Solutions to Math54 Sample Midterm I

1. (15 Points) Solve linear systems of equations $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 6 \\ 7 \\ 4 \end{pmatrix}. $$

**Solution:** The following is a sequence of row elementary operations that reduce $A$ to triangular form.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & -3 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & 1 \end{pmatrix}. $$

The solution is

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. $$

2. (15 Points) Compute the LU factorization for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}. $$

**Solution:**

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}. $$

3. (25 Points) Let $\mathcal{P}$ be the set of all functions of the form $c_0 + c_1 \sin(x) \cos(x) + c_2 \cos^2(x) + c_3 \sin^2(x)$, where the $c$’s are arbitrary real constants. It is known that $\mathcal{P}$ is a vector space under the usual function addition and scalar multiplication. Find the dimension and a basis for $\mathcal{P}$.

**Solution:**

$$\cos^2(x) = 1 - \sin^2(x). $$

The dimension is 3 and one basis is $\{1, \sin(x) \cos(x), \sin^2(x)\}$. 
4. (25 Points) Find the dimensions and bases for the null space and the range space of the matrix

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \]

**Solution:** Dimension for range space = 1, and dimension for null space = 3. Basis for range space

\[ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \]

Basis for null space

\[ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]

5. (20 Points) Let

\[ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \]

Show that \( A^{-1} \) exists if and only if \( \alpha \delta - \beta \gamma \neq 0 \). Find the formula for \( A^{-1} \) when it does exist.

**Solution:** If \( \alpha \delta - \beta \gamma \neq 0 \), then the inverse exists and

\[ A^{-1} = \frac{1}{\alpha \delta - \beta \gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}. \]

On the other hand, if \( \alpha \delta - \beta \gamma = 0 \), we want to show that \( A^{-1} \) does not exist.

In fact, if both \( \delta = \gamma = 0 \). Then the second row of \( A \) is zero. Hence for any \( 2 \times 2 \) matrix \( B \), the second row of \( AB \) must be zero as well. Hence the matrix equation

\[ AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

can not hold for any \( 2 \times 2 \) matrix \( B \). Hence \( A^{-1} \) can not exist. On the other hand, if at least one of \( \delta \) and \( \gamma \) is non-zero, then

\[ A \begin{pmatrix} \delta \\ -\gamma \end{pmatrix} = 0, \quad \text{since} \quad \alpha \delta - \beta \gamma = 0. \]

In other words, \( Ax = 0 \) has a non-zero solution, which implies that \( A^{-1} \) does not exist. Otherwise the solution to \( Ax = 0 \) would be \( x = A^{-1}0 = 0 \).