Math 55 Spring 2002
Final Solutions

The order of the problems in these solutions may be different than in the exam you took.

1. A computer password is 6-8 characters long. Each character must be a digit, an uppercase letter, or a lowercase letter. Each password must contain at least one digit, one uppercase letter, and one lowercase letter. How many passwords are possible?

Solution. Using inclusion/exclusion, the number of passwords of length 6 is

$$6^6 - (36^6 + 36^6 + 52^6) + (10^6 + 26^6 + 26^6).$$

The number of passwords of length 7 is

$$6^7 - (36^7 + 36^7 + 52^7) + (10^7 + 26^7 + 26^7).$$

The number of passwords of length 8 is

$$6^8 - (36^8 + 36^8 + 52^8) + (10^8 + 26^8 + 26^8).$$

Add all three of these together to get the total number of possible passwords.

2. A full-adder has three binary inputs (in1, in2, and carry-in) and two binary outputs (sum and carry-out), whose values represent the two-bit sum of the three binary inputs. For example, if in1, in2 and carry-in are 1, 1, and 0 respectively, then sum and carry-out are 0 and 1 respectively. Draw a truth table for the full-adder.

Solution.

<table>
<thead>
<tr>
<th>in 1</th>
<th>in 2</th>
<th>carry-in</th>
<th>carry-out</th>
<th>sum</th>
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3. Which of the following are set identities for all sets A, B, C and the universal set U?
(a) \( A \cup B = B \cup A \).
(b) \((A \cup B) \cup C = A \cup (B \cup C)\).
(c) \(\overline{A \cup B} = \overline{A} \cup \overline{B}\).

Solution.

(a) This identity is true.

\[
x \in A \cup B \iff x \in A \lor x \in B \iff x \in B \lor x \in A \iff x \in B \cup A.
\]

(b) This identity is true.

\(\subseteq\): If \(x \in (A \cup B) \cup C\), then either \(x \in C\) or \(x \in (A \cup B)\). If \(x \in C\), then \(x \in B \cup C \subseteq A \cup (B \cup C)\). If \(x \in (A \cup B)\) then either \(x \in A\) or \(x \in B\). If \(x \in A\) then \(x \in A \cup (B \cup C)\). If \(x \in B\) then \(x \in B \cup C\), so \(x \in A \cup (B \cup C)\).

\(\supseteq\): If \(x \in A \cup (B \cup C)\), then either \(x \in A\) or \(x \in (B \cup C)\). If \(x \in A\), then \(x \in A \cup B\). If \(x \in (B \cup C)\) then either \(x \in B\) or \(x \in C\). If \(x \in C\) then \(x \in (A \cup B) \cup C\). If \(x \in B\) then \(x \in A \cup B\), so \(x \in (A \cup B) \cup C\).

(c) This identity is false. It is not enough to prove that \(\overline{A \cup B} = \overline{A} \cup \overline{B}\), since it is not necessarily true that \(\overline{A} \cap \overline{B} = \overline{A \cup B}\).

Instead we may simply give a counterexample: Let \(A = \{a\}\), let \(B = \{b\}\), and let \(U = \{a, b\}\). Then \(A \cup B = \emptyset\), but \(\overline{A} \cup \overline{B} = \{a, b\}\).

Thus, the identity is not true for all \(A, B\).

4.

(a) Find the number of solutions to the equation

\[x_1 + x_2 + x_3 = n\]

where \(n, x_1, x_2,\) and \(x_3\) are integers satisfying \(n \geq 0\) and \(0 \leq x_1, x_2, x_3 \leq 1\).

(b) Prove that the sum of the cubes of any three consecutive integers is divisible by 9.

Solution.

(a) For \(n = 0\) there is 1 solution: \(x_1 = x_2 = x_3 = 0\).

For \(n = 1\) there are 3 solutions: exactly one of the \(x_i\) is 1, the rest are 0.

For \(n = 2\) there are 3 solutions: exactly one of the \(x_i\) is 0, the rest
are 1. 
For \(n = 3\) there is 1 solution: \(x_1 = x_2 = x_3 = 1\). 
For \(n \geq 4\) there are no solutions. 

(b) We will prove using induction that 
\[
9 | \left( n^3 + (n + 1)^3 + (n + 2)^3 \right)
\]
for all integers \(n\). Since we need to prove this for all integers, we will do induction in two directions from our base case. 
Let \(n = 0\) for our base case. The statement holds since 
\[
0^3 + 1^3 + 2^3 = 9,
\]
which is divisible by 9. Suppose the statement holds for a nonnegative integer \(k\), meaning 
\[
k^3 + (k + 1)^3 + (k + 2)^3 = 9m
\]
for some integer \(m\). Then 
\[
(k + 1)^3 + (k + 2)^3 + (k + 3)^3 = 9m + (k + 3)^3 - k^3 \\
= 9m + 9k^2 + 27k + 27 \\
= 9(m + k^2 + 3k + 3).
\]
Thus, 
\[
9 | \left( (k + 1)^3 + (k + 2)^3 + (k + 3)^3 \right)
\]
and the statement holds for all \(n \geq 0\) by induction. Now suppose that the statement holds for some non-positive integer \(k\): meaning 
\[
k^3 + (k + 1)^3 + (k + 2)^3 = 9m
\]
for some integer \(m\). Then 
\[
(k - 1)^3 + k^3 + (k + 1)^3 = 9m + (k - 1)^3 - (k + 2)^3 \\
= 9m + (k^3 - 3k^2 + 3k - 1) - \\
(k^3 + 6k^2 + 12k + 8) \\
= 9m - 9k^2 - 9k - 9 \\
= 9(m - k^2 - k - 1).
\]
Thus, 
\[
9 | \left( (k - 1)^3 + k^3 + (k + 1)^3 \right)
\]
and the statement holds for all \(n \leq 0\) by induction. Therefore, the statement holds for all integers \(n\).
(a) Four soldiers each choose a card from a standard deck. The highest ranked card must lead the charge to the front of the battlefield. One of the soldiers chooses the 3 of diamonds and that ends up being the highest ranked card, and he’s off to the front. On his way the soldier wonders, “what was the chance of the 3 of diamonds being the highest card?” Assume that the suits are ordered clubs, diamonds, hearts and spades, and the ranks are ordered 2 through Ace (for example, spade 2 is ranked lower than spade Queen but higher than diamond Ace). While he is busy ducking bullets, answer his question for him.

(b) Sam and John each take 3 shots at a basket. Sam’s success probability is 1/3 on each shot, and John’s is 3/4. All shots are independent. Find

i. The probability that at least one of the 6 shots is a success.

ii. The expected total number of successes.

Solution.

(a) There are 14 cards with lower rank than the 3 of diamonds consisting of the 2 of diamonds and the 13 clubs. In order for the 3 of diamonds to be the highest card, other three soldiers must draw cards from these 14 when picking out of the 51. Thus the probability is

\[
\frac{C(14, 3)}{C(51, 3)} = \frac{14 \cdot 13 \cdot 12}{51 \cdot 50 \cdot 49}.
\]

(b) In this problem each sample in the sample space, \( S \), corresponds to either a success or a failure for each shot Sam takes and either a success or a failure for each shot John takes. For example one sample is \((FSS, SSS)\), where each \( F \) is a failure and each \( S \) is a success.

i. We want to calculate the probability of the event \( E_1 \) which consists of all samples where there is at least one success. This event is equal to \( S - E_2 \), where \( E_2 \) is the event consisting of the sample with all misses. Thus,

\[
P(E_1) = P(S - E_2) = 1 - P(E_2) = 1 - \left(\frac{2}{3}\right)^3 \left(\frac{1}{4}\right)^3 = 1 - \frac{1}{216} = \frac{215}{216}.
\]
ii. Let $S_i$ be a random variable which is 0 if Sam’s $i$-th shot is a failure and 1 if his $i$-th shot is a success. Similarly let $J_i$ be a random variable which is 0 if John’s $i$-th shot is a failure and 1 if his $i$-th shot is a success. Then we want to find the expected value of the random variable

$$X = S_1 + S_2 + S_3 + J_1 + J_2 + J_3.$$ 

Now, we have

$$E(S_i) = 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3}$$

and

$$E(J_i) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{3}{4}.$$ 

Thus,

$$E(X) = E(S_1) + E(S_2) + E(S_3) + E(J_1) + E(J_2) + E(J_3)$$

$$= 3 \cdot \frac{1}{3} + 3 \cdot \frac{3}{4} = \frac{13}{4}. \quad \square$$

6.

(a) Compute $11^{79} \mod 5$.

(b) Compute $11^{79} \mod 7$.

(c) Compute $11^{79} \mod 5 \cdot 7$.

Solution.

(a)

$$11^{79} \mod 5 = 11^{4 \cdot 19 + 3} \mod 5$$

$$= (11^4)^{19} \cdot 11^3 \mod 5$$

$$= 1^{19} \cdot 1331 \mod 5 \text{ (by Fermat’s Little Theorem)}$$

$$= 1331 \mod 5$$

$$= 1.$$

(b)

$$11^{79} \mod 7 = 11^{6 \cdot 13 + 1} \mod 7$$

$$= (11^6)^{13} \cdot 11 \mod 7$$

$$= 1^{13} \cdot 11 \mod 7 \text{ (by Fermat’s Little Theorem)}$$

$$= 11 \mod 7$$

$$= 4.$$
(c) We want to solve the system of congruences
\[ x \equiv 1 \pmod{5} \]
\[ x \equiv 4 \pmod{7}. \]
3 is the inverse to 5 modulus 7 and 3 is also the inverse to 7 modulus 5. Thus, the solution is given by
\[ 1 \cdot 7 \cdot 3 + 4 \cdot 5 \cdot 3 \equiv 21 + 60 \equiv 81 \equiv 11 \pmod{35}. \]
Therefore \(11 \pmod{35} = 11\).

7.
(a) Show that for any integer \(n \geq 2\),
\[ \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 - \frac{1}{n}. \]
(b) Find a closed form solution for the recurrence equation
\[ a_0 = 0, \ a_1 = 1, \ a_n = 5a_{n-1} - 6a_{n-2} + 2^n, \ n \geq 2. \]

Solution.
(a) We will prove this by induction on \(n\). Let \(P(n)\) be the statement that
\[ \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 - \frac{1}{n}. \]
We have
\[ \sum_{k=2}^{2} \frac{1}{k^2} = \frac{1}{4} \leq \frac{1}{2} = 1 - \frac{1}{2}, \]
and so \(P(2)\) is true. Now, suppose \(P(n)\) is true, then we have
\[ \sum_{k=2}^{n+1} \frac{1}{k^2} = \sum_{k=2}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \]
\[ \leq 1 - \frac{1}{n} + \frac{1}{(n+1)^2} \]
\[ = 1 - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n} + \frac{1}{(n+1)^2} \]
\[ = 1 - \frac{1}{n+1} + \frac{n - (n+1)}{n(n+1)} + \frac{1}{(n+1)^2} \]
\[ = 1 - \frac{1}{n+1} + \frac{(n+1)^2 - n(n+1)}{n(n+1)} \]
\[ \leq 1 - \frac{1}{n+1} + \frac{1}{n(n+1)} - \frac{1}{n(n+1)} \]
\[ = 1 - \frac{1}{n+1}. \]
So, $P(n + 1)$ is true. Therefore $P(n)$ is true for all $n \geq 2$. □

(b) First, we need to find the homogeneous solution. The characteristic equation is

$$r^2 - 5r + 6 = (r - 2)(r - 3) = 0.$$ 

Thus, the homogeneous solution is

$$a_n^{\text{homogeneous}} = \alpha 2^n + \beta 3^n.$$ 

Next, we need to find the particular solution. Since the nonhomogeneous part of the recurrence relation is $2^n$ and $2$ is a solution to the characteristic equation, the particular solution that we need to try is

$$a_n^{\text{particular}} = p_0 n 2^n.$$ 

Plugging this into the recurrence relation gives

\[
\begin{align*}
p_0 n 2^n &= 5p_0 (n - 1)2^{n - 1} - 6p_0 (n - 2)2^{n - 2} + 2^n \\
2p_0 n &= 5p_0 (n - 1) - 3p_0 (n - 2) + 2 \\
2p_0 n &= 2p_0 n + p_0 + 2 \\
p_0 &= -2.
\end{align*}
\]

Thus, the particular solution is

$$a_n^{\text{particular}} = -2n 2^n = -n 2^{n + 1}.$$ 

The general solution comes from the sum of the homogeneous and particular solutions and so it is given by

$$a_n = a_n^{\text{homogeneous}} + a_n^{\text{particular}} = \alpha 2^n + \beta 3^n - n 2^{n + 1}.$$ 

The first initial condition gives us

$$0 = a_0 = \alpha + \beta \implies \alpha = -\beta.$$ 

The second initial condition gives us

$$1 = a_1 = 2\alpha + 3\beta - 4 \implies \beta = 5, \alpha = -5.$$ 

Therefore a closed form for the solution is given by

$$a_n = 5 \cdot 3^n - 5 \cdot 2^n - n 2^{n + 1}.$$ 

□