§ 5.1 Recurrence Relations

2. Show that the sequence \( \{a_n\} \) is a solution of the recurrence relation \( a_n = -3a_{n-1} + 4a_{n-2} \) if

(a) \( a_n = 0 \).

\[ 0 = -3 \cdot 0 + 4 \cdot 0. \]

(b) \( a_n = 1 \).

\[ 1 = -3 \cdot 1 + 4 \cdot 1. \]

(c) \( a_n = (-4)^n \).

\[ (-4)^n = -4 \cdot (-4)^{n-1} = -3 \cdot (-4)^{n-1} + 4 \cdot (-4)^{n-2}. \]

(d) \( a_n = 2(-4)^n + 3 \).

\[ -3 \cdot [2(-4)^{n-1} + 3] + 4 \cdot [2(-4)^{n-2} + 3] = (-12 + 8) \cdot (-4)^{n-2} + 3 = 2(-4)^n + 3. \]

10. An employee joined a company in 1987 with a starting salary of $50,000. Every year this employee receives a raise of $1000 plus 5% of the salary of the previous year.

a) Set up a recurrence relation for the salary of this employee \( n \) years after 1987.

\[ a_0 = 50000, \ a_n = 1000 + 1.05a_{n-1} \text{ for } n \geq 1. \]
b) What is the salary of this employee in 1995.

Solution. It is \( a_8 = 34421.88108 \approx 34422 \) dollars.


c) Find an explicit formula for the salary of this employee \( n \) years after 1987.

Solution. Note that \( a_n - a_{n-1} = 1.05(a_{n-1} - a_{n-2}) \) for \( n \geq 2 \), we may let \( b_n = a_n - a_{n-1} \). Then \( b_n = 1.05b_{n-1} \) for \( n \geq 2 \) and \( b_1 = a_1 - a_0 = 1000 + 0.05a_0 = 3500 \). It’s easy to get that \( b_n = a_n - a_{n-1} = 3500 \cdot 1.05^{n-1} \) for \( n \geq 1 \). Observe that \( (a_n - a_{n-1}) + \cdots + (a_2 - a_1) + (a_1 - a_0) = a_n - a_0 \), so \( a_n = b_n + \cdots + b_1 + a_0 = 3500(1.05^{n-1} + \cdots + 1.05 + 1) + 1000 = 70000(1.05^n - 1) + 1000 \).

16. a) Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and \( n \) as their last term were \( n \) is a positive integer. That is, sequences \( a_1, a_2, \ldots, a_k \) where \( a_1 = 1, a_k = n \) and \( a_j < a_{j+1} \) for \( j = 1, 2, \ldots, k-1 \).

Solution. Let \( f(n) \) denote the number of strictly increasing sequences of positive integers that have 1 as their first term and \( n \) as their last term. Note that given a such sequence, the second last element of this sequence could be any integers in \( [1, n] \), so \( f(n) = f(n-1) + \cdots + f(2) + f(1) \) for \( n \geq 1 \). However, further observation of this recurrence relation gives \( f(n) = 2f(n-1) \) when \( n \geq 3 \), this is because all those sequences with the second last element less than \( n - 1 \) can be extended by adding \( n - 1 \) between the second last and the last elements, which then becomes one with the second last element being \( n - 1 \).

b) What are the initial conditions?

Solution. \( f(1) = f(2) = 1 \).

c) How many sequences of the type described in (a) are there when \( n \) is a positive integer with \( n \geq 2 \)?

Solution. Using the second recurrence relation in (a), it’s not hard to get that \( f(n) = 2^{n-2} \), for \( n \geq 2 \).

22. a) Find a recurrence relation for the number of ways to climb \( n \) stairs if the person climbing the stairs can take one, two, or three stairs at a time.

Solution. Let \( a_n \) be the number of ways to climb \( n \) stairs. How can such a climb end? There are \( a_{n-1} \) that end with a one-stair step, \( a_{n-2} \) that end with a two-stair jump, and \( a_{n-3} \) that end in a three-stair leap. Thus \( a_n = a_{n-1} + a_{n-2} + a_{n-3} \), and this is valid for all \( n \geq 3 \).

b) What are the initial conditions?

Solution. We need three: \( a_0 = 1, a_1 = 1, a_2 = 2 \).

c) How many ways can this person climb a flight of eight stairs?

Solution. Computing successive values of \( a_n \) using the recurrence relation, we get \( a_3 = 4, a_4 = 7, a_5 = 13, a_6 = 24, a_7 = 44, a_8 = 81 \).

24. a) Find a recurrence relation for the number of ternary strings (of length \( n \)) that contain two consecutive 0s.
Let \( f(n) \) denote the number of ternary strings that contain two consecutive 0s. Imagine we are constructing a ternary string of length \( n \). If the first digit is 1 or 2, then we need to get 00 from the rest \( n - 1 \) digits. Suppose the first digit is 0, look at the second digit, if it is 1 or 2, as before, we can only hope to get 00 from the rest \( n - 2 \) digits. But if both the first and the second digits are 0 then the rest \( n - 2 \) digits can be any combination, since we’ve got two consecutive 0s from the first two digits. Hence, the recurrence relation is
\[
f(n) = 2f(n - 1) + 2f(n - 2) + 3^{n-2}, \text{ for } n \geq 2.
\]

b) **What are the initial conditions.**

Solution. \( f(1) = 0, \ f(2) = 1. \)

C) **How many ternary strings of length six contain two consecutive 0s?**

Solution. \( f(3) = 4, \ f(4) = 16, \ f(5) = 48, \ f(6) = 144. \)

34. **Find a recurrence relation for the number of bit sequences of length \( n \) with an even number of 0s.**

Solution. Let \( a_n \) denote the number of bit sequences of length \( n \) with an even number of 0s, \( b_n \) denote the number of bit sequences of length \( n \) with an odd number of 0s. Clearly \( a_n + b_n = 2^n \). Now consider bit strings of length \( n \) that contain an even number of 0s. There are two cases. Case 1, the first bit is 1, there are \( a_{n-1} \) such strings. Case 2, the first bit is 0, since we need an odd number of 0s from the rest \( n - 1 \) bits, there are \( b_{n-1} = 2^{n-1} - a_{n-1} \) such strings. So \( a_n = a_{n-1} + (2^{n-1} - a_{n-1}) = 2^{n-1} \) for \( n \geq 1. \)

38. **Show that the Fibonacci numbers satisfy the recurrence relation \( f_n = 5f_{n-4} + 3f_{n-5} \) for \( n = 5, 6, 7, \ldots, \) together with the initial conditions \( f_0 = 0, \ f_1 = 1, \ f_2 = 1, \ f_3 = 2, \) and \( f_4 = 3. \) Use this recurrence relation to show that \( f_{5n} \) is divisible by 5, for \( n = 1, 2, 3, \ldots. \)**

Solution. We prove by induction. **Basis Step:** \( f_5 = f_3 + f_4 = 5, \) and \( 5f_1 + 3f_0 = 5. \) So it is true for \( n = 1. \)

**Inductive Step:** suppose it is true for \( k \leq n, \) then
\[
f_{n+1} = f_n + f_{n-1} = (5f_{n-5} + 3f_{n-4}) + (5f_{n-6} + 3f_{n-5}) = 5f_{n-5} + f_{n-6} + 3f_{n-4} + f_{n-5} = 5f_{n-4} + 3f_{n-3}
\]

To see that \( 5|f_{5n} \) for \( n = 1, 2, \ldots. \) We also prove by induction. \( f_5 = 5, \) so it’s true for \( n = 1. \) Suppose this is true for \( k = n, \) since \( f_{5(n+1)} = 5f_{5n-4} + 3f_{5n}, \) \( f_{5(n+1)} \) is divisible by 5. This finishes the proof.
§ 5.2 Solving Recurrence Relations

2. Determine which of the following are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) \( a_n = 3a_{n-2} \)

b) \( a_n = 3 \)

c) \( a_n = a_{n-1}^2 \)

d) \( a_n = a_{n-1} + 2a_{n-3} \)

e) \( a_n = a_{n-1}/n \)

f) \( a_n = a_{n-1} + a_{n-2} + n + 3 \)

g) \( a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7} \)

Solution. a) Linear and homogeneous with constant coefficients. It has degree 2.

b) Linear with constant coefficients, but not homogeneous.

c) Not linear.

d) Linear and homogeneous with constant coefficients. It has degree 3.

e) Linear and homogeneous, but without constant coefficients.

f) Linear and with constant coefficients, but not homogeneous.

g) Linear and homogeneous with constant coefficients. It has degree 7.

4. Solve the following recurrence relation together with the initial conditions given.

a) \( a_n = a_{n-1} + 6a_{n-2} \) for \( n \geq 2, a_0 = 3, a_1 = 6 \)

Solution. The characteristic equation is \( r^2 - r - 6 = 0 \), which has distinct roots \(-2, 3\). Using Theorem 1, we see that \( a_n = \alpha_1(-2)^n + \alpha_23^n \) for some \( \alpha_1, \alpha_2 \). Using our initial conditions, we find that we must have \( 3 = \alpha_1 + \alpha_2 \) and \( 6 = -2\alpha_1 + 3\alpha_2 \), so \( \alpha_1 = 3/5, \alpha_2 = 12/5 \), and \( a_n = (3(-2)^n+4\cdot3^{n+1})/5 \).

b) \( a_n = 7a_{n-1} - 10a_{n-2} \) for \( n \geq 2, a_0 = 2, a_1 = 1 \)

Solution. The characteristic equation is \( r^2 - 7r + 10 = 0 \), which has distinct roots \( 5, 2 \). Using Theorem 1, we see that \( a_n = \alpha_15^n + \alpha_22^n \) for some \( \alpha_1, \alpha_2 \). Using our initial conditions, we find that we must have \( 2 = \alpha_1 + \alpha_2 \) and \( 1 = 5\alpha_1 + 2\alpha_2 \), so \( \alpha_1 = -1, \alpha_2 = 3 \), and \( a_n = -5^n + 3 \cdot 2^n \).

c) \( a_n = 6a_{n-1} - 8a_{n-2} \) for \( n \geq 2, a_0 = 4, a_1 = 10 \)

Solution. The characteristic equation is \( r^2 - 6r + 8 = 0 \), which has distinct roots \( 4, 2 \). Using Theorem 1, we see that \( a_n = \alpha_14^n + \alpha_22^n \) for some \( \alpha_1, \alpha_2 \). Using our initial conditions, we find that we must have \( 4 = \alpha_1 + \alpha_2 \) and \( 10 = 4\alpha_1 + 2\alpha_2 \), so \( \alpha_1 = 1, \alpha_2 = 3 \), and \( a_n = 4^n + 3 \cdot 2^n \).

d) \( a_n = 2a_{n-1} - a_{n-2} \) for \( n \geq 2, a_0 = 4, a_1 = 1 \)

Solution. The characteristic equation is \( r^2 - 2r + 1 = 0 \), which has one multiple root \( 1 \). Using Theorem 2 we see that \( a_n = \alpha_1 + \alpha_2n \) for some \( \alpha_1, \alpha_2 \). Using our initial conditions, we find that we must have \( 4 = \alpha_1 + \alpha_2 \) and \( 1 = \alpha_1 + 2\alpha_2 \), so \( \alpha_1 = 4, \alpha_2 = -3 \), and \( a_n = 4 - 3n \).

e) \( a_n = a_{n-2} \) for \( n \geq 2, a_0 = 5, a_1 = -1 \)

Solution. The characteristic equation is \( r^2 - 1 = 0 \), which has distinct roots \( 1, -1 \). Using Theorem 1, we see that \( a_n = \alpha_1 + \alpha_2(-1)^n \) for some \( \alpha_1, \alpha_2 \). Using our initial conditions, we find that we must have \( 5 = \alpha_1 + \alpha_2 \) and \( -1 = 1 - \alpha_2 \), so \( \alpha_1 = 2, \alpha_2 = 3 \), and \( a_n = 2 + 3 \cdot (-1)^n \).

f) \( a_n = -6a_{n-1} - 9a_{n-2} \) for \( n \geq 2, a_0 = 3, a_1 = -3 \)

Solution. The characteristic equation is \( r^2 + 6r + 9 = 0 \), which has one multiple root \(-3\). Using Theorem 2 we see that \( a_n = (\alpha_1 + \alpha_2n)(-3)^n \) for some \( \alpha_1, \alpha_2 \). Using our initial conditions, we find that we must have \( 3 = \alpha_1 \) and \(-3 = -3(\alpha_1 + \alpha_2) \), so \( \alpha_1 = 3, \alpha_2 = -2 \), and \( a_n = (3 - 2n)(-3)^n \).
g) \[ a_{n+2} = -4a_{n+1} + 5a_n \text{ for } n \geq 0, a_0 = 2, a_1 = 8 \]

**Solution.** The characteristic equation is \( r^2 + 4r - 5 = 0 \), which has distinct roots \(-5, 1\). Using Theorem 1, we see that \( a_n = \alpha_1(-5)^n + \alpha_2(1)^n \) for some \( \alpha_1, \alpha_2 \). Using our initial conditions, we find that we must have \( 2 = \alpha_1 + \alpha_2 \) and \( 8 = -5\alpha_1 + \alpha_2 \), so \( \alpha_1 = -1, \alpha_2 = 3 \), and \( a_n = (-5)^n + 3 \).

8. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

a) Find a recurrence relation for \( L_n \), where \( L_n \) is the number of lobsters caught in year \( n \), under the assumption for this model.

**Solution.** \( L_n = (L_{n-1} + L_{n-2})/2 \) for \( n \geq 2 \).

b) Find \( L_n \) if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.

**Solution.** The characteristic equation is \( 2r^2 - r - 1 = 0 \), which has two distinct roots \(-\frac{1}{2}, 1\). Using Theorem 1 we see that \( a_n = \alpha_1 + \alpha_2(-\frac{1}{2})^n \) for some \( \alpha_1, \alpha_2 \). Using our initial conditions, we find that we must have \( 100000 = \alpha_1 - \frac{\alpha_2}{2} / 2 \) and \( 300000 = \alpha_1 + \frac{3\alpha_2}{2} / 2 \), so \( \alpha_1 = 700000/3 \), \( \alpha_2 = 800000/3 \), and \( L_n = (700000 - 800000 \cdot (-\frac{1}{2})^n)/3 \), for \( n \geq 2 \).

22. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots \(-1, -1, -2, 2, 5, 7\)?

**Solution.** By Theorem 4, the general form is

\[ a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)(-1)^n + (\alpha_{2,0} + \alpha_{2,1}n)2^n + (\alpha_{3,0} + \alpha_{3,1}n)5^n + \alpha_{4}7^n \]

28. a) Find all solutions of the recurrence relation \( a_n = 2a_{n-1} + 2n^2 \).

**Solution.** First, the associated recurrence relation is \( a_n = 2a_{n-1} \). Its general solution is \( a_n = a_02^n \) for \( n \geq 1 \). Theorem 6 tells us that a particular solution has the form \( b_2n^2 + b_1n + b_2 \). Inserting this into the recurrence relation gives \( b_0n^2 + b_1n + b_2 = 2(b_0(n-1)^2 + b_1(n-1) + b_2) + 2n^2 \). Simplifying, we see that \( (-b_0)n^2 + (4b_0 - b_1)n + (-2b_0 + 2b_1 - b_2) = 2n^2 \). So \( b_0 = -2, b_1 = -8 \) and \( b_2 = -12 \). Therefore the general solutions of the original recurrence relation is \( a_n = a_02^n - 2n^2 - 8 - 12 \).

b) Find the solution of the recurrence relation in part (a) with initial condition \( a_1 = 4 \).

**Solution.** If \( a_1 = 4 \), then \( a_0 = 1 \). So \( a_n = 2^n - 2n^2 - 8 - 12 \).

42. Show that if \( a_n = a_{n-1} + a_{n-2}, a_0 = s \) and \( a_1 = t \), where \( s \) and \( t \) are constants, then \( a_n = sf_{n-1} + tf_n \) for all positive integers \( n \).

**Solution.** We prove by induction. Basis Step: the Fibonacci numbers starts at 1, we may assume \( f_0 = 0 \) to keep \( f_2 = f_1 + f_0 \) valid. Then \( a_1 = t = s \cdot 0 + t \cdot 1 = sf_0 + tf_1 \). So it is true for \( n = 1 \).

Inductive Step: suppose it is true for \( k \leq n \), we proceed to show that \( a_{n+1} = sf_n + tf_{n+1} \). This is true since

\[
\begin{align*}
\text{LHS} &= a_n + a_{n-1} = (sf_{n-1} + tf_n) + (sf_{n-2} + tf_{n-1}) \\
&= s(f_{n-1} + f_{n-2}) + t(f_n + f_{n-1}) = sf_n + tf_{n+1} = \text{RHS}
\end{align*}
\]