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Math55: Discrete Mathematics Practice Final

Your Name:  
Your GSI:  
Your SID:  

This is a closed book, closed notes exam. You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Look over the whole exam to find problems that you can do quickly. You need not simplify your answers unless you are specifically asked to do so.

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1. (a) (5 points). How many onto functions are there from a set with 7 elements to one with 3 elements?
   (b) (5 points). What is the coefficient of $a^7b^5$ in the expansion of $(2a - 3b)^{12}$?

2. (10 points) Each box of C++ Cereal comes with a small plastic toy, which is shaped either like “C” or like a “+”. A given box contains a “+” with probability $\frac{2}{3}$ and a “C” with probability $\frac{1}{3}$. Suppose someone buys $n \geq 3$ boxes of cereal. What is the probability that this person has at least one “C” and at least two “+”?

3. Consider the first 250 Fibonacci numbers: $f_0 = 0, f_1 = 1, f_2 = 1, \cdots, f_{249} \approx 5 \times 10^{51}$.
   (a) (5 points) Show that there are at least 84 of them which have the same remainder when divided by 3.
   (b) (5 points) How many of them are divisible by 2?

4. (a) (5 points) Show that $\frac{1}{\sqrt{n+1}} \geq 2 \left( \sqrt{n+2} - \sqrt{n+1} \right)$ for all integers $n \geq 1$.
   (b) (5 points) Show that for $n \geq 1$,
   \[ 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq 2 \left( \sqrt{n+1} - 1 \right).\]

5. (a) (5 points) Use the Euclidean Algorithm to compute $\gcd(21, 13)$.
   (b) (5 points) Find integers $s$ and $t$ such that $\gcd(21, 13) = s \cdot 21 + t \cdot 13$.
   (c) (5 points) Use the Chinese Remainder Theorem to find an integer $x$ with $0 \leq x \leq 21 \cdot 13 = 273$ such that
   \[ x \equiv 5 \mod 21 \quad \text{and} \quad x \equiv 3 \mod 13.\]

6. Find a formula for the number of triples $(x, y, z)$ of non-negative integers satisfying
   \[ x + y + z = 16 \]
   (a) (5 points) and no other restrictions.
   (b) (5 points) subject to the conditions that $x \geq 4, y \geq 5,$ and $z \leq 2$.

7. Suppose that a randomly chosen child is male with probability $\frac{1}{2}$ and female with probability $\frac{1}{2}$. Consider two families 1 and 2 with two children each. Let $A_1$ be the event that family 1 has at least one male child and $A_2$ be the event that the \textit{oldest} child in family 2 is male. For $i = 1, 2$, let $C_i$ be the event that family number $i$ has two male children.
   (a) (5 points) What is the sample space $S$? What is the probability $P(x)$ of each point $x \in S$? What is the total expected number of male children in both families?
   (b) (5 points) Calculate the probabilities $P(A_1)$ and $P(A_2)$. 
(c) (5 points) Calculate the conditional probabilities $P(C_1|A_1)$ and $P(C_2|A_2)$. Given that $A_1$ and $A_2$ take place, which is more likely: $C_1$ or $C_2$?

(d) (5 points) Define what it means for two events $A$ and $B$ to be independent. For which $i$ and $j$ are $A_i$ and $C_j$ independent?