & a_5 & a_6 \\ 0 & 0 & b_1 & 0 & b_2 & b_3 & 0 & b_4 & b_5 & b_6 \\ 0 & 0 & c_1 & 0 & c_2 & c_3 & 0 & c_4 & c_5 & c_6 \\ 0 & 0 & d_1 & 0 & d_2 & d_3 & 0 & d_4 & d_5 & d_6 \end{pmatrix} \tag{24}$$

with columns labeled $x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3$.

The choice of d as the Gotzmann number ensures that $\widehat{\Omega}$ has an invertible minor of order $r' - h'$ whenever Ω has an invertible maximal minor. The natural determinantal equations (19) defining H_S^h as a closed subscheme of G_r^h are the minors of order $r' - h' + 1$ of the matrix $\widehat{\Omega}$. They are the Iarrobino–Kleiman equations expressed in Stiefel coordinates, and are exactly the equations which ensure that $R[\mathbf{x}]_{d+1}/\mathbf{x} \cdot L_d$ is locally free of rank h'. In our example, we are looking at $2,200 = \binom{12}{9} \times \binom{10}{9}$ polynomials of degree 9. We wish to replace these by a smaller number of cubic polynomials in the 4×4 minors of the matrix Ω.

In general, our problem is this: Let J be the Fitting ideal generated by the minors of order $r' - h' + 1$ of the matrix $\widehat{\Omega}$. This is an ideal in the polynomial ring $k[\Omega]$ generated by entries of Ω, that is, by the Stiefel coordinates, viewed as indeterminates. We seek an ideal J' generated by polynomials of degree n in the Plücker coordinates, or maximal minors of Ω, such that J and J' define systems of equations which have the same solutions Ω over any local ring R.

We now give Bayer's construction and show that it solves the above problem. Let $\Omega \otimes S_1$ be the matrix representing the submodule $S_1 \otimes_k L_d$ of the tensor product $S_1 \otimes_k R[\mathbf{x}]_d$. Thus $\Omega \otimes S_1$ is a matrix with $n(r-h)$ rows and nr columns. The row labels of $\Omega \otimes S_1$ coincide with the row labels of $\widehat{\Omega}$. We form their concatenation

$$\big(\widehat{\Omega} \mid \Omega \otimes S_1\big).$$

Bayer's equations are certain maximal minors of this matrix. Each column of $\widehat{\Omega}$ is a sum of columns of $\Omega \otimes S_1$, and these sums involve distinct leading columns. Therefore we may—for the sake of efficiency—pick a submatrix $(\Omega \otimes S_1)_{\mathrm{red}}$ of $(\Omega \otimes S_1)$ of format $n(r-h) \times (nr - r')$ such that the maximal minors of

$$\big(\widehat{\Omega} \mid (\Omega \otimes S_1)_{\mathrm{red}}\big) \tag{25}$$

have the same $\mathbb{Z}$-linear span as those of $\big(\widehat{\Omega} \mid \Omega \otimes S_1\big)$. Note that the matrix (25) has $n(r-h)$ rows and nr columns. Each maximal minor of (25) is a homogeneous polynomial of degree $n(r-h)$ in $k[\Omega]$, and, by Laplace expansion, it can be written as a homogeneous polynomial of degree n in the Plücker coordinates (22). The *Bayer equations* are those maximal minors of (25) gotten by taking any set of $r' - h' + 1$ columns of $\widehat{\Omega}$ and any set of $n(h-r) - r' + h' - 1$ columns of $(\Omega \otimes S_1)_{\mathrm{red}}$.

In our running example, we take the reduced tensor product matrix as follows:

$$(\Omega \otimes S_1)_{\mathrm{red}} = \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & 0 & a_6 & 0 & 0 \\ b_2 & b_3 & b_4 & b_5 & 0 & b_6 & 0 & 0 \\ c_2 & c_3 & c_4 & c_5 & 0 & c_6 & 0 & 0 \\ d_2 & d_3 & d_4 & d_5 & 0 & d_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & a_5 & a_6 \\ 0 & 0 & 0 & 0 & b_3 & 0 & b_5 & b_6 \\ 0 & 0 & 0 & 0 & c_3 & 0 & c_5 & c_6 \\ 0 & 0 & 0 & 0 & d_3 & 0 & d_5 & d_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

The matrix (25) has format 12×18, and each of its maximal minors is a homogeneous polynomial of degree 3 in the 15 Plücker coordinates $[x^2, xy]$, $[x^2, xz]$, ..., $[yz, z^2]$. There are $560 = \binom{10}{9} \times \binom{8}{3}$ Bayer equations, gotten by taking any 9 columns from (24) and any 3 columns from (26).

Proof of Theorem 4.4: Clearly, every Bayer equation belongs to the Fitting ideal $I_{r'-h'+1}(\widehat{\Omega})$. We must show that (for R local) the vanishing of the Bayer minors implies that $I_{r'-h'+1}(\widehat{\Omega}) = 0$. This would be obvious if the matrix $(\Omega \otimes S_1)_{\mathrm{red}}$ contained an identity matrix as a maximal square submatrix. But the Bayer ideal is unchanged if we use $\Omega \otimes S_1$ in place of $(\Omega \otimes S_1)_{\mathrm{red}}$, and it is $GL_{n(r-h)}(R)$-invariant. Hence it suffices that $\Omega \otimes S_1$ have some maximal minor invertible in R. This follows from the fact that Ω has such a minor. $\square$

While the Bayer equations do define the correct scheme structure on the Hilbert scheme, they are far from minimal with this property. For instance, in our example, there are 560 Bayer cubics which, together with the 15 quadratic Plücker relations for G_6^2, define the Hilbert scheme H_S^h as a closed subscheme of dimension 4 and degree 21 in $\mathbb{P}^{14}$. However, H_S^h is irreducible and its prime ideal is the ideal of algebraic relations on the 2×2 minors of the matrix

$$\begin{pmatrix} x_1^2 & x_1y_1 & x_1z_1 & y_1^2 & y_1z_1 & z_1^2 \\ x_2^2 & x_2y_2 & x_2z_2 & y_2^2 & y_2z_2 & z_2^2 \end{pmatrix}$$

This prime ideal is minimally generated by 45 quadrics.

5. Toric Hilbert schemes and their Chow morphisms

In this section we examine Hilbert schemes which arise in toric geometry. Our goals are to describe equations for the *toric Hilbert scheme*, and to define the *toric Chow morphism*. In the process we answer some questions left open in earlier investigations by Peeva, Stillman, and the second author. We fix an A-grading of the polynomial ring $S = k[\mathbf{x}]$ and consider the constant Hilbert function

$$h(a) = 1 \quad \text{for all } a \in A_+. \tag{27}$$

The multigraded Hilbert scheme H_S^1 defined by this Hilbert function is called the *toric Hilbert scheme*. Its functor $H_S^1(R)$ parametrizes ideals $I \subseteq R[\mathbf{x}]$ such that $(R[\mathbf{x}]/I)_a$ is a rank-one locally free R-module for all $a \in A_+$.

Assuming that the elements $a_i = \deg(x_i)$ generate A, we have a presentation

$$0 \to M \to \mathbb{Z}^n \to A \to 0, \tag{28}$$

which induces a surjective homomorphism of group algebras over k,

$$k[\mathbf{x}, \mathbf{x}^{-1}] = k[\mathbb{Z}^n] \to k[A], \tag{29}$$

and a corresponding closed embedding of $G = \operatorname{Spec} k[A]$ as an algebraic subgroup of the torus $\mathbb{T}^n = \operatorname{Spec} k[\mathbf{x}, \mathbf{x}^{-1}]$. The torus $\mathbb{T}^n$ acts naturally on $\mathbb{A}^n$ as the group of invertible diagonal matrices, and so its subgroup G also acts on $\mathbb{A}^n$. An ideal $I \subseteq R \otimes S$ is homogeneous for our grading if and only if the closed subscheme defined by I in $\mathbb{A}^n_R = \mathbb{A}^n \times_k \operatorname{Spec} R$ is invariant under the action of $G_R = G \times_k \operatorname{Spec} R$.

Remark 5.1. If A is a finite abelian group then the toric Hilbert scheme H^1_S coincides with Hilbert scheme $\operatorname{Hilb}^G(\mathbb{A}^n)$ of regular G-orbits studied by Nakamura [25]. If the group A is free abelian and the grading is positive, then H^1_S coincides with the toric Hilbert scheme studied by Peeva and Stillman [26, 27]. □

There is a distinguished point on the toric Hilbert scheme H^1_S, namely, the ideal

$$I_M \quad = \quad \langle\, x^u - x^v \;:\; u, v \in \mathbb{N}^n,\, \deg(u) = \deg(v)\,\rangle.$$

Note that $\deg(u) = \deg(v)$ means that $u - v$ lies in the sublattice M in (28). Restricting the ring map in (29) to $S = k[\mathbb{N}^n]$, its kernel is I_M. Hence, identifying $\mathbb{A}^n = \operatorname{Spec} S$ with the space of $n \times n$ diagonal matrices, and $\mathbb{T}^n$ with its open subset of invertible matrices, I_M is the ideal of the closure in $\mathbb{A}^n$ of the subgroup $G \subseteq \mathbb{T}^n$.

A nonzero binomial $x^u - x^v \in I_M$ is called *Graver* if there is no other binomial $x^{u'} - x^{v'}$ in I_M such that $x^{u'}$ divides x^u and $x^{v'}$ divides x^v. The degree $a = \deg(u) = \deg(v)$ of a Graver binomial is a *Graver degree*. The set of Graver binomials is finite. The finite set of all Graver degrees can be computed using Algorithm 7.2 in [31].

Proposition 5.2. *The set of Graver degrees is supportive, and the natural determinantal equations* (19) *for this set coincide with the determinantal equations for the toric Hilbert scheme given by Peeva and Stillman in* [27, Definition 3.3].

Proof. The Graver degrees are supportive by [26, Proposition 5.1]; the proof given there for positive gradings works for nonpositive gradings as well. The Fitting equations in [27, Definition 3.3] are precisely our Fitting equations (19), in the special case when the Hilbert function h is the constant 1. □

In the positively graded case, a doubly-exponential bound was given in [30, Proposition 5.1] for a set of degrees which is very supportive for the toric Hilbert scheme. Peeva [26, Corollary 5.3] improved the bound to single-exponential and gave an explicit description of a very supportive set E in [26, Theorem 5.2].

Proposition 5.3. *Let* $\deg\colon \mathbb{N}^n \to A$ *be a positive grading and* $E \subseteq A_+$ *a finite, very supportive set of degrees for the toric Hilbert scheme. Then the natural quadratic equations* (18) *are precisely the quadratic binomials given in* [30, Equation (5.3)].

Proof. We only need to make explicit the equations expressing condition (18):

$$\text{for all } a < b \in E \text{ and all } x^u \text{ with } \deg(u) = b - a\text{: } x^u L_a \subseteq L_b.$$

Let R be a local ring. For the constant Hilbert function $h = 1$, the ambient graded Grassmann scheme is a product of projective spaces, one for each degree:

$$G^1_{S_E} = \prod_{a \in E} \mathbb{P}(S_a).$$

For each monomial x^u in S_a there is a coordinate z^a_u on the projective space $\mathbb{P}(S_a)$, such that the z^a_u for $\deg(u) = a$ are the Plücker coordinates on $\underline{\mathbb{P}(S_a)}(R)$. The submodule L_a of $R \otimes S_a$ represented by a point (z^a_u) in $\underline{\mathbb{P}(S_a)}(R)$ is generated by

$$z^a_u \cdot x^v - z^a_v \cdot x^u \quad \text{for all } \deg(u) = \deg(v) = a.$$

For R local, condition (18) is thus equivalent to the system of binomial equations

$$(30) \quad z^a_u \cdot z^b_{v+w} = z^a_v \cdot z^b_{u+w} \quad \text{for } a, b \in E,\ \deg(u) = \deg(v) = a,\ \deg(w) = b - a,$$

which are precisely the equations in [30, (5.3)]. A closed subscheme cut out by equations in any scheme is determined by the evaluation of its subfunctor on local rings R. Hence $H^1_{S_E}$, the closed subscheme of $\prod_{a\in E} \mathbb{P}(S_a)$ whose subfunctor is characterized by condition (18), is cut out by equations (30). □

In view of our general theory, Propositions 5.2 and 5.3 show that Peeva and Stillman's determinantal equations in [27] define the same scheme structure as the binomial quadrics in [30, Equation (5.3)]. This question had been left open in [27].

It is instructive to examine Theorem 3.16 in the case of the toric Hilbert scheme H^1_S. Suppose the grading of S is positive, let $D \subseteq A_+$ be the set of Graver degrees and E the very supportive set in [26, Theorem 5.2]. Then the toric Hilbert scheme H^1_S is defined by the quadratic binomials (30) on D together with the Fitting equations (19), where e runs over E. From this it follows that the infinite sum in [27, Definition 3.3] over all degrees $e \in A_+$ can be replaced by the finite sum over $e \in E$. The resulting finite set of determinantal equations still defines H^1_S.

We now turn to the construction of the toric Chow morphism. It was conjectured in [30, Problem 6.4] that there exists a natural morphism from the toric Hilbert scheme to a certain inverse limit of toric GIT quotients, and this is what we shall now construct. In [30] it was assumed that the action of G on $\mathbb{A}^n$ is the linearization of an action on projective space, or, equivalently, that $(1, 1, \ldots, 1) \in M^\perp$, but this hypothesis is not needed. Our notation concerning toric varieties follows [8] and [15]. For compatibility with the standard toric variety setting, one should take $k = \mathbb{C}$, although in fact the construction below makes sense for any k.

In (29) we identified $\operatorname{Spec} k[\mathbf{x}, \mathbf{x}^{-1}]$ with the torus $\mathbb{T}^n$ of diagonal matrices acting on $\mathbb{A}^n$. Each Laurent monomial x^u is thus a regular function on $\mathbb{T}^n$, and this identifies the lattice $\mathbb{Z}^n$ in (28) with the lattice of linear characters of $\mathbb{T}^n$. The sublattice M consists of those characters which are trivial on the subgroup $G \subseteq \mathbb{T}^n$, so the grading group A consists of linear characters of the group G. In particular, S_0 is the ring of G-invariants in S, so $\operatorname{Spec} S_0$ is the affine quotient $\mathbb{A}^n/G$.

The GIT quotient $\mathbb{A}^n/_a\, G$ in the sense of Mumford [23], for a G-linearization of the trivial line bundle on $\mathbb{A}^n$ using a G-character $a \in A$, is given as

$$\mathbb{A}^n/_a\, G \quad = \quad \operatorname{Proj} \bigoplus_{r=0}^{\infty} S_{rka},$$

where ka is multiple of a for which the ring on the right-hand side is generated in degrees $r = 0, 1$. These GIT quotients, including the affine quotient $\mathbb{A}^n/G = \mathbb{A}^n/_0\, G$, are toric varieties, whose description in terms of fans we pause to review. Let $N = \operatorname{Hom}(M, \mathbb{Z})$ be the lattice dual to M, so (28) yields an exact sequence

$$(31) \qquad 0 \leftarrow \operatorname{Ext}^1(A, \mathbb{Z}) \leftarrow N \leftarrow \mathbb{Z}^n \leftarrow \operatorname{Hom}(A, \mathbb{Z}) \leftarrow 0.$$

The map $N \leftarrow \mathbb{Z}^n$ supplies a tuple $\Pi = (v_1, \ldots, v_n)$ of distinguished vectors in N. The lattice $\mathbb{Z}^n$ in (31) is dual to the one in (28). The latter can be identified with the set $\mathbb{Z}^\Pi$ of functions $\Pi \to \mathbb{Z}$. Then the degree map $\mathbb{Z}^\Pi \to A$ induces a map

$$\phi : \mathbb{R}^\Pi \longrightarrow A \otimes_\mathbb{Z} \mathbb{R}. \tag{32}$$

To each degree $a \in A$ is associated a regular subdivision Σ_a, as in [5]. Namely, Σ_a is the fan of cones in N spanned by subsets of the form $\sigma = f^{-1}(0) \subseteq \Pi$, for functions $f \in (\mathbb{R}_{\geq 0})^\Pi \cap \phi^{-1}(a)$. The fans $\Sigma = \Sigma_a$ arising this way are called *compatible fans*.

With this notation, we have $X_{\Sigma_a} = \mathbb{A}^n /_a G$, and we can identify S with the Cox homogeneous coordinate ring [8] of X_{Σ_a}. More precisely, for each $\sigma \in \Sigma_a$ as above, let x_σ be the product of the variables x_i with $v_i \in \Pi \setminus \sigma$. Then the semistable locus $U_a = \mathbb{A}^n_{ss(a)}$ is the union of the principal open affines U_{x_σ}, and the ring of G-invariants $S[x_\sigma^{-1}]_0$ is the semigroup algebra of $\sigma^\vee \cap M$, so $U_a/G = X_{\Sigma_a}$. It is natural at this point to make the following definition.

Definition 5.4. A degree $a \in A_+$ is *integral* if the inclusion of convex polyhedra

$$\operatorname{conv}\big(\mathbb{N}^n \cap \deg^{-1}(a)\big) \subseteq (\mathbb{R}_{\geq 0})^n \cap \phi^{-1}(a) \tag{33}$$

is an equality, where ϕ is the map in (32).

For every degree $a \in A_+$, the monomials x_σ for $\sigma \in \Sigma_a$ are just the square-free parts of all monomials x^u whose degree is a positive multiple of a. Integrality of a means that every x_σ already occurs as the square-free part of a monomial of degree a. Equivalently, a is integral if the semistable locus U_a is equal to the complement of the closed subset $V(J_a)$, where J_a is the ideal in S generated by S_a. We remark that every degree $a \in A_+$ has some positive multiple ka which is integral, and that the fan Σ_a is the normal fan to the polyhedron on the right-hand side in (33).

The set of all compatible fans, ordered by refinement, can be identified with the poset of chambers in the Gale dual of Π, by [5, Theorem 2.4]. If Σ' is a refinement of Σ then the construction in [15, §1.4] gives a projective morphism of toric varieties

$$X_{\Sigma'} \to X_\Sigma. \tag{34}$$

The varieties X_Σ for all compatible fans Σ form an inverse system. Their inverse limit in the category of k-schemes is called the *toric Chow quotient* and denoted by

$$\mathbb{A}^n /_C G = \varprojlim \{X_\Sigma : \Sigma \text{ a compatible fan in } N\}. \tag{35}$$

Example 5.5. The fan Σ_0 giving the affine quotient $\mathbb{A}^n/G$ is just the cone $\mathbb{R}_{\geq 0}\Pi$. If the grading is positive then Π positively spans N and all compatible fans are complete. In this case, the affine quotient $\mathbb{A}^n/G = X_{\Sigma_0}$ is a point, and the toric GIT quotients are projective. At the opposite extreme, if A is finite, then Π is a basis of $N \otimes \mathbb{R}$, the only toric GIT quotient is the affine one, and our Chow morphism below coincides with Nakamura's Chow morphism $\mathrm{Hilb}^G(\mathbb{A}^n) \to \mathbb{A}^n/G$. □

The following theorem provides the solution to Problem 6.4 in [30].

Theorem 5.6. *There is a canonical morphism*

$$H^1_{S_{\mathrm{int}(A)}} \to \mathbb{A}^n /_C G \tag{36}$$

from the toric Hilbert scheme restricted to the set of integral degrees to the toric Chow quotient $\mathbb{A}^n/_C G$*, which induces an isomorphism of the underlying reduced schemes. In particular, composing* (36) *with the degree restriction morphism, we* *obtain a canonical* Chow morphism *from the toric Hilbert scheme to the toric Chow quotient*

$$H^1_S \to \mathbb{A}^n/_C\, G. \tag{37}$$

For the proof of Theorem 5.6 we need to recall some facts about the Proj of a graded ring. Let T be an $\mathbb{N}$-graded k-algebra, generated over T_0 by finitely many elements of T_1, so $T = T_0[\mathbf{y}]/I$ for generators $y_0, \ldots, y_m$ of T_1 and a homogeneous ideal $I \subseteq T_0[\mathbf{y}]$. Then $\operatorname{Proj} T$ is the closed subscheme $V(I)$ of $\mathbb{P}^m \times \operatorname{Spec} T_0$. Its functor $\underline{\operatorname{Proj} T}$ is defined as follows: $\underline{\operatorname{Proj} T}(R)$ is the set of homogeneous ideals $L \subseteq R \otimes T$ such that L_d is a locally free R-module of rank 1 for all d. In symbols, $\operatorname{Proj} T = H^1_T$. Setting $T^{(d)} = \oplus_r T_{rd}$, the degree restriction morphism $\operatorname{Proj} T = H^1_T \to H^1_{T^{(d)}} = \operatorname{Proj} T^{(d)}$ is an isomorphism. More generally, suppose that T_1 does not necessarily generate T, but that the following weaker conditions hold:

(i) T is finite over the T_0-subalgebra T' generated by T_1; or equivalently,
(ii) there exists d_0 such that $T_{d+1} = T_1 T_d$ for all $d \geq d_0$.

In this case it remains true that the degree restriction morphism $H^1_T \to \operatorname{Proj} T^{(d)}$ is an isomorphism for $d \geq d_0$. There is a canonical morphism $\operatorname{Proj} T^{(d)} \to \operatorname{Proj} T'$, which is finite, but need not be an isomorphism.

One sees easily that $a \in A_+$ is integral if and only if the ring

$$S^{(a)} \underset{\text{def}}{=} \bigoplus_{r=0}^{\infty} S_{ra} \tag{38}$$

satisfies conditions (i) and (ii) above. In particular, for $a \in \operatorname{int}(A)$ we have

$$H^1_{S^{(a)}} = \mathbb{A}^n/_a\, G = X_{\Sigma_a},$$

so restriction of degrees yields a canonical morphism $H^1_S \to X_{\Sigma_a}$.

Lemma 5.7. *The morphism $H^1_S \to X_\Sigma$ above depends only on the compatible fan Σ and not on the choice of degree a with $\Sigma_a = \Sigma$. Moreover these morphisms commute with the morphisms $X_{\Sigma'} \to X_\Sigma$ given by refinement of fans as in (34).*

Proof. We will describe the morphisms $H^1_S \to X_\Sigma$ geometrically. Because H^1_S represents the Hilbert functor, it comes with a universal family $F \subseteq H^1_S \times_k \mathbb{A}^n$ (where $\mathbb{A}^n = \operatorname{Spec} S$), and the group $G = \operatorname{Spec} k[A]$ acts on F so that the projections

$$H^1_S \leftarrow F \to \mathbb{A}^n$$

are equivariant. To the character a of G corresponds a GIT quotient $F/_a\, G = F_{ss}/G$. The a-semistable locus F_{ss} is the preimage of $\mathbb{A}^n_{ss}$, and we have induced morphisms

$$H^1_S \leftarrow F/_a\, G = F_{ss}/G \to \mathbb{A}^n_{ss}/G = X_\Sigma. \tag{39}$$

Now, $F/_a\, G$, considered as a scheme over H^1_S, is just $\operatorname{Proj} \bigoplus_{r=0}^{\infty} (\mathcal{S}/\mathcal{L})_{ra}$, where $\mathcal{S}$ is the sheaf of A-graded algebras $\mathcal{O}_{H^1_S} \otimes_k S$, and $\mathcal{L}$ is the universal ideal sheaf. But $(\mathcal{S}/\mathcal{L})_{ra}$ is locally free of rank 1 over $\mathcal{O}_{H^1_S}$ for all r, which implies $F/_a\, G \cong H^1_S$. Hence in (39) there is a composite morphism $H^1_S \to X_\Sigma$, which is easily seen to coincide with the degree restriction morphism. The morphism in (39) depends on a only through $\mathbb{A}^n_{ss}$, which in turn depends only on Σ_a. Furthermore, if Σ' refines Σ, with corresponding semi-stable loci $\mathbb{A}^n_{ss'}$ and $\mathbb{A}^n_{ss}$, then $\mathbb{A}^n_{ss'} \subseteq \mathbb{A}^n_{ss}$ and the morphism $X_{\Sigma'} \to X_\Sigma$ is just the morphism $\mathbb{A}^n_{ss'}/G \to \mathbb{A}^n_{ss}/G$ induced by the inclusion. This makes the lemma obvious. $\square$

Proof of Theorem 5.6. We already saw that restriction of degrees yields morphisms

$$H^1_S \to H^1_{S_{\mathrm{int}(A)}} \to \mathbb{A}^n/_C\, G.$$

For every R, the natural map

$$H^1_{S_{\mathrm{int}(A)}}(R) \to \underline{(\mathbb{A}^n/_C\, G)}(R) \tag{40}$$

is injective, since to give the restriction of $L \in H^1_S(R)$ to integral degrees is the same as to give its image in $H^1_{S^{(a)}}(R) = \underline{X_{\Sigma_a}}(R)$ for each integral a. In general, a morphism of schemes $X \to Y$ induces an isomorphism of reduced schemes $X_{\mathrm{red}} \to Y_{\mathrm{red}}$ if and only if the natural map $\underline{X}(R) \to \underline{Y}(R)$ is an isomorphism for all reduced rings R. Hence it remains to show that the map (40) is surjective when R is reduced.

Suppose for each $a \in \mathrm{int}(A)$ we are given $L_a \subseteq R \otimes S_a$ with $(R \otimes S_a)/L_a$ locally free of rank 1. We may assume the L_a are consistent in the following sense: first, $\bigoplus_r L_{ra}$ is an ideal in $R \otimes S^{(a)}$ for each a, so it represents a point of $\underline{X_{\Sigma_a}}(R)$, and second, these points are compatible with the morphisms $X_{\Sigma_a} \to X_{\Sigma_b}$ whenever Σ_a refines Σ_b. Then we are to show:

$$\text{for all } a, b \in \mathrm{int}(A) \text{ and all } x^u \text{ with } \deg(u) = b - a\text{: } x^u L_a \subseteq L_b, \tag{41}$$

so L represents a point of $H^1_{S_{\mathrm{int}(A)}}(R)$.

Let $D_\Sigma = \{a \in \mathrm{int}(A) : \Sigma_a = \Sigma\}$. There is a subdivision of $A \otimes_{\mathbb{Z}} \mathbb{R}$ into rational convex polyhedral cones C_Σ such that D_Σ is the preimage of the relative interior of C_Σ via the canonical map $\psi\colon\ \mathrm{int}(A) \to A \otimes_{\mathbb{Z}} \mathbb{R}$. The fan Σ' refines Σ if and only if C_Σ is a face of $C_{\Sigma'}$. For a given Σ, the ring $S_{\psi^{-1}(C_\Sigma)}$ can be identified with the multi-homogeneous coordinate ring of X_Σ with respect to the various line bundles $\mathcal{O}(a)$ pulled back via refinement homomorphisms $X_\Sigma \to X_{\Sigma_a}$. Our consistency hypotheses amount to saying that (41) holds whenever $\psi(a)$ and $\psi(b)$ both lie in a common cone C_Σ.

Consider now the general case of (41). For any $d > 0$, the points $\psi(da + k\deg(u))$, for $0 \le k \le d$, lie along the line segment l from $\psi(da)$ to $\psi(db)$. For a suitably chosen large d, every cone C_Σ that meets l will contain at least one point $\psi(da + k_i \deg(u))$ with $da + k_i \deg(u)$ an integral degree. Then (41) holds for each consecutive pair of integral degrees $da + k_i \deg(u)$, $da + k_{i+1} \deg(u)$ in this arithmetic progression. Hence it holds for da, db and the monomial x^{du}, so $f \in L_a$ implies $x^{du} f^d \in L_{db}$. But $R \otimes S^{(b)}/(\bigoplus_r L_{rb})$ is an $\mathbb{N}$-graded R-algebra, locally (on $\operatorname{Spec} R$) isomorphic to a polynomial ring in one variable over R. Hence it is a reduced ring, that is, $\bigoplus_r L_{rb}$ is a radical ideal in $R \otimes S^{(b)}$, and therefore $x^u f \in L_b$. $\square$

Because the natural map in (40) is always injective, our proof of Theorem 5.6 gives a bit more. Namely, if the toric Chow quotient $\mathbb{A}^n/_C\, G$ happens to be reduced, then its isomorphism with $(H^1_S)_{\mathrm{red}} \subseteq H^1_S$ provides a right inverse to the map in (40), showing that the latter map is bijective. Hence we have the following improvement.

Corollary 5.8. *If the toric Chow quotient $\mathbb{A}^n/_C\, G$ is reduced, then the morphism $H^1_{S_{\mathrm{int}(A)}} \to \mathbb{A}^n/_C\, G$ in Theorem 5.6 is an isomorphism.*

The toric Chow morphism is generally neither injective nor surjective, see e.g. [31, Theorem 10.13]. However, there is an important special case, namely, the supernormal case, when it is bijective, and in fact induces an isomorphism of the underlying reduced schemes. A degree $a \in A$ is called *prime* if there is no variable x_i which divides every monomial of degree a. If every degree is integral, $\mathrm{int}(A) = A$, then the sublattice M of $\mathbb{Z}^n$ is said to be *unimodular*. If every prime degree is integral, the A-grading is said to be *supernormal*. This terminology is consistent with [18].

Corollary 5.9. *If the A-grading of S is supernormal then the toric Chow morphism (37) induces an isomorphism of reduced schemes $(H^1_S)_{\mathrm{red}} \to (\mathbb{A}^n/_C\, G)_{\mathrm{red}}$. If in addition, $\mathbb{A}^n/_C\, G$ is reduced, then the toric Chow morphism is an isomorphism.*

Proof. We need only show that if D is the set of prime degrees, then the degree restriction morphism $H^1_S \to H^1_{S_D}$ is an isomorphism. Fix a degree $a \in A\backslash D$ and let x^u be the greatest common divisor of all monomials of degree a. Then $a' = a - \deg(u)$ is a prime degree, and multiplication by x^u defines an R-module isomorphism between $R \otimes S_{a'}$ and $R \otimes S_a$, for every k-algebra R. For any element $I \in H^1_S(R)$, we have $I_a = x^u \cdot I_{a'}$ and hence the restriction map $H^1_S(R) \to H^1_{S_D}(R)$ is injective. But it is also surjective because every element J in $H^1_{S_D}(R)$ lifts to an element of $H^1_S(R)$ by setting $J_a = x^u \cdot J_{a'}$ for degrees $a \in A\backslash D$. □

Example 5.10. Give $S = \mathbb{C}[x_1, x_2, x_3, x_4]$ the $\mathbb{Z}^2$-grading $\deg(x_1) = \deg(x_2) = (1,0)$, $\deg(x_3) = (0,1)$, $\deg(x_4) = (2,1)$. The configuration $\Pi \subseteq N$ can be represented by the four vectors $\{(-1,1), (1,1), (0,1), (0,-1)\}$ in $\mathbb{Z}^2$. There is a unique finest compatible fan Σ, and $\mathbb{A}^2/_C\, G = X_\Sigma$ is a smooth projective toric surface. The prime degrees are $(\alpha, \beta) \in \mathbb{N}^2$ with $\alpha \geq 2\beta$. The integral degrees are those for which $\alpha \geq 2\beta$ or α is even. Hence this example is supernormal, but not unimodular. By Corollary 5.9, its toric Hilbert scheme is isomorphic to X_Σ. □

We remark that Corollary 5.9 can be used to give an alternative and more conceptual proof of Theorem 1.2 in [18], which states that in the supernormal case, the $\mathbb{T}^n$-fixed points on the toric Hilbert scheme ("virtual initial ideals") are in natural bijection with the $\mathbb{T}^n$-fixed points on the toric Chow quotient ("virtual chambers").

We next discuss the distinguished component of the toric Hilbert scheme H^1_S. Here we will fix $k = \mathbb{C}$ and describe the distinguished component in set-theoretic terms. The distinguished point $I_M \in H^1_S(\mathbb{C})$ is the ideal of the closure of G. More generally, the ideal I of the closure in $\mathbb{A}^n$ of any G-coset $G \cdot \tau \subseteq \mathbb{T}^n$ is a point of $H^1_S(\mathbb{C})$. In fact, $\mathbb{T}^n$ acts on H^1_S and the $\mathbb{T}^n$-orbit of I_M consists of all such ideals I. Moreover, I is the only point of $H^1_S(\mathbb{C})$ for which $V(I)$ contains $G \cdot \tau$. Now $\mathbb{T}^n/G$ is the open torus orbit in each of the toric varieties X_Σ, and so is naturally embedded as an open set U in the inverse limit $\mathbb{A}^n/_C\, G = \varprojlim_\Sigma X_\Sigma$. The observations above show that the toric Chow morphism restricts to a bijection from the preimage of U in H^1_S to U. Hence the preimage of U is an irreducible open subset of H^1_S, and its closure is an irreducible component of H^1_S, which we call the *coherent component*.

The closure of U in $\mathbb{A}^n/_C\, G$ is the toric variety X_Δ defined by the common refinement Δ of all compatible fans Σ_a. Thus we have a canonical morphism from the coherent component of H^1_S to X_Δ. For a supernormal grading, it is an isomorphism.

Example 5.11. There is a nice connection between toric Hilbert schemes and recent work by Brion [6]. Consider the (Grothendieck) Hilbert scheme associated with the diagonal embedding $X \to X \times X$ of a projective variety X. Brion shows that, if X is a homogeneous space, then the diagonal is a smooth point on a unique component of the Hilbert scheme, the associated Chow morphism is an isomorphism, and all degenerations of the diagonal in $X \times X$ are reduced and Cohen-Macaulay. Our theory implies corresponding results when X is a unimodular toric variety [3, §6]. Indeed, by Proposition 6.1 in [3], the Hilbert scheme of X in $X \times X$ is the toric Hilbert scheme for $S = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ graded via a unimodular Lawrence lattice $M \subseteq \mathbb{Z}^{2n}$. The Lawrence ideal I_M is the distinguished point X; the unique component it lies on is the distinguished component defined above, and its toric degenerations are reduced and Cohen-Macaulay by [3, Thm. 1.2 (b)]. $\square$

Variants of our results on H^1_S apply to the *m-th toric Hilbert scheme* H^m_S defined by the Hilbert function

$$h(a) = \min(m, \mathrm{rk}_k(S_a)).$$

The m-th toric Hilbert scheme is a common generalization of two objects of recent interest in combinatorial algebraic geometry, namely, the Hilbert scheme of m points in affine n-space (the case $A = \{0\}$) and the toric Hilbert scheme (the case $m = 1$). Again, H^m_S has a distinguished *coherent component*, and it admits a natural morphism to a certain Chow variety.

We briefly outline how to extend our constructions to the case $m > 1$. The appropriate Chow quotient is the inverse limit of symmetric powers $\varprojlim_\Sigma \mathrm{Sym}^m X_\Sigma$. There is a Chow morphism, which factors as

$$H^m_S \to \prod_\Sigma \mathrm{Hilb}^m(X_\Sigma) \to \varprojlim_\Sigma \mathrm{Sym}^m X_\Sigma. \tag{42}$$

Here $\mathrm{Hilb}^m(X_\Sigma)$ is the Hilbert scheme of m points in X_Σ. For sufficiently general a such that $\Sigma = \Sigma_a$, we have $\mathrm{Hilb}^m(X_\Sigma) = H^m_{S(a)}$, and the Chow morphism is degree restriction composed with the usual Chow morphisms $\mathrm{Hilb}^m(X_\Sigma) \to \mathrm{Sym}^m X_\Sigma$. The analog of Theorem 5.6 no longer holds, however.

The coherent component is the unique component of H^m_S which maps birationally on the open subset $\mathrm{Sym}^m(\mathbb{T}^n/G)$ of the Chow quotient. A typical point in the coherent component is the ideal of the union of m closures in $\mathbb{A}^n$ of G-cosets in $\mathbb{T}^n$. One difference from the $m = 1$ case is that for $m > 1$, not every ideal of a union of G-coset closures belongs to H^m_S. If the m cosets are specially chosen, it can happen that the monomials of some degree a define fewer than $h(a)$ linearly independent functions on them, a possibility that does not occur when $m = 1$.

6. Constructing other Hilbert schemes

The theory of graded k-modules with operators developed in Section 2 allows us to construct many interesting Hilbert schemes in addition to the multigraded Hilbert schemes of Theorem 1.1. This final section lists some noteworthy examples.

6.1. **Partial multigraded Hilbert schemes.** Take S an A-graded polynomial ring, as before, $T = S_D$ for any subset $D \subset A$, and any function $h : D \mapsto \mathbb{N}$. The operators are multiplications by monomials. Using Remark 2.1, we see that the "monomial ideals" in the system (T, F) are just the restrictions to S_D of monomial ideals in S. This given, it is easy to see that the analog of Proposition 3.2 holds, and the proof of Theorem 1.1 goes through to show that $H^h_T = H^h_{S_D}$ is represented by a quasiprojective scheme, projective if the grading is positive. (We actually used this result already in Section 5 when we implicitly assumed that the integral degrees Hilbert functor $H^1_{S_{\mathrm{int}(A)}}$ is represented by a scheme.) Whenever $D \subset E \subset A$, we have a degree restriction morphism $H^h_{S_D} \to H^h_{S_E}$. Such Hilbert schemes occur naturally in parametrizing subschemes of a toric variety X. Here S is the homogeneous coordinate ring [8] of X, the grading group A is the Picard group of X and D is a suitable translate of the semigroup of ample divisors in A. This application is currently being studied by Maclagan and Smith [22].

6.2. **Quot schemes.** Take $T = S^r/M$ to be a finitely generated A-graded module over the A-graded polynomial ring S. The scheme representing H_T^h is a *Quot scheme*, used in algebraic geometry to parametrize vector bundles or sheaves on a given scheme. The arguments in Section 3 extend easily to show that the multigraded Quot scheme is always a quasiprojective scheme over k. The special case $A = 0$, $M = 0$ is already interesting: here H_T^h parametrizes artinian S-modules with r generators having length $m = h(0)$. For $n = 2$, this scheme is closely related to the space $\mathcal{M}(m, r)$ of Nakajima, which plays the lead character in his work on Hilbert schemes of points on surfaces; see [24, Chapter 2].

6.3. **Universal enveloping algebras.** Let $\mathfrak{g} = \bigoplus_a \mathfrak{g}_a$ be an A-graded Lie algebra over k, free and finitely generated as a k-module. Take $T = U(\mathfrak{g})$ to be its universal enveloping algebra, or, more generally, if $c \in \mathfrak{g}$ is a central element of degree 0, take T to be the reduced enveloping algebra $U_c(\mathfrak{g}) = U(\mathfrak{g})/I$, where I is the two-sided ideal $\langle c - 1\rangle$. Then T is an A-graded associative algebra. Taking our system of operators F to be generated by the left multiplications by elements of $\mathfrak{g}$, or the right multiplications, or both, we obtain Hilbert functors H_T^h parametrizing homogeneous left, right or two-sided ideals with Hilbert function h. Similarly, there are Quot functors H_T^h, when T is a finitely generated module over $U(\mathfrak{g})$ or $U_c(\mathfrak{g})$.

All these functors are represented by quasiprojective schemes over k. To see this, recall the *arithmetic filtration* of $U = U(\mathfrak{g})$ or $U_c(\mathfrak{g})$ given by $\alpha^i U = \sum_{j \le i} \mathfrak{g}^j$. The associated graded algebra $\mathrm{gr}_\alpha U = S$ is an A-graded commutative polynomial ring in variables $x_1, \ldots, x_n$ forming a homogeneous k-basis of $\mathfrak{g}$ (or of $\mathfrak{g}/kc$). If $I \subseteq U$ is a left, right or two-sided ideal, then $\mathrm{gr}_\alpha I$ is an ideal in S. When the ground ring is a field, we may define the *initial ideal* $\mathrm{in}(I) = \mathrm{in}(\mathrm{gr}_\alpha I)$. Fixing a Poincaré-Birkhoff-Witt basis in U, the basis elements corresponding to standard monomials for $\mathrm{in}(I)$ form a basis of U/I. Given these observations, it is easy to adapt the proof of Theorem 1.1 in Section 3 to show that in this more general setting, H_T^h is still a closed subscheme of the quasiprojective scheme $H_{T_D}^h$, where D is a finite supportive set for S and h. It can also be shown, using Gröbner basis theory for U, that H_T^h is isomorphic to $H_{T_D}^h$ if D is very supportive for S and h.

An interesting example is the *Weyl algebra* $W = k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$, which is the reduced enveloping algebra of a Heisenberg algebra $\mathfrak{g}$. Any A-grading of $k[\mathbf{x}]$ extends to a grading of W with $\deg(\partial_i) = -\deg(x_i)$. Then we have a Hilbert scheme H_W^h parametrizing homogeneous left ideals with Hilbert function h in the Weyl algebra. It would be interesting to relate these Hilbert schemes to the work of Berest and Wilson [4] in the case $n = 1$. We note that, unlike in the case of the polynomial ring $S = k[\mathbf{x}]$, the finest possible grading, $A = \mathbb{Z}^n$, $\deg(x_i) = -\deg(\partial_i) = e_i$, gives rise to highly non-trivial Hilbert schemes H_W^h. Namely, H_W^h parametrizes all *Frobenius ideals* with Hilbert function h. Frobenius ideals appear in Gröbner-based algorithms for solving systems of linear partial differential equations [29, §2.3].

6.4. **Quivers, posets and path algebras.** Here is a nice example where the set A of "degrees" is not a group. Fix a finite poset Q. We identify Q with its Hasse diagram and we interpret it as an acyclic quiver. Let $T = kQ$ denote its *path algebra*. This is the free k-module spanned by all directed paths in Q modulo the obvious concatenation relations. We take A to be the set of all intervals $[u, v]$ in the poset Q. Then the "graded component" $T_{[u,v]}$ is the k-vector space with basis consisting of all chains from u to v in the poset Q. Fixing a Hilbert function

$h : A \to \mathbb{N}$, we get the scheme H_T^h which parametrizes all homogeneous quotients of the path algebra kQ modulo a certain number of linearly independent relations for each interval $[u, v]$. The case when h attains only the values 0 and 1 deserves special attention. In this case the scheme H_T^h is binomial, and it parametrizes the *Schurian algebras* in the sense of [7]. When $h = 1$ is the constant one function then we get a non-commutative analogue to the toric Hilbert scheme. The distinguished point on this Hilbert scheme H_T^1 is the *incidence algebra* of the poset Q, and it would be interesting to study its deformations from this point of view. Question: What is the smallest poset Q for which the scheme H_T^1 has more than one component?

6.5. **Exterior Hilbert schemes.** The use of the exterior algebra as a tool for (computational) algebraic geometry has received considerable attention in recent years; see e.g. [12]. Let T denote the exterior algebra in n variables $x_1, \ldots, x_n$ over our base ring k, again with a grading by an abelian group A. As in the preceding example, T is of finite rank over k, so the Hilbert scheme H_T^h is a closed subscheme of a product of Grassmann schemes (as are the associated Quot schemes). The torus fixed points on such Hilbert schemes are precisely the simplicial complexes on $\{x_1, \ldots, x_n\}$, and it would be interesting to study the combinatorial notion of "shifting of simplicial complexes" in the framework of exterior Hilbert schemes.

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Department of Mathematics, University of California, Berkeley, CA 94720