§ 4.2. The Pigeonhole Principle

2. Show that if there are 30 students in a class, then at least 2 have last names that begin with the same letter.

Solution. There are 26 letters. By the Pigeonhole Principle, there are \( \lceil \frac{30}{26} \rceil = 2 \) students having last names that begin with the same letter.

6. Let \( d \) be a positive integer. Show that among any group of \( d + 1 \) (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by \( d \).

Solution. There are only \( d \) possible remainders modulo \( d \): 0, 1, \ldots, \( d - 1 \). If there are \( d + 1 \) integers, by the Pigeonhole Principle, among their remainders with respect to \( d \) at least \( \lceil \frac{d+1}{d} \rceil = 2 \) of them must be the same, which means the corresponding integers have the same remainder when they are divided by \( d \).

26. Show that if there are 100,000,000 wage earners in the United States who earn less than 1,000,000 dollars, then there are two who earned exactly the same amount of money, to the penny, last year.

Solution. From 1 cent to 999,999 dollar 99 cents, there are 99,999,999 possible amounts, if there are 100,000,000 wage earners in the United States who earn less than 1,000,000 dollars, then by the Pigeonhole Principal, there are at least \( \lceil \frac{100,000,000}{99,999,999} \rceil = 2 \) persons earned exactly the same amount of money, to the penny, last year.

*30. Prove that at a party where there are at least two people, there are two people who know the same number of other people there.
Solution. First, it is important to note that "knowing" someone is a symmetric relationship. That is to say that if Person A knows Person B then Person B knows Person A.

Now consider the people who know nobody else at the party. Obviously, if there are two of these, then we are finished. If there are fewer than two, we may omit them as nobody knows them (who invited these people anyways?). This leaves us with a finite number of people, all of whom know at least one other person in the group (it should be pointed out that we still know n ≥ 2). Each of these n people know between 1 and n − 1 other people at the party. Thus, by the pigeonhole principle, two people must know the same number of people at the party.

What if there are some strangers, which may happen in the real life. Well, the conclusion still holds. Suppose there are k strangers, who know nobody in the party. If k > 1, then those k strangers know the same amount of people in the party. If k = 1, then consider the rest n − 1 people in the party, applying the previous argument for n − 1 people, we get the same conclusion.

36. Let n₁, n₂, . . . , nᵣ be positive integers. Show that if \( n₁ + n₂ + \cdots + Nᵣ - t + 1 \) objects are placed into t boxes, then for some i, i = 1, 2, . . . , t, the ith box contains at least \( nᵢ \) objects.

Solution. We prove by contradiction. Toward to a contradiction, suppose that for each i, the ith box contains at most \( nᵢ - 1 \) objects. Then the total number of objects in all the t boxes are \( \leq (n₁ - 1) + (n₂ - 1) + \cdots + (nᵣ - 1) = n₁ + n₂ + \cdots + nᵣ - t \), less than the given number of objects. This is a contradiction.

§ 4.3 Permutations and Combinations

2. How many permutations are there of the set \{a, b, c, d, e, f, g\}?

Solution. 7!.

10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

Solution. The permutation of 6 different names is 6!.

26. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with 6 members if it must have more women than men.

Solution. There are three cases: (1) 4 women and 2 men, \( C(10, 4) \cdot C(15, 2) = 22050 \); (2) 5 women and 1 man, \( C(10, 5) \cdot C(15, 1) = 3780 \); and (3) all 6 members are women, \( C(10, 6) = 210 \). So, altogether there are 22050 + 3780 + 210 = 26040 ways to form a desired committee.

34. Show that if \( p \) is a prime and \( k \) is an integer such that \( 1 \leq k \leq p - 1 \), then \( p \) divides \( C(p, k) \).

Solution.

\[
C(p, k) = \frac{p(p-1)(p-2)\cdots(p-k+1)}{1\cdot2\cdots k}.
\]

Since \( p \) is prime and \( 1 \leq k \leq p - 1 \), none of the \( k \) numbers in the denominator divides \( p \). On the other hand, \( C(p, k) \) is an integer, so the denominator divides the rest of the numerator, i.e. \( (p-1)(p-2)\cdots(p-k+1) \), therefore \( C(p, k) \) is a multiple of \( p \), which means that \( p \) divides \( C(p, k) \).
∗42. Give a formula for the coefficient of \(x^k\) in the expansion of \((x + 1/x)^{100}\), where \(k\) is an integer.

Solution.

\[
(x + 1/x)^{100} = (x^2 + 1)^{100}/x^{100}
\]

So the coefficient of \(x^k\) in the expansion of \((x + 1/x)^{100}\) equals the coefficient of \(x^{100+k}\) in \((x^2 + 1)^{100}\). Since the powers of \(x\) in the expansion of \((x^2 + 1)^{100}\) are all even, while \(k\) is odd, the coefficient of \(x^k\) in the expansion of \((x + 1/x)^{100}\) is 0. When \(k\) is even, then the coefficient of \(x^{100+k}\) in the expansion of \((x^2 + 1)^{100}\) equal \(C(100, (100 + k)/2)\), which by the argument above gives the coefficient of \(x^k\) in the expansion of \((x + 1/x)^{100}\).

44. The row of Pascal’s triangle containing the binomial coefficients \(C(10, k)\), \(0 \leq k \leq 10\), is:

\[
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 1 \quad 1
\]

Use Pascal’s identity to produce the row immediately following this row in Pascal’s triangle.

Solution. The next row \(C(11, k)\), \(0 \leq k \leq 11\), is:

\[
1 \quad 11 \quad 55 \quad 165 \quad 330 \quad 462 \quad 462 \quad 330 \quad 165 \quad 55 \quad 11 \quad 1
\]

∗52. Give a combinatorial proof that \(\sum_{k=1}^{n} kC(n, k)^2 = nC(2n-1, n-1)\).

Solution. Following the hint, we shall give two ways to count the number of ways to select a committee, with \(n\) members from a group of \(n\) mathematics professors and \(n\) computer science professors, such that the chairperson of the committee is a mathematics professor. One way to do this is to: first choose \(k\) mathematics professor and \(n-k\) computer science professors, and then among those \(k\) mathematics professor, appoint one to be the chair. So for each \(k\), there are \(C(n,k) \cdot C(n-k,k) \cdot k\) ways to do this. And \(k\) can range from 1 to \(n\). So the left hand side of the identity reflect this method of selecting committee members.

Another way to do this is to: appoint one mathematics professor as the chair first, and then select the rest \(n-1\) members from the rest \(2n-1\) mathematics professors or computer science professors. One can easily see that this method corresponds to the right hand side of the identity. Since both ways do the same job, the left hand side equals the right hand side.

Lenstra’s Notes on Probability Theory

1. Andrew, Beatrix and Charles are playing with a crown. If Andrew has the crown, he throws it to Charles. If Beatrix has the crown, she throws it to Andrew or to Charles, with equal probabilities. If Charles has the crown, he throws it to Andrew or to Beatrix, with equal probabilities.

At the beginning of the game the crown is given to one of Andrew, Beatrix and Charles, with equal probabilities. What is the probability that, after the crown is thrown once, Andrew has it? that Beatrix has it? that Charles has it?
Solution.

\[ P(\text{Andrew has it}) = P(\text{Andrew start}) \times P(\text{Andrew kept it}) + P(\text{Beatrix start}) \times P(\text{Beatrix \rightarrow Andrew}) + P(\text{Charles start}) \times P(\text{Charles \rightarrow Andrew}) = \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{3} \]

\[ P(\text{Beatrix has it}) = P(\text{Andrew start}) \times P(\text{Andrew \rightarrow Beatrix}) + P(\text{Beatrix start}) \times P(\text{Beatrix kept it}) + P(\text{Charles start}) \times P(\text{Charles \rightarrow Beatrix}) = \frac{1}{3} \times 0 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} \]

\[ P(\text{Charles has it}) = P(\text{Andrew start}) \times P(\text{Andrew \rightarrow Charles}) + P(\text{Beatrix start}) \times P(\text{Beatrix \rightarrow Charles}) + P(\text{Charles start}) \times P(\text{Charles kept it}) = \frac{1}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 0 = \frac{1}{2} \]

3. Let \( S \) be a sample space, and let \( A, B \subseteq S \) be two independent events. Let \( A' = S - A \) and \( B' = S - B \). Prove that \( A' \) and \( B' \) are independent events. Are \( A \) and \( B' \) independent? And \( A \) and \( A' \)?

Solution.

\[ P(A' \cap B') = P(S - (A \cup B)) = 1 - P(A \cup B) \quad \text{(set arithmetic)} \]

\[ = 1 - P(A) - P(B) + P(A \cap B) \quad \text{(play with definition of \( P \))} \]

\[ = 1 - P(A) - P(B) + P(A) \times P(B) \quad \text{(inclusion-exclusion)} \]

\[ = (1 - P(A)) \times (1 - P(B)) \]

\[ = P(A') \times P(B') \]

So, \( A' \) and \( B' \) are independent.

\[ P(A \cap B') = P(A - (A \cap B)) = P(A) - P(A \cap B) = P(A) - P(A) \times P(B) = P(A) \times (1 - P(B)) = P(A) \times P(B') \]

So, \( A \) and \( B' \) are independent.

\[ P(A \cap A') = P(\text{empty set}) = 0 \]

So \( A \) and \( A' \) are independent if and only if \( P(A) = 0 \) or \( P(A) = 1 \).

4. A random number is drawn from \( \{1, 2, 3, \ldots, 899, 900\} \); each number has probability \( \frac{1}{900} \). Are the events “the number is divisible by 3” and “the number is divisible by 5” independent? What is the probability that the number has no factor in common with 15?

Solution.

\[ A = \{3, 6, 9, \ldots, 897, 900\}, \]

\[ B = \{5, 10, 15, \ldots, 895, 900\} \]
and

\[ C = \{ \text{numbers relatively prime to } 15 \} = S - (A \cup B). \]

\[ P(A) = \frac{1}{3}, \quad P(B) = \frac{1}{5} \quad \text{and} \quad P(A \cap B) = \frac{1}{15}, \]

so they are independent.

\[ P(C) = 1 - P(A) - P(B) + P(A \cap B) = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{15} = \frac{8}{15}. \]