5.4: 10,16,20,24,25
5.5: 1,2,3,4,9,10,11,16,17,18,19,20,25,26,28,29,34,35

10. the characteristic polynomial is \( x^3 - 6x^2 + 9x - 4 = 0 \). This factors into \( (x - 1)^2(x - 4) \).

So the eigenvalues are 1, 1, 4. Eigenvectors are given by

\[
\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},
\]

Thus if we let \( Q = [ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 ] \) and \( \Lambda \) be the diagonal matrix with 1, 1, 4 on the diagonals, then \( A = Q\Lambda Q^{-1} \).

16. Let \( Q = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \) and \( \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \).

20. Let \( Q = \frac{1}{\sqrt{42}} \begin{bmatrix} 3\sqrt{3} & \sqrt{14} & 1 \\ 2\sqrt{3} & -\sqrt{14} & -4 \\ \sqrt{3} & -\sqrt{14} & 5 \end{bmatrix} \) and \( \Lambda = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \). Then \( A = Q\Lambda Q^{-1} \).

21. Assume \( A = Q\Lambda Q^{-1} \). Then since \( Q^{-1} = Q^T \), we have \( A = Q\Lambda Q^T \). Then \( A^T = (Q^T)^T\Lambda^T Q^T = Q\Lambda Q^T \) as desired.

24. By definition, \( \sqrt{\vec{z} \ast \vec{\bar{z}}} = (z_1\bar{z}_1 + ... + z_n\bar{z}_n)^{1/2} \). Now, by exercise 23, \( z_i\bar{z}_i = |z|^2 \) and the result follows immediately.

25. Obvious from definition of \( \vec{z} \ast \vec{\bar{w}} \).

5.5

1. 1,1,2,3,5,8

2. Using the method on pp. 359-360 and the formula they obtain for \( F_k \) we conclude that at the end of 36 months we will have 14,930,352 rabbits. Stew anyone?

3. 1,3,4,7,11,18

4. using similar reasoning to p. 360, changing \( \vec{x}_0 = [3\ 1]^T \), we conclude \( F_k = 5^{-1/2}(\lambda_1^3 - \lambda_2^3) - \lambda_2^3(\lambda_1 - 3)) \) and thus, \( F_{100} = 1.2816 \times 10^2 \).

9. \( F_{k+1} = 6T_k, S_{k+1} = F_k/2, T_{k+1} = S_k/3 \)

10. \( A = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \).

11. a. verify directly, \( A^3 = I \).
b. \([6000 \ 0 \ 0], [0 \ 3000 \ 0], [0 \ 0 \ 1000], [6000 \ 0 \ 0]\), ...not very exciting population dynamics.

16. We need to show that there is a non-zero vector in \(NS(A - I)\). Since each column of \(A\) sums to 1, each column of \(A - I\) must sum to zero. Thus each column of \(A\) belongs to the \(n-1\) dimensional subspace of \(\mathbb{R}^n\) that is orthogonal to the vector \(\vec{v} = [11...1]^T\). Any \(n\) vectors in an \(n-1\) dimensional subspace are linearly dependent. QED

17. We need to characterize \(NS(A - \lambda I)\). Since the columns of \(A\) sum to 1, the columns of \((A - \lambda I)\) sum to \(1 - \lambda\). Following the hint, adding the rows together yields the equation \((1 - \lambda)(x_1 + ... + x_n) = 0\). Since \(\lambda\) is not 1, we conclude that the components of \(\vec{x}\) add to zero.

18. If a bunch of (not all zero) numbers add up to 0, then at least one of them must be negative. Therefore, \(\vec{x}\) cannot "represent a realistic distribution", i.e., not all components can be non-negative.

3.1

1. Assume \(y = e^{rt}\). Then the differential equation implies \(r^2 + 2r - 3 = 0\) or \((r + 3)(r - 1) = 0\). So the general solution is \(y = c_1e^{-3t} + c_2e^t\).

2. Assume \(y = e^{rt}\). Then the differential equation implies \(r^2 + 3r + 2 = 0\) or \((r + 2)(r + 1) = 0\). So the general solution is \(y = c_1e^{-2t} + c_2e^{-t}\).

4. Assume \(y = e^{rt}\). Then the differential equation implies \(2r^2 - 3r + 1 = 0\) or \((2r - 1)(r - 1) = 0\). So the general solution is \(y = c_1e^{t/2} + c_2e^{t}\).

9. General solution is \(y = c_1e^{-2t} + c_2e^t\). Plugging into the initial values implies \(c_1 + c_2 = 1\) and \(-2c_1 + c_2 = 1\). Solving the equations gives \(y = e^t\). We all know what the graph looks like.

10. General solution is \(y = c_1e^{-3t} + c_2e^{-t}\). Plugging in for the initial values and solving yields \(y = \frac{1}{2}(5e^{-t} - e^{-3t})\). \(y\) goes to zero as \(t\) goes to infinity.

15. same drill, solution is given by \(y = \frac{1}{10e}(e^{-9t} + 9e^t)\). \(y\) goes to infinity as \(t\) goes to infinity.

16. same drill, solution given by \(y = \frac{e}{2}(-e^{t/2} + 3e^{-t/2})\). \(y\) goes to negative infinity as \(t\) goes to infinity.

17. we take \((r - 2)(r + 3) = r^2 + r - 6\), and obtain the equation \(y'' + y' - 6y = 0\).

18. similar drill to obtain \(y'' + 5y' + y = 0\).

19. solution is \(y = e^{t/4} + e^{-t}\). Differentiating and setting equal to 0, we see that an extremum occurs at \(e^t = 4e^{-t}\) or \(t = ln2\) yielding \(y(ln2) = 1\).

20. solving the equation yields \(y = -e^t + 3e^{t/2}\). Differentiating, setting equal to zero and solving for \(t\) yields \(t = ln(9/4)\). To find where the solution is equal to zero, we just solve \(y = 0\) for \(t\) and obtain \(t = ln9\).

25. \(y = \frac{1}{2}((1 + 2\beta)e^{-2t} + (4 - 2\beta)e^{t/2})\)

28. Integrating with respect to \(t\) yields \(t^2y' = t + c_1\) or \(y' = 1/t + c_1/t^2\). Integrating term by term yields \(y = ln|t| + c_1/t + c_2\).

29. letting \(v = y'\) and rearranging terms, we obtain \(tv' + v = 1\). Integrating yields \(v = y' = 1 + c_1/t\). Integrating again yields \(y = t + c_1 ln(t) + c_2\).

34. making the substitution as suggested we obtain \(yvv' + v^2 = 0\). We know treat \(y\) as the independent variable (like \(t\)). Integrating, we obtain \(v = 1/y + c_1\). Now remembering that \(v = y'\)
we integrate again obtaining $y = c_2 \sqrt{t + c_1}$.

35. This equation says $y'' = -y$. By inspection we realize that $\sin$ and $\cos$ satisfy this equation. thus the general solution is $y = c_1 \cos t + c_2 \sin t$. 