1. **Section 4.1**

1. By theorem 4.1.1, solutions exist on \( \mathbb{R} \).
2. By theorem 4.1.1, solutions exist on the intervals \(( -\infty, 0)\) and \((0, \infty)\).
4. By theorem 4.1.1, solutions exist on \((0, \infty)\).
6. By theorem 4.1.1, solutions exist on \((-\infty, -2), (-2, 2),\) and \((2, \infty)\).
7. Suppose that there are constants \(c_i\) such that
   \[ c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \]
   Then
   \[
   0 = c_1(2t - 3) + c_2(t^2 + 1) + c_3(2t^2 - t) \\
   = (-3c_1 + c_2) + t(2c_1 - c_3) + t^2(c_2 + 2c_3).
   
   This is equivalent to
   \[
   \begin{pmatrix}
   -3 & 1 & 0 \\
   2 & 0 & -1 \\
   0 & 1 & 2
   \end{pmatrix}
   \begin{pmatrix}
   c_1 \\
   c_2 \\
   c_3
   \end{pmatrix}
   = \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}.
   
   But the above matrix is invertible, hence the only solution is \(c_1 = c_2 = c_3 = 0\). Thus
   \(\{f_1, f_2, f_3\}\) is linearly independent.
9. Each of the functions belongs to the vector space \(P_2\) of second-degree polynomials. Since
   \(\dim \ P_2 = 3\), the set \(\{f_1, f_2, f_3, f_4\}\) is linearly dependent.
10. We proceed as in problem 7, and reduce the problem to the invertibility of the matrix
    \[
    A = \begin{pmatrix}
    -3 & 1 & 0 & 1 \\
    2 & 0 & -1 & 1 \\
    0 & 0 & 2 & 1 \\
    0 & 1 & 0 & 0
    \end{pmatrix}.
    
    To see that \(A\) is invertible, compute \(\det(A) = 13\).
11. It is routine to check that the given functions solve the differential equation. The Wronskian is
    \[
    \begin{vmatrix}
    1 & \cos t & \sin t \\
    0 & -\sin t & \cos t \\
    0 & -\cos t & -\sin t
    \end{vmatrix} = 1.
    \]
14. It is routine to check that the given functions solve the differential equation. The Wronskian is

\[
\begin{vmatrix}
1 & t & e^{-t} & te^{-t} \\
0 & 1 & -e^{-t} & e^{-t} - te^{-t} \\
0 & 0 & e^{-t} & -2e^{-t} + te^{-t} \\
0 & 0 & -e^{-t} & 3e^{-t} - te^{-t}
\end{vmatrix} = e^{-2t}.
\]

17. Since \(10\sin^2 t + 5\cos 2t = 5 \sin^2 t - 5(\cos^2 t - \sin^2 t) = 5 = 0\), \(W(5, \sin^2 t, \cos 2t) = 0\).

18. We have

\[
L[c_1y_1 + c_2y_2] = (c_1y_1 + c_2y_2)(^{(n)}\! + p_1(t)(c_1y_1 + c_2y_2)(^{(n-1)}\! + \cdots + p_n(t)(c_1y_1 + c_2y_2) \\
= (c_1y_1^{(n)} + c_2y_2^{(n)}) + p_1(t)(c_1y_1^{(n-1)} + c_2y_2^{(n-1)}) + \cdots + p_n(t)(c_1y_1 + c_2y_2) \\
= c_1y_1^{(n)} + p_1(t)y_1^{(n-1)} + \cdots + p_n(t)y_1 + c_2y_2^{(n)} + p_1(t)y_2^{(n-1)} + \cdots + p_n(t)y_2 \\
= c_1L[y_1] + c_2L[y_2].
\]

Hence if \(L[y_1] = L[y_2] = \cdots = L[y_n] = 0\), \(L[c_1y_1 + c_2y_2 + \cdots + c_ny_n] = c_1L[y_1] + c_2L[y_2] + \cdots + c_nL[y_n] = 0\).

19. (a) Note that for \(k \leq n\),

\[
(t^n)^{(k)} = \binom{n}{k} t^{n-k}.
\]

Hence

\[
L[t^n] = \sum_{m=0}^{n} a_m \binom{n}{m} t^{n-m}.
\]

(b) Clearly \(L[e^{rt}] = e^{rt}(a_0r^n + a_1r^{n-1} + \cdots + a_n)\).

(c) One checks that each of the following are solutions: \(e^t, e^{-t}, e^{2t}, e^{-2t}\). To verify that they are a fundamental set of solutions (they are), one checks that the Wronskian is not zero.

20. (a) Expand \(W\), use the product rule, collect terms, and note that a matrix that has two identical rows has determinant equal zero.

(b) Substitute the formula

\[
y'''' = -p_1(t)y'' - p_2(t)y' - p_3(t)y
\]

into each entry of the third row of the matrix obtained in part (a), and note that a matrix that has two identical rows has determinant equal zero. Note also that when an row of a matrix is multiplied by a constant, the determinant is multiplied by the same constant.

(c) Solve the differential equation obtained in part (b).

21. We have that \(\int p_1(t) \, dt = \int 2 \, dt = 2t\). Hence by problem 20, the Wronskian is

\(Ce^{-2t}\).
2. Section 4.2

2. \(-1 + \sqrt{3}i = 2e^{2\pi i/3}\)
4. \(-i = e^{3\pi i/2}\)
8. \(1 - i = \sqrt{2}e^{-\pi i/4}\). So the square roots of \(1 - i\) are \(\sqrt{2}e^{-\pi i/8}\) and \(\sqrt{2}e^{7\pi i/8}\).

26. We need to find the roots of the characteristic polynomial:

\[ r^4 - 7r^3 + 6r^2 + 30r - 36 = 0. \]

By inspection, we see that \(r = 2, -3\) both solve the above system. To find the other two roots, we factor out \((r - 2)(r + 3)\) from above and use the quadratic formula to get \(r = 3 \pm \sqrt{3}\) as the other roots. Hence the general solution of our ODE is

\[ c_1e^{3t} + c_2e^{-2t} + c_3e^{(3+\sqrt{3})t} + c_4e^{(3-\sqrt{3})t}. \]

28. We need to find \(r\) such that

\[ r^4 + 6r^3 + 17r^2 + 22r + 14 = 0. \]

I don’t know how to do this other than ask a computer, or guess (or look in the back of the book). In any case, the roots turn out to be \(\{-1 \pm i, -2 \pm \sqrt{3}i\}\), so the general solution is

\[ c_1e^{-t}\cos t + c_2e^{-t}\sin t + c_3e^{-2t}\cos(\sqrt{3}t) + c_4e^{-2t}\sin(\sqrt{3}t). \]

29. The characteristic polynomial is \(r^3 + r = r(r^2 + 1)\), so its roots are \(\{0, \pm i\}\). Hence the general solution is

\[ y(t) = c_1 + c_2\sin t + c_3\cos t. \]

The initial conditions give

\[
\begin{align*}
0 &= y(0) = c_1 + c_3 \\
1 &= y'(0) = c_2 \\
2 &= y''(0) = -c_3
\end{align*}
\]

and solving this we obtain \(c_1 = 2, c_2 = 1,\) and \(c_3 = -2\). Thus the solution of our IVP is

\[ y(t) = 2 + \sin t - 2\cos t. \]

As \(t \to \infty\), the solution doesn’t converge as it follows the closed path \(\{2 + \sin t - 2\cos t|0 \leq t \leq 2\pi\}\).

33. We need to find \(r\) so that

\[ 0 = 2r^4 - r^3 - 9r^2 + 4r + 4 = (r - 1)(r - 2)(2r + 1)(r + 2). \]

Thus the solutions are \(r = 1, 2, -\frac{1}{2}, -2\). Thus the general solution of our ODE is

\[ y(t) = c_1e^t + c_2e^{2t} + c_3e^{-t/2} + c_4e^{-2t}. \]
The initial conditions give

\[ -2 = y(0) = c_1 + c_2 + c_3 + c_4 \]
\[ 1 = y'(0) = c_1 + 2c_2 - \frac{1}{2}c_3 - 2c_4 \]
\[ -1 = y''(0) = c_1 + 4c_2 + \frac{1}{4}c_3 + 4c_4 \]
\[ 0 = y'''(0) = c_1 + 8c_2 - \frac{1}{8}c_3 - 8c_4 \]

Solving this system we obtain

\[ c_1 = -\frac{2}{3}, c_2 = -\frac{1}{10}, c_3 = -\frac{16}{15}, c_4 = -\frac{1}{6}. \]

Therefore, the solution of our IVP is

\[ y(t) = -\frac{2}{3}e^t - \frac{1}{10}e^{2t} - \frac{16}{15}e^{-t/2} - \frac{1}{6}e^{-2t}. \]

We see that \( y(t) \to -\infty \) as \( t \to \infty \).

37. The general solution of our ODE is given by

\[ y(t) = c_1 \cos t + c_2 \sin t + c_3 e^t + c_4 e^{-t}. \]

Using \( 2 \cosh t = e^t + e^{-t} \) and \( 2 \sinh t = e^t - e^{-t} \) and a little algebra, we see that the general solution can be put into the form

\[ y(t) = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t. \]

The solution of the IVP (same routine as above) is

\[ y(t) = -\frac{1}{2} \cos t - \frac{1}{2} \sin t + \frac{1}{2} \cosh t + \frac{1}{2} \sinh t. \]

39. Follow the outline.

40. Follow the outline.