4.

a) \(- (1 - x^8)/(1 - x)\)
b) Geometric series: \(1/(1 - 3x)\)
c) \(x^2/(1 + 3x)\)
e) \((1 + 2x)^7\).

8.

a) \((\frac{3}{n^{1/2}})\) if \(n\) is even, 0 otherwise.

b) \(- (3)^n (\frac{3}{n})\)

g) Observe that \(\frac{x^n}{1 + x^n + x^2} = \frac{x^2}{1 - x^2} = (x - x^2) \sum_{i=0}^{\infty} x^{3n}\). Thus the coefficient \(a_n\) is 1 if \(n\) is congruent to 1 (mod 3), -1 if \(n\) is congruent to 2 (mod 3) and 0 otherwise.

30.

a) \(2G(x)\)
b) \(xG(x)\)
c) \(x^2(G(x) - G(0) - G'(0)x) = x^2(G(x) - a_0 G(x) - a_1 x G(x))\)
d) Answer for (c) divided by \(x^4\).
e) \(G'(x)\)
f) \((G(x)^2)\)

36. Let \(G(x) = \sum_{i=0}^{\infty} a_k x^k\). Then,

\[ G(x) - 4 - 12x = \sum_{i=2}^{\infty} a_k x^k = \sum_{i=2}^{\infty} (a_{k-1} + 2a_{k-2} + 2^k)x^k = xG(x) + 2x^2G(x) + 1/(1 - 2x). \]

Thus, \(G(x) = (4 + 12x + 1/(1 - 2x))/(1 - x - 2x^2)\). One can then find an explicit formula for the coefficients (omitted).

40. Observe first that \(2^n n! (1)(3)(5)\ldots(2n - 1) = (2n)!\), since the \(2^n n!\) is exactly the product of the even terms \((2)(4)\ldots(2n)\). Thus,

\[ \frac{(2n)!}{(-4)^n} = \frac{(2n)!}{(-4)^n n!} = \frac{2^n (1)(3)(5)\ldots(2n - 1)}{n! (-4)^n} = \frac{(-1/2)(-3/2)(-5/2)\ldots(-2n - 1)/2}{n!} = \left(-\frac{1/2}{n}\right) \]

Then,

\[ (1 - 4x)^{-1/2} = \sum_{n=0}^{\infty} \left(-\frac{1/2}{n}\right)(-4x)^n. \]

The result is now obvious from the first part.

50. Let \(S\) denote the set of all byte strings of length \(n\). Let \(c_0, c_1 : S \rightarrow \{0, 1\}\) denote the functions whose output given a string \(x\) is number of 0’s in the string mod 2, and 1’s in the string mod 2, respectively. Let \(A_n, B_n, C_n, D_n\) denote the set of strings with \(c_0 = a, c_1 = b\) for \(ab = 00, 01, 10, 11\), respectively. Then \(a_n = |A_n|, b_n = |B_n|, c_n = |C_n|, d_n = |D_n|\). Now, clearly, \(A_n \cup B_n \cup C_n \cup D_n = S\), and the sets \(A_n, B_n, C_n, D_n\) are clearly disjoint (elements in two different sets have different \((c_0, c_1)\) values), so \(4^n = |S| = a_n + b_n + c_n + d_n\).

For the recurrence relations, say the first one, the strings in \(A_{n+1}\). We can partition \(A_{n+1}\) into the strings in \(A_{n+1}\) with last byte a 0, 1, 2, 3, separately. If the last byte is a 2 or 3, the first \(n\) bytes can be any string with an even number of 0’s and an even number of 1’s, hence there are \(a_n\) many of each. If the last byte is a 0, the first \(n\) bytes are a string with an even number of 1’s and an odd number of 0’s, that is a string in \(C_n\). Similarly, there are \(b_n\) many strings with a last byte equal to 0. Thus, \(a_{n+1} = b_n + c_n + 2a_n\). The same thing
can be done for $b_{n+1}, c_{n+1}$, and we get the desired relations by using the formula $d_n = 4^n - a_n - b_n - c_n$ to get rid of any $d_n$ terms which appear.

We will omit the routine calculation of what the small values of the sequence are.

Now, let $A(x) = \sum_{i=0}^{\infty} a_n x^n$, similarly for $B(x), C(x)$. [NOTE: $a_0 = 1, b_0 = 0, c_0 = 0, d_0 = 0$, since THE empty string has no zeros and no ones]. Thus,

$$A(x) = 1 + x \sum_{i=0}^{\infty} a_{n+1} x^n = 1 + x \sum_{i=0}^{\infty} (2a_n + c_n + b_n) x^n = 1 + 2x A(x) + x C(x) + x B(x).$$

So, $(1 - 2x)A(x) = 1 + x B(x) + x C(x)$. Similarly, $B(x) = x B(x) - x C(x) + 1/(1 - 4x)$, $C(x) = x C(x) - x B(x) + x/(1 - 4x)$. Using these last two equations, $(1 - x) B(x) + x C(x) = x/(1 - 4x) = x B(x) + (1 - x) C(x)$.

Equating the far left and right terms, we find that $(1 - 2x) B(x) = (1 - 2x) C(x)$. So, if we restrict to $|x| < 1/2$, $B(x) = C(x)$. Plugging this back in to either of the last two equations, we find that $B(x) = x/(1 - 4x)$. Finally, $A(x) = (1 + x^2)/(1 - 4x)(1 - x) = 1/(1 - 2x) + x^2/(1 - 2x) - 2/(1 - 4x)$. We use a partial fractions decomposition on the last term, to find $A(x) = 1/(1 - 2x) + 2x^2/(1 - 2x) + 2/(1 - 4x))$. The explicit formula for $a_n, b_n, c_n$ can be found now routinely. This is omitted.

4. 1250000 + 650000 = 1450000.

10. Let $S$ denote the set of positive integers not exceeding 100, that are not divisible by 5, and $T$ those that are not divisible by 7. We are looking for $|S \cap T|$. Using Demorgan’s Laws and Inclusion-Exclusion,

$$|S \cap T| = |(S^C \cup T^C)| = 100 - |S^C \cup T^C| = 100 - |S^C| - |T^C| + |S^C \cap T^C| = 100 - 20 - 14 + 2 = 68.$$

14. Let $X, Y, Z$ denote the set of permutations of the English alphabet which contain ”fish”, ”rat”, and ”bird”, respectively. We seek $|X^C \cap Y^C \cap Z^C|$. Once again, using Demorgan’s Laws and Inclusion-Exclusion,

$$|X^C \cap Y^C \cap Z^C| = |(X \cup Y \cup Z)^C| = 26! - |X| - |Y| - |Z| + |X \cap Y| + |X \cap Z| + |Y \cap Z| - |X \cap Y \cap Z|.$$

The number of permutations which contain ”fish” is just 23! (treat ”fish” as a single letter), so $|X| = 23!$. Similarly, $|Y| = 24!, |Z| = 23!$. There 2! permutations which contain ”fish” and ”rat” (the letters used are all distinct), but no strings which simultaneously contain ”fish” and ”bird”, or ”rat” and ”bird”; hence all other double and triple intersections are empty. Thus, the total number is 26! - 24! - 23! - 23! + 2!.

4. Let $a_1 = 3, a_2 = 4, a_3 = 5, a_4 = 8$. Let us define a few sets:

$$S = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 + x_3 + x_4 = 17, x_i \geq 0 \text{ for all } i\}$$

$$S_i = \{(x_1, x_2, x_3, x_4) \in S \mid x_i \leq a_i\}.$$

The number we seek is $R = |S_1 \cap S_2 \cap S_3 \cap S_4|$, since this is the number of 4-tuples $(x_1, x_2, x_3, x_4)$ whose sum is 17, and $x_1$ is a nonnegative integer less than or equal to 3, etc. Thus, we use inclusion exclusion (the alternate form):

$$R = |S| - \sum_i |S_i^C| + \sum_{i<j} |S_i^C \cap S_j^C| - \sum_{i<j<k} |S_i^C \cap S_j^C \cap S_k^C| + |S_1^C \cap S_2^C \cap S_3^C \cap S_4^C|.$$

Observe that $S_i^C = \{(x_1, x_2, x_3, x_4) \in S \mid x_i \geq a_i + 1\}$, so $|S_i^C|$ is the number of ways to solve $x_1 + x_2 + x_3 + x_4 = 17$ with nonnegative $x_i's$, and $x_i \geq a_i + 1$. Let $y_i = x_i - (a_i + 1)$, $y_j = x_j$ for $i \neq j$, we see that the number of such solutions is the same as the number of solutions to $y_1 + y_2 + y_3 + y_4 = 17 - (a_i + 1)$ with nonnegative integers, which is $\binom{17 - (a_i + 1) + 4 - 1}{4}$. One performs an analogous calculation for multiple intersections (which amounts to having to subtract off multiple $a_i$), to obtain the final answer:

$$R = \left(\begin{array}{c}
17 + 4 - 1 \\
4 - 1
\end{array}\right) - \left(\begin{array}{c}
17 - 4 + 4 - 1 \\
4 - 1
\end{array}\right) - \left(\begin{array}{c}
17 - 5 + 4 - 1 \\
4 - 1
\end{array}\right) - \left(\begin{array}{c}
17 - 9 + 4 - 1 \\
4 - 1
\end{array}\right) + \left(\begin{array}{c}
17 - 9 + 4 - 1 \\
4 - 1
\end{array}\right) + \left(\begin{array}{c}
17 - 10 + 4 - 1 \\
4 - 1
\end{array}\right) + \left(\begin{array}{c}
17 - 13 + 4 - 1 \\
4 - 1
\end{array}\right) + \left(\begin{array}{c}
17 - 11 + 4 - 1 \\
4 - 1
\end{array}\right) + \left(\begin{array}{c}
17 - 14 + 4 - 1 \\
4 - 1
\end{array}\right) + \left(\begin{array}{c}
17 - 15 + 4 - 1 \\
4 - 1
\end{array}\right).$$
\[
-\left( \frac{17 - 15 + 4 - 1}{4 - 1} \right) - \left( \frac{17 - 18 + 4 - 1}{4 - 1} \right) - \left( \frac{17 - 19 + 4 - 1}{4 - 1} \right) - \left( \frac{17 - 20 + 4 - 1}{4 - 1} \right) + \left( \frac{17 - 24 + 4 - 1}{4 - 1} \right).
\]

Note that the last 4 terms are 0.

18.

We shall show that both sides count the number of derangements of \( n \) objects (that is permutations \( \phi : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\} \); by definition the left hand side does. Now, for \( i = 2, \ldots, n \), let \( A_i \) be the set of derangements \( \phi \), with \( \phi(1) = i \). Clearly, the \( A_i \) are disjoint, and the union of the \( A_i \) is equal to the set of all derangements, since \( \phi(1) \neq 1 \), for \( \phi \) a derangement. Now, \( A_i = B_i \cup C_i \), where \( B_i \) is then those \( \phi \) with \( \phi(1) = 1 \), \( \phi(i) = 1 \), and \( C_i \) is the complement of \( B_i \) in \( A_i \). Then, \( |B_i| = D_{n-2} \), since any such \( \phi \) can be completed to a derangement of all \( n \) numbers by simply deranging the numbers other than 1 and \( i \). We claim that \( |C_i| = D_{n-1} \). To see this, we note that \( \phi \) is a map which doesn’t send 2 to 2, doesn’t send 3 to 3, \( \ldots \), doesn’t send \( i \) to 1 (since \( \phi \notin B_i \), \( \ldots \), doesn’t send \( n \) to \( n \). Viewing the object \( i \) in the domain as 1, this is just a derangement of \( n-1 \) objects, as desired. Since \( i \) was arbitrary, we conclude that \( A_i = D_{n-1} + D_{n-2} \) for all \( i \), hence,

\[
D_n = A_2 + \ldots + A_n = (n - 1)(D_{n-1} + D_{n-2}).
\]

22. First, we see that if \( x \) and \( pq \) are relatively prime, if and only if neither \( p \) nor \( q \) divide \( x \). We see this as follows. Certainly, if \( x \) and \( pq \) are relatively prime, then, \( x \) cannot be divisible by either \( p \) or \( q \) since each of them divides \( pq \). Conversely, if \( x \) and \( pq \) are not relatively prime, then there is a nonidentity divisor of \( pq \) which divides \( x \); the only divisors of \( pq \) are 1, \( p \), \( q \), \( pq \), so we see if \( p \), \( q \) or \( pq \) divides \( x \), then, either \( p \) divides \( x \) or \( q \) divides \( x \). Now, let \( R \) be the set of integers between 1 and \( pq \), \( S \) the set of integers between 1 and \( pq \) which are divisible by \( p \), \( T \) the set of integers between 1 and \( pq \) which are divisible by \( q \). We see that \( |S| = q, |T| = p \), e.g. the multiples of \( p \) between 1 and \( pq \) are \( p, 2p, \ldots, qp \). Now, the set of numbers between 1 and \( pq \) which are relatively prime to \( pq \) is exactly \( S^C \cap T^C \), where \( S^C = R \setminus S \) is the complement of \( S \) with respect to \( R \), similarly for \( T \). By Demorgan’s Laws, then inclusion exclusion,

\[
|S^C \cap T^C| = |(S \cup T)^C| = |R| - |S \cup T| = |R| - |S| - |T| + |S \cap T|.
\]

Now, \( |R| = pq \), and \( S \cap T \) is the set of integers between 1 and \( pq \) divisible by both \( p \) and \( q \), hence by \( pq \), since \( p \) and \( q \) are distinct primes, hence are relatively prime. Thus, \( S \cap T = \{pq\} \), so \( |S \cap T| = 1 \). Thus, \( \phi(pq) = pq - p - q + 1 = (p - 1)(q - 1) = \phi(p)\phi(q) \).