1 Section 5.1

(2) The determinant of a $1 \times 1$ matrix is just the number in it; in this case it is $-5.7$. Since this is not zero, the matrix is invertible.

(8) The matrix is: $egin{pmatrix} -1.1 & 2.9 \\ -2.2 & 5.8 \end{pmatrix}$

It’s determinant is just $-1.1 \times 5.8 - 2.9 \times (-2.2) = 0$. Since this is zero the matrix is not invertible.

(12) We have:

$$
\begin{vmatrix} 3 & -1 & -2 \\ 3 & 7 & 0 \\ 8 & 1 & 1 \end{vmatrix} = 3 \begin{vmatrix} 7 & 0 \\ 8 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 3 & 0 \\ 8 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 7 \\ 8 & 1 \end{vmatrix}
$$

This amounts to 131. Since this is not zero, the matrix is invertible.

(26) The matrix is $egin{pmatrix} 4x & 5 \\ x & x - 1 \end{pmatrix}$.

Its determinant is $4x \times (x - 1) - 5x$. This is zero exactly when $x=0$ or $x=9/4$, so for these values the matrix is singular, and for all others it is invertible.

(30) Evaluate the determinant as in problem 12 (or however you like) to get the polynomial $det(A) = x^3 - x^2 - 19x + 19$. There actually is a formula for the roots of a cubic but it is rather nasty and anytime you get one of these in the book there is usually an obvious factor by which you can divide to get a simpler equation; in this case the polynomial factors as $(x - 1) \times (x^2 - 19)$. Thus the values of $x$ for which the matrix is not invertible are $x=1$ and $x = \sqrt{19}$ and $x = -\sqrt{19}$.

(32) Usually if something is not true in linear algebra in general it is almost never true and consequently guessing simple examples is a highly effective strategy. In this $egin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ works. Det of the left hand side is 0+0 and det of the right hand side is 1.

(34) We have: $det(AB^{-1}C^2) = det(A)det(B^{-1})det(C^2) = det(A)(det(B))^{-1}(det(C))^2 = ab^{-1}c^2$.

(39) The (5.3) definition just says is is $ad-bc$. So we check what the other gives us: It is $ad-bc$ also because the cofactor $C_{11}$ is just $d$ and $C_{21}$ is $-c$. 

1
2 Section 5.2

(2) There is no point doing the simple calculations here. The eigenvalues you should get are 3 and 8.

(7) In this case we have $Av = 0 = 0v$, so $v$ is an eigenvector with eigenvalue 0.

(8) Multiplying a matrix by the n by 1 vector $[11...1]^T$ just gives the sum of the columns, in this case the same vector but with all n’s instead of ones, so the eigenvalue is n.

(9) Each of the vectors when multiplied by the matrix just takes a row of the matrix and subtracts from it the previous row. Since these are the same, the result is zero. They are linearly independent because if we have $a_2v_2 + ... + a_nv_n = 0$ then looking at the first component gives $a_2 = 0$, then look at the second component, etc, so get that all the a’s must be zero.

(10) The vector from (8) has eigenvalue n and the vectors from (9) have eigenvalue 0.

(23) We have $A^2v = \lambda Av = \lambda^2v$, so the eigenvalue is $\lambda^2$.

(24) We have $A^k v = \lambda A^{k-1}v = \lambda^2 A^{k-2}v = ... = \lambda^k v$, so $\lambda^k$ is the eigenvalue.

(29) Suppose a is an eigenvalue of B with eigenvector v. Then by (24) $B^k v = a^k v$, but since $B^k$ is zero and v is not, we must have $a^k = 0$, so a is zero too.

(30) Answer: 0 and 1. Reason: Let $v$ be any eigenvector of C with eigenvalue a. Then $C^2$ has v as an eigenvector with eigenvalue $a^2$ by (23). But $C^2 v = Cv = av$ so $a^2 = a$, so a must be 1 or 0.

(34) The general matrix solves the problem for the specific matrix so I will just do that. I will assume a and b are not both zero (then all eigenvalues are 0 and all nonzero vectors are eigenvectors). Look for roots of det(xI-A)=0, which in this case is the polynomial in x given by $x^2 - (a^2 + b^2)$. This has roots $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$. Thus the first eigenvector is any nonzero vector in the nullspace of $\begin{pmatrix} \sqrt{a^2 + b^2} & -b \\ -b & \sqrt{a^2 + b^2} \end{pmatrix}$. Chose the second component to be equal to b and then for the first component we are forced to pick $\sqrt{a^2 + b^2} + a$. The other one is similar: it is $(\sqrt{a^2 + b^2} - a, -b)$. 
3 Section 5.3

(2) The matrix $S$ is the matrix whose columns are the linearly independent eigenvectors, in this case $S = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$.

The matrix $\Lambda$ is the matrices with eigenvalues on the diagonal - it is important that they be listed in the same order as the corresponding eigenvectors. In this case $\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$.

Finding $A$ is just a calculation using the formula given, $A = \begin{pmatrix} 14 & -30 \\ 5 & -11 \end{pmatrix}$.

(8) A 2 by 2 matrix is diagonalizable exactly if we can find 2 linearly independent eigenvectors. We when try to do this we first look for eigenvalues, i.e., solutions to the characteristic equation which is this case is just $x^2$ and the only root is 0, so this eigenspace is just the nullspace of the matrix. But the nullspace is evidently not everything and hence has dimension less than the required two, so the matrix is not diagonalizable.

(10) Like (8) but now we just need to know that there are less than 3 linearly independent eigenvectors. The characteristic polynomial is just $(x - 5)^3$, so 5 is the only eigenvalue. Its eigenspace is the nullspace of $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Since the rank is two, the dim of the NS is 1, which is less than the required three so the matrix is not diagonalizable.

(14) I assume they really mean to say they want the new $\Lambda$ also to be diagonal. The diagonal matrix must have eigenvalues on the diagonal, so the only thing we can change here is the order. The matrix $S$ must then have columns eigenvectors so these can be changed by choosing different eigenvectors with the same eigenvalue. Thus a new pair is, for example, $S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Lambda_1 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$.

(18) A matrix is diagonalizable exactly if there exists a basis of eigenvectors. In this case we need two linearly independent eigenvectors. First we find eigenvalues. The characteristic equation is $(x - 2)^2 = 0$, so the only eigenvalue is 2. But matrix $2I-A$ is not zero, so has nullspace of dim less than 2, and we cannot have enough eigenvectors, so the matrix is not diagonalizable.

(27) We have $A^k = (S\Lambda S^{-1})^k = (S\Lambda S^{-1})^*(S\Lambda S^{-1})^* \cdots (S\Lambda S^{-1})^*$. Now
all of the $S$ and $S^{-1}$ terms in the middle cancel out and we are left with $SA^kS^{-1}$.

(32) Let $S$ be the matrix which diagonalizes both - it works for both because it is the matrix of eigenvectors and we assume $A$ and $B$ have the same eigenvectors. Thus $A = SA_1S^{-1}$ and $B = SA_2S^{-1}$, and $AB = SA_1S^{-1}SA_2S^{-1} = SA_1A_2S^{-1}$ = (because diagonal matrices always commute) $SA_2A_1S^{-1} = SA_2S^{-1}SA_1S^{-1} = BA$. 
